PHYSICS 410

THE MATLAB ode45 INTEGRATOR AND APPLICATIONS

- As with Runge-Kutta-Fehlberg, uses two RK methods, one $O(h^4)$, one $O(h^5)$; specifically, the Dormand-Prince pair, which minimizes the $O(h^5)$ solution error, whereas the Fehlberg technique minimizes the $O(h^4)$ error

- Recommended "first choice" integrator: if solution with this proves problematic, MATLAB has many more algorithms available

- Calling sequences (partial list):

```
[tout, yout] = ode45(odefun, tspan, y0)
[tout, yout] = ode45(odefun, tspan, y0, options)
```

```
[TOUT,YOUT] = ODE45(ODEFUN,TSPAN,Y0) with TSPAN = [T0 TFINAL]
integrates the system of differential equations y' = f(t,y)
from time T0 to TFINAL with initial conditions Y0. ODEFUN is
a function handle. For a scalar T and a vector Y, ODEFUN(T,Y)
must return a column vector corresponding to f(t,y). Each row
in the solution array YOUT corresponds to a time returned in
the column vector TOUT.  To obtain solutions at specific times
T0,T1,...,TFINAL (all increasing or all decreasing), use TSPAN =
[T0 T1 ... TFINAL].
```

```
[TOUT,YOUT] = ODE45(ODEFUN,TSPAN,Y0,OPTIONS) solves as above
with default integration properties replaced by values in
OPTIONS, an argument created with the ODESET function. See
ODESET for details. Commonly used options are scalar relative
error tolerance 'RelTol' (1e-3 by default) and vector of
absolute error tolerances 'AbsTol' (all components 1e-6
by default).  If certain components of the solution must be
non-negative, use ODESET to set the 'NonNegative' property to
the indices of these components.
```

- `Options` example (for ODE system with two components):

```
>> options = odeset('AbsTol', [1.0e-5, 1.0e-6], ...
                    'RelTol', [1.0e-5])
```

Example: Simple Harmonic Motion

- Governing differential equation (unit angular frequency)

