PHYSICS 410

THE WAVE EQUATION

1-d Wave Equation

- Continuum equation (non-dimensionalized, $c = 1$)

$$u(x,t)_{tt} = u_{xx}\,, \quad u(x,0) = u_0(x)\,, \quad u_t(x,0) = v_0(x)\,, \quad u(0,t) = u(1,t) = 0$$

- Interior FDA

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \qquad j = 2, 3, \cdots, J-1$$

- Truncation error

$$\tau = \frac{1}{12}\Delta t^2 (u_{tttt})_j^n - \frac{1}{12}\Delta x^2 (u_{xxxx})_j^n + O(\Delta t^4) + O(\Delta x^4) = O(\Delta t^2, \Delta x^2) = O(h^2)$$

- Discrete boundary conditions

$$u_1^{n+1} = u_J^{n+1} = 0$$

- Discrete initial conditions

$$
\begin{aligned}
u_j^1 \quad &, \qquad j = 1, 2, \cdots, J \\
u_j^2 \quad &, \qquad j = 1, 2, \cdots, J
\end{aligned}
$$

- First time level comes from $u_0(x)$

$$u_j^1 = u_0(x_j)$$

- u_j^2 must be initialized up to and including terms of order $O(\Delta t^2)$:

$$
\begin{aligned}
u_j^2 &= u_j^1 + \Delta t\,(u_t)_j^1 + \frac{1}{2}\Delta t^2\,(u_{tt})_j^1 + O(\Delta t^3) \\
&= u_j^1 + \Delta t\,(u_t) + \frac{1}{2}\Delta t^2\,(u_{xx})_j^1 + O(\Delta t^3) \\
&\approx u_0(x_j) + \Delta t\, v_0(x_j) + \frac{1}{2}\Delta t^2 u_0''(x_j)
\end{aligned}
$$

- Stability analysis

- First rewrite difference equation in "first order" form; introduce $v_j^n = u_j^{n-1}$:

$$\begin{aligned} u_j^{n+1} &= 2u_j^n - v_j^n + \lambda^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right), \\ v_j^{n+1} &= u_j^n, \end{aligned}$$

or, in matrix form

$$\left[\begin{array}{c} u \\ v \end{array}\right]^{n+1} = \left[\begin{array}{cc} 2+\lambda^2 D^2 & -1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right]^{n}$$

- Under Fourier transformation this becomes

$$\left[\begin{array}{c} \tilde{u} \\ \tilde{v} \end{array}\right]^{n+1} = \left[\begin{array}{cc} 2-4\lambda^2 \sin^2 \xi/2 & -1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} \tilde{u} \\ \tilde{v} \end{array}\right]^{n}$$

- We must now determine conditions under which above matrix has eigenvalues that lie within or on the unit circle

- Characteristic equation (whose roots are the e.v.'s) is

$$\begin{vmatrix} 2 - 4\lambda^2 \sin^2(\xi/2) - \mu & -1 \\ 1 & -\mu\,, \end{vmatrix} = 0$$

or

$$\mu^2 + \left(4\lambda^2 \sin^2 \frac{\xi}{2} - 2\right)\mu + 1 = 0\,.$$

- Equation has roots

$$\mu(\xi) = \left(1 - 2\lambda^2 \sin^2 \frac{\xi}{2}\right) \pm \left(\left(1 - 2\lambda^2 \sin^2 \frac{\xi}{2}\right)^2 - 1\right)^{1/2}.$$

- Need sufficient conditions for

$$|\mu(\xi)| \le 1,$$

or equivalently

$$|\mu(\xi)|^2 \le 1.$$

- Can write
$$\mu(\xi) = (1-Q) \pm ((1-Q)^2 - 1)^{1/2}\,,$$
where
$$Q \equiv 2\lambda^2 \sin^2\frac{\xi}{2}\,,$$
is *real* and *non-negative* ($Q \geq 0$).

- Three cases to consider:

1. $(1-Q)^2 - 1 = 0$,
2. $(1-Q)^2 - 1 < 0$,
3. $(1-Q)^2 - 1 > 0$.

- Case 1: $Q = 0$ or $Q = 2$; in both cases $|\mu(\xi)| = 1$

- Case 2: $((1-Q)^2 - 1)^{1/2}$ is purely imaginary, so

$$|\mu(\xi)|^2 = (1-Q)^2 + 1 - (1-Q)^2 = 1 \qquad (1)$$

- Case 3: $(1-Q)^2 - 1 > 0 \longrightarrow (1-Q)^2 > 1 \longrightarrow Q > 2$, then

$$1 - Q - ((1-Q)^2 - 1)^{1/2} < -1\,,$$

so stability criterion will *always* be violated.

- Thus, necessary condition for Von-Neumann stability is

$$(1-Q)^2 - 1 \leq 0 \longrightarrow (1-Q)^2 \leq 1 \longrightarrow Q \leq 2\,.$$

- But $Q \equiv 2\lambda \sin^2(\xi/2)$ and $\sin^2(\xi/2) \leq 1$, so have

$$\lambda \equiv \frac{\Delta t}{\Delta x} \leq 1\,,$$

for stability of our scheme