PHYSICS 410

THE RUNGE-KUTTA-FEHLBERG METHOD

- Uses two RK methods, one $O(h^4)$, one $O(h^5)$

- As noted previously, for a given order there are in general an infinite number of possible RK methods

- Clever approach due to Fehlberg is to choose methods so that they use identical intermediate function evaluations f_0, f_1, f_2, f_3, f_4 and, for the $O(h^5)$ method, f_5

$$
\begin{aligned}
f_0 &= f(x_0, y_0) \\
f_1 &= f\left(x_0 + \frac{h}{4}, y_0 + \frac{h}{4} f_0\right) \\
f_2 &= f\left(x_0 + \frac{3h}{8}, y_0 + \frac{3h}{32} f_0 + \frac{9h}{32} f_1\right) \\
f_3 &= f\left(x_0 + \frac{12h}{13}, y_0 + \frac{1932h}{2197} f_0 - \frac{7200h}{2197} f_1 + \frac{7296h}{2197} f_2\right) \\
f_4 &= f\left(x_0 + h, y_0 + \frac{439h}{216} f_0 - 8h f_1 + \frac{3680h}{513} f_2 - \frac{845h}{4104} f_3\right) \\
f_5 &= f\left(x_0 + \frac{h}{2}, y_0 - \frac{8h}{27} f_0 + 2h f_1 - \frac{3544h}{2565} f_2 + \frac{1859h}{4104} f_3 - \frac{11h}{40} f_4\right)
\end{aligned}
$$

- With these definitions, the $O(h^4)$ approximation is

$$\boxed{y = y_0 + h\left(\frac{25}{216}f_0 + \frac{1408}{2565}f_2 + \frac{2197}{4104}f_3 - \frac{1}{5}f_4\right)}$$

and the $O(h^5)$ approximation is

$$\boxed{\tilde{y} = y_0 + h\left(\frac{16}{135}f_0 + \frac{6656}{12825}f_2 + \frac{28561}{56430}f_3 - \frac{9}{50}f_4 + \frac{2}{55}f_5\right)}$$

- Can compute the approximate error directly

$$e_{\text{est}} = \tilde{y} - y = h\left(\frac{1}{360}f_0 - \frac{128}{4275}f_2 - \frac{2197}{75240}f_3 + \frac{1}{50}f_4 + \frac{2}{55}f_5\right)$$