$$\frac{d^2y(t)}{dt^2} = -y$$

- Cast in canonical first-order form by defining dependent variables

$$y_1(t) \equiv y(t) \qquad y_2(t) \equiv \frac{dy_1}{dt}$$

- Then equation of motion becomes the system

$$\frac{dy_1}{dt} = y_2 \qquad \frac{dy_2}{dt} = -y_1$$

subject to initial conditions

$$y_1(0) = y(0) \qquad y_2(0) = \frac{dy}{dt}(0)$$

Function `fcn_sho`

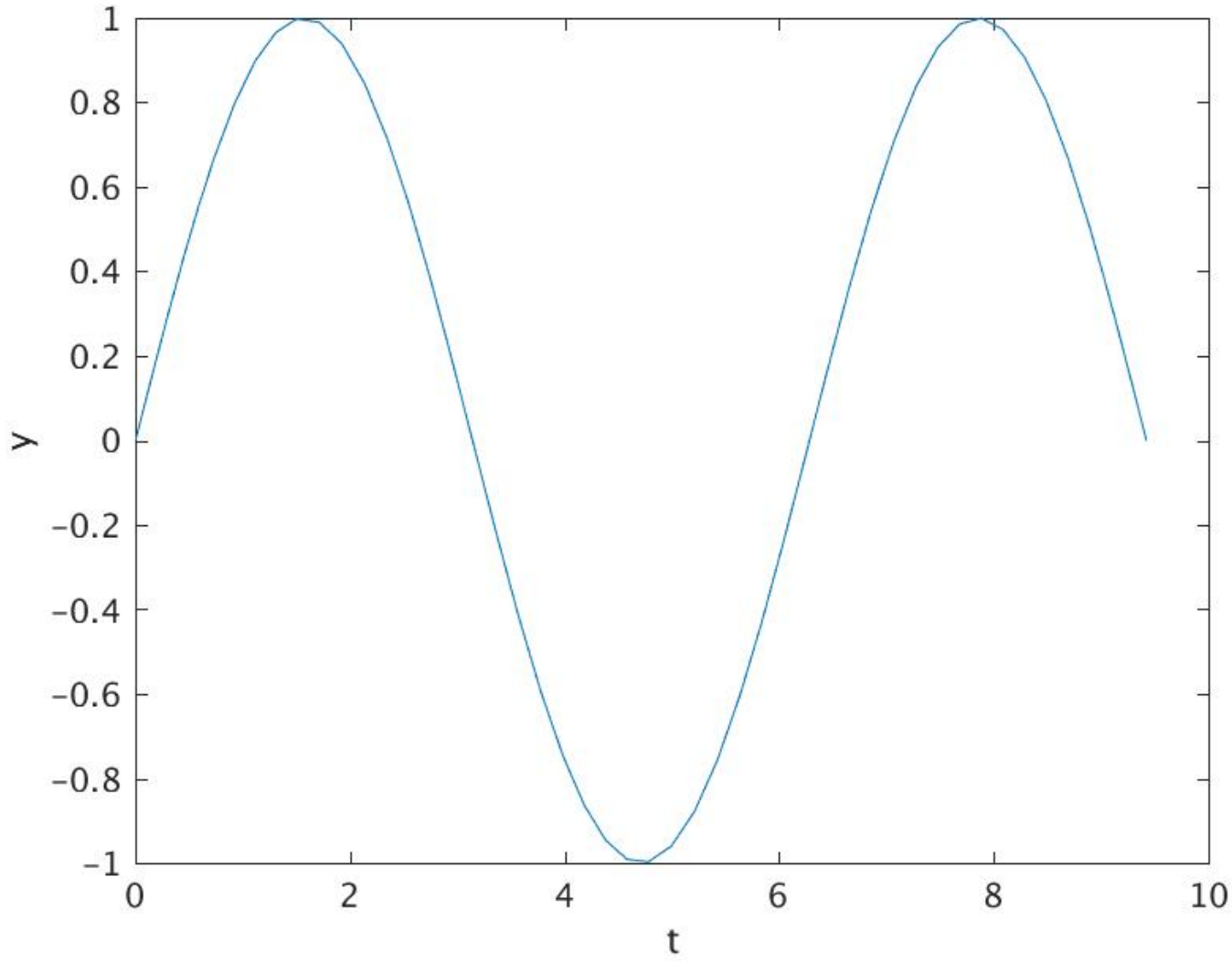
```
function dydt = fcn_sho(t, y)
% fcn_sho: Derivatives function for simple harmonic motion with unit
% angular frequency. Note that the function must return a COLUMN
% vector.
   dydt = zeros(2,1);
   dydt(1) =  y(2);
   dydt(2) = -y(1);
end
```

Script ode45_sho.m

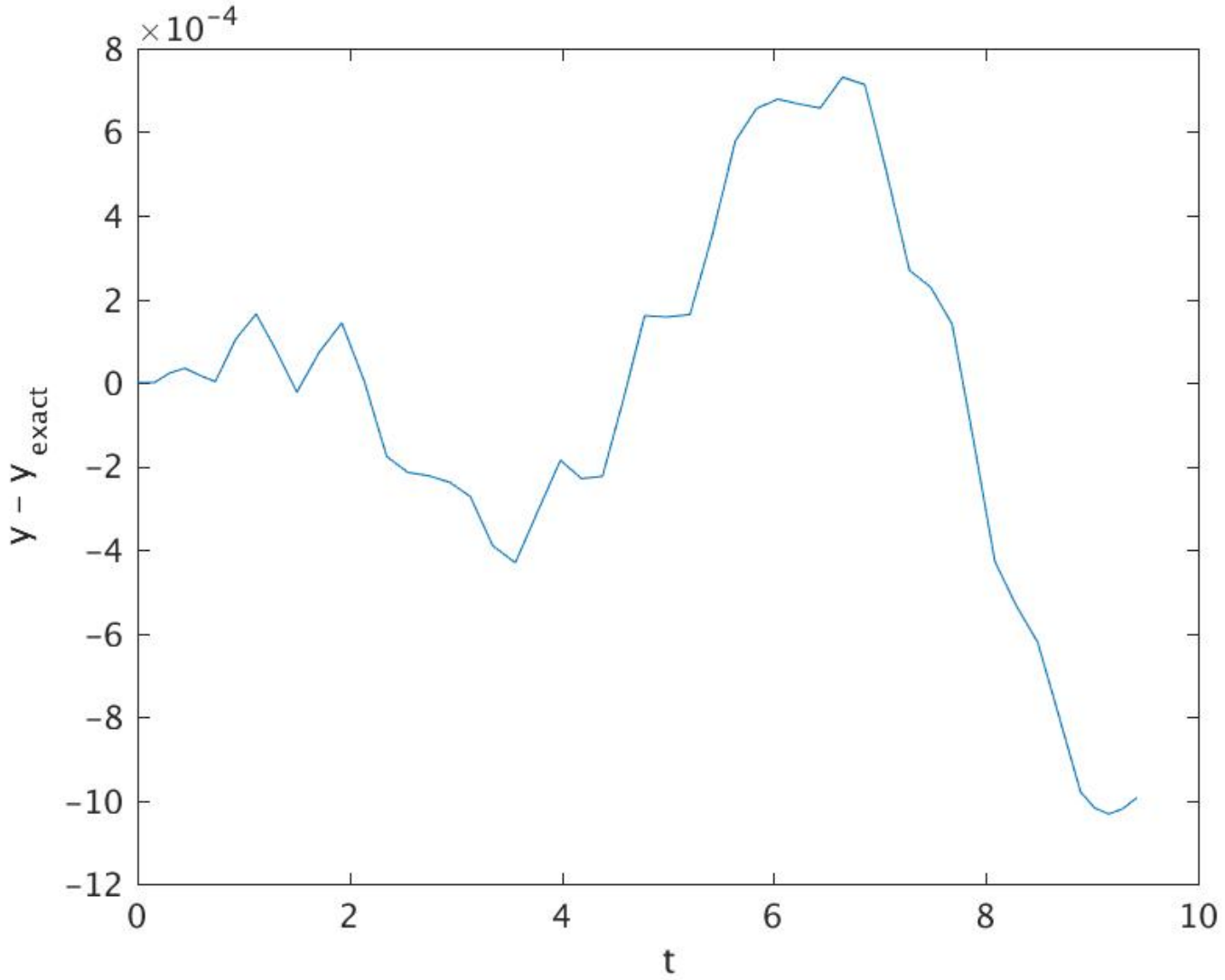
```
% ode45_sho: Integrates equations of motion for simple harmonic oscillator
% using ODE45

% Integrate on the domain 0 <= t <= 3 pi, with initial conditions
%
% y_1(0) = x(0) = 0
% y_2(1) = v(0) = 1
%
% corresponding to the exact solution
%
% y(t) = sin(t)

% Integrate with default parameters ...
[tout yout] = ode45(@fcn_sho, [0.0 3.0*pi], [0.0 1.0]);

figure(1);
plot(tout, yout(:,1));
xlabel('t');
ylabel('y');
print('sho_y.jpg', '-djpeg');
```

```
figure(2);
yxct = sin(tout);
plot(tout, yout(:,1) - yxct);
xlabel('t');
ylabel('y - y_{\rm exact}');
print('sho_error.jpg', '-djpeg');

figure(3);
iplot = 1;
% Integrate with a variety of error tolerances ...
for tol = [1.0e-3, 1.0e-5, 1.0e-7, 1.0e-9]
   options = odeset('AbsTol', [tol, tol], 'RelTol', tol);
   [tout yout] = ode45(@fcn_sho, [0.0 3.0*pi], [0.0 1.0], options);
   yxct = sin(tout);
   subplot(2,2,iplot);
   plot(tout, yout(:,1) - yxct);
   xlabel('t');
   ylabel('Error');
   label = sprintf('Tolerance = %g', tol);
   legend(label,'Location','South');
   iplot = iplot + 1;
end
print('sho_var_tolerance.jpg', '-djpeg');
```

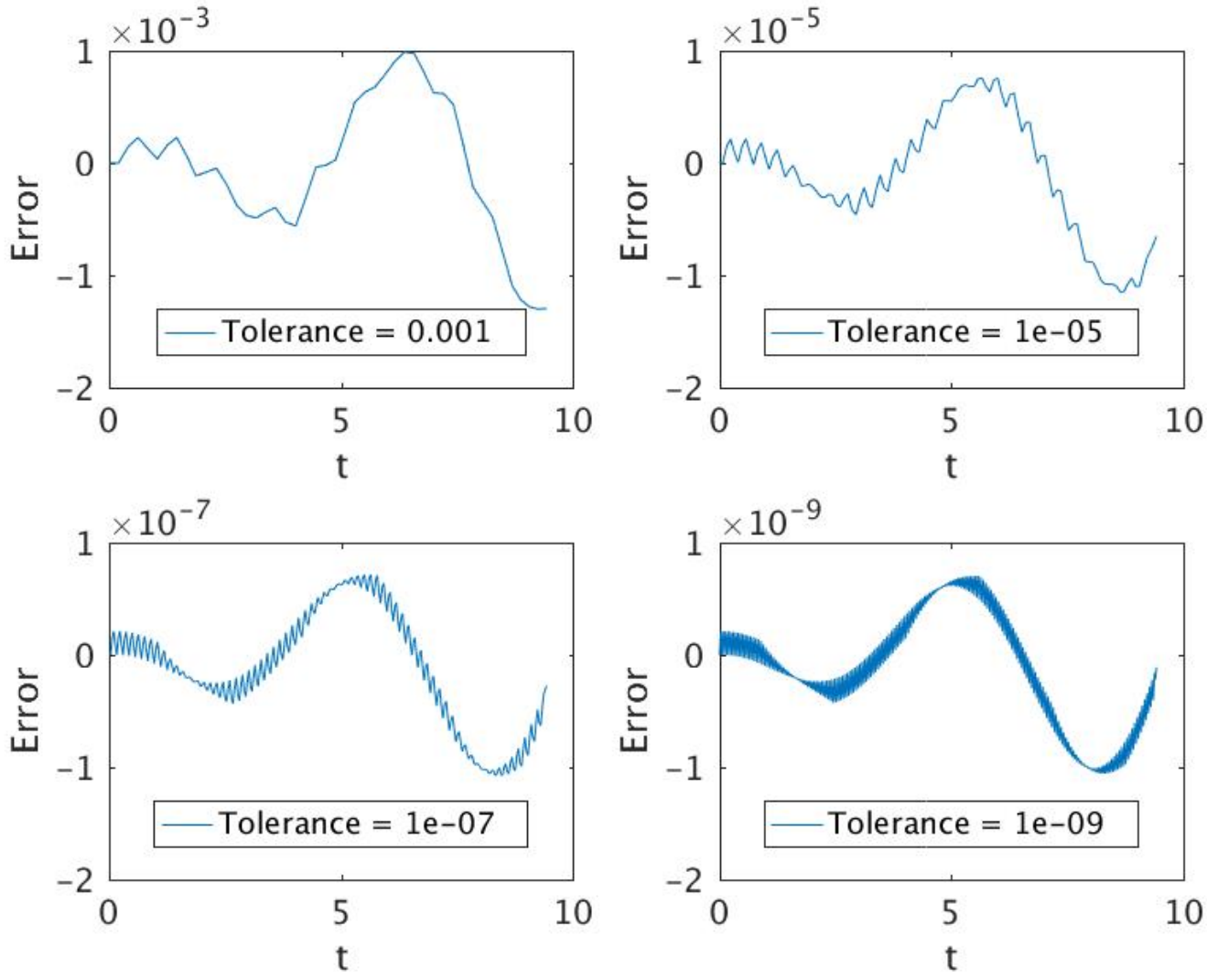
- Plot of approximate solution versus time



- Plot of error versus time (using default `ODE45` parameters: `AbsTol = 1.0e-6`, `RelTol = 1.0e-3`)



- Plot of error versus time using various tolerances: `AbsTol = RelTol = Tolerance`)



Independent residual evaluation

- First, rewrite ODE in form

$$\frac{d^2y(t)}{dt^2} + y(t) = 0 \tag{1}$$

- Next, using `ODE45`, generate solution $\hat{u}(t^h, \epsilon)$ on a level-ℓ uniform mesh:

$$t_n^h = 0, h, 2h, \cdots t_{\max}$$

with (assuming $t_{\min} = 0$)

$$h = \frac{t_{\max}}{2^\ell}$$

- Then, apply $O(h^2)$ finite-difference discretization of (1) to $\hat{y}$ to compute *residual* R_n:

$$R_n \equiv \frac{\hat{y}_{n+1} - 2\hat{y}_n + \hat{y}_{n-1}}{h^2} + \hat{y}_n \qquad n = 1, 2, \cdots 2^\ell - 1$$

- Should find that RMS value (ℓ_2 norm) of R_n is an $O(h^2)$ quantity:

$$\left[\frac{\sum_n |R_n|^2}{2^\ell - 1}\right]^{\frac{1}{2}} \equiv \|\mathbf{R}\|_2 = O(h^2)$$

- In particular, consider computing R_n at three separate levels of 2:1-related discretization, $h_1 = h_\ell, h_2 = 2h_1 = h_{\ell-1}, h_3 = 4h_1 = h_{\ell-2}$. From a single level ℓ computation in MATLAB, these can be generated by taking every element, every second element and every fourth element of the output vector from the ODE integrator.

- Then, by plotting

$$16R^\ell, 4R^{\ell-1}, R^{\ell-2}$$

on a single graph, convergence of the independent residual evalulator will appear as near-coincidence of the 3 curves.

Independent residual evaluation: script file

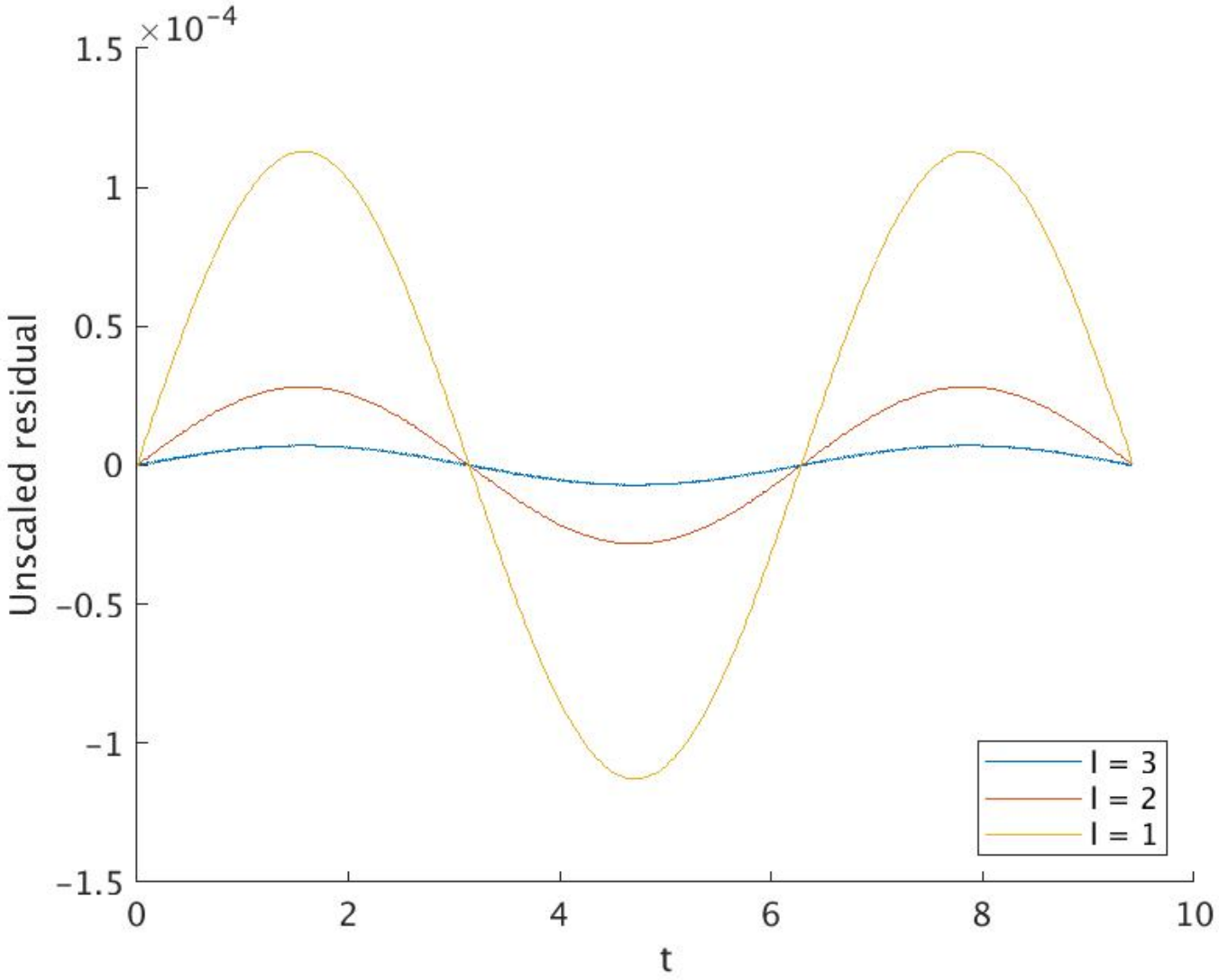
ode45_sho_ir.m

```
% ode45_sho_ir: Integrates equations of motion for simple harmonic
% oscillator using ODE45 and evaluates independent residuals using
% a finite difference approximation

% Integrate on the domain 0 <= t <= 3 pi, with initial conditions
%
% y_1(0) = x(0) = 0
% y_2(1) = v(0) = 1
%
% corresponding to the exact solution
%
% y(t) = sin(t)

% Integrate with default parameters, except use an uniform mesh of
% output times of length 2^10 + 1 and a stringent error tolerance
tol = 1.0e-10;
options = odeset('AbsTol', [tol, tol], 'RelTol', tol);
[tout yout] = ode45(@fcn_sho, linspace(0.0,3.0*pi,1025), [0.0 1.0], ...
   options);
```

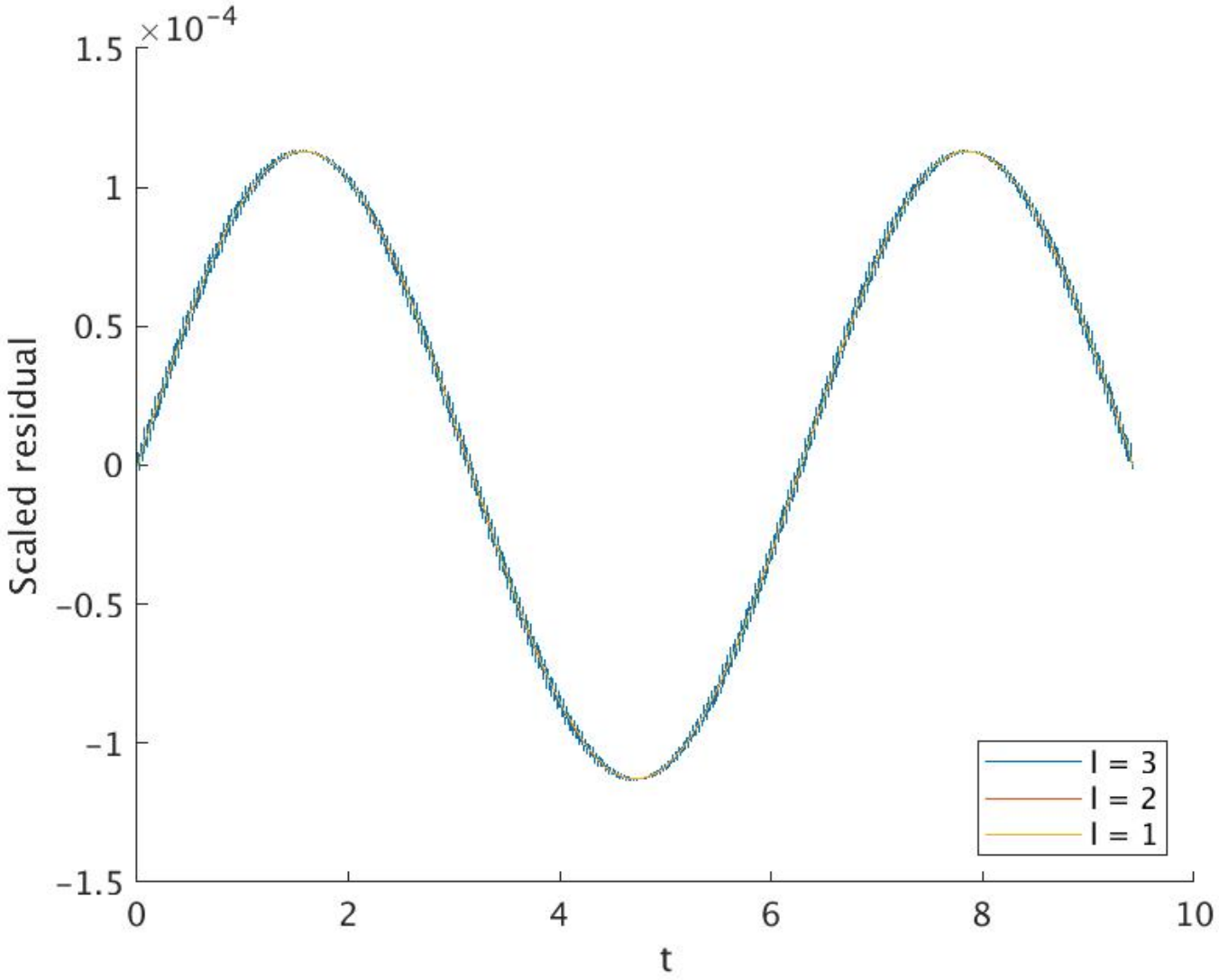
```
% Compute and plot three levels of scaled independent residuals
figure(1); clf; figure(2); clf
yir = yout(:,1);
tir = tout;
for ll = 1 : 3
   dt = tir(2) - tir(1);
   ir = zeros(1,length(yir));
   for j = 2 : length(yir) - 1
      ir(j) = (yir(j-1) - 2.0 * yir(j) + yir(j+1)) / dt^2 + yir(j);
   end
   % Plot unscaled residuals
   figure(1);
   hold on;
   plot(tir, ir);
   % Plot scaled residuals
   figure(2);
   hold on;
   plot(tir, 4.0^(3 - ll) * ir);
   yir = yir(1:2:end);
   tir = tir(1:2:end);
end;
```

```
figure(1);
xlabel('t');
ylabel('Unscaled residual');
legend('l = 3', 'l = 2', 'l = 1', 'Location', 'SouthEast');
print('sho_ir_unscaled.jpg', '-djpeg');
figure(2);
xlabel('t');
ylabel('Scaled residual');
legend('l = 3', 'l = 2', 'l = 1', 'Location', 'SouthEast');
print('sho_ir_scaled.jpg', '-djpeg');
```

- Plot of unscaled independent residuals (`AbsTol = 1.0e-10, RelTol = 1.0e-10`)



- Plot of scaled independent residuals (`AbsTol = 1.0e-10, RelTol = 1.0e-10`)

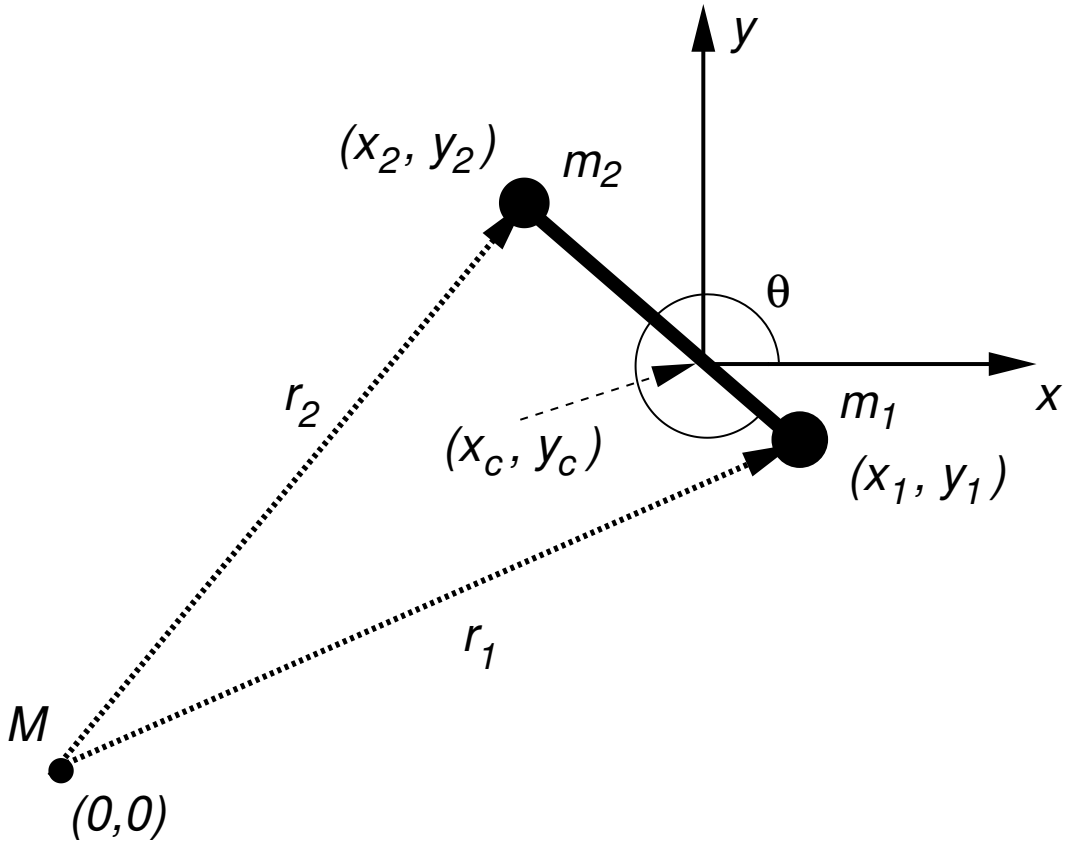


Orbiting Dumbell

- Following Giordano *Computational Physics*, Section 4.6

- *Background:* With the exception of Hyperion, which is one of Saturn's satellites, all of the moons in the solar system are "spin-locked"; a moon which is spin-locked has a rotational frequency, ω about its own spin axes which is the same as its orbital frequency, Ω. The supposed mechanism by which the spin-locking comes about is somewhat involved; however the point is that Hyperion is somehow exceptional—study of its $\omega(t)$ suggests that it is tumbling *chaotically* in its orbit about Saturn, which is presumably due to both to its peculiar shape (like that of an egg), and the fact that it is in an elliptical orbit about Saturn.

- To investigate the effects of a non-spherical distribution of mass on a satellite's spin as it orbits its parental body, we consider the model of an "orbiting dumbbell".



- Consider two test masses, m_1, m_2 connected by a massless rigid rod of length d, in orbit about a mass, $M \gg m_1, m_2$

- Let $(x_i, y_i), i = 1, 2$ be the coordinates of the two test masses, let (x_c, y_c) be the coordinates of the dumbbell's center of mass, and let θ be the angle the rod makes with the x-axis.

- Defining

$$\mu \equiv \frac{m_2}{m_1 + m_2}$$

the distances of the masses from the center of mass are

$$d_1 = \mu\, d \qquad\qquad d_2 = (1 - \mu) d$$

then

$$x_i = x_c \pm d_i \cos\theta \qquad\qquad y_i = y_c \pm d_i \sin\theta$$

- The moment of inertia of the dumbbell about (x_c, y_c) is

$$I = m_1 {d_1}^2 + m_2 {d_2}^2 = \frac{m_1 {m_2}^2}{(m_1 + m_2)^2} d^2 + \frac{m_2 {m_1}^2}{(m_1 + m_2)^2} d^2 = \frac{m_1 m_2}{m_1 + m_2} d^2$$

- The equations of motion for the body are

$$(m_1 + m_2)\, \mathbf{a_c} = (m_1 + m_2)\, \ddot{\mathbf{r}}_c = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$I\alpha = I\ddot{\theta} = \sum \tau = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_2 \times \mathbf{F}_2$$

where

$$\mathbf{F}_1 = -\frac{GMm_1}{{r_1}^3}\left[x_1, y_1\right]$$

$$\mathbf{F}_2 = -\frac{GMm_2}{{r_2}^3}\left[x_2, y_2\right]$$

are the gravitational forces acting on m_1 and m_2 respectively.

- The translational equations yield:

$$(m_1 + m_2)\ \ddot{x}_c = -GM\left(\frac{m_1}{{r_1}^3}x_1 + \frac{m_2}{{r_2}^3}x_2\right)$$

$$(m_1 + m_2)\ \ddot{y}_c = -GM\left(\frac{m_1}{{r_1}^3}y_1 + \frac{m_2}{{r_2}^3}y_2\right)$$

or

$$\ddot{x}_c = -GM\left(\frac{1-\mu}{{r_1}^3}x_1 + \frac{\mu}{{r_2}^3}x_2\right)$$

$$\ddot{y}_c = -GM\left(\frac{1-\mu}{{r_1}^3}y_1 + \frac{\mu}{{r_2}^3}y_2\right)$$

- The rotational equation gives:

$$
\begin{aligned}
I\ddot{\theta} &= \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_2 \times \mathbf{F}_2 \\
&= -GM\frac{m_1}{{r_1}^3}d_1\left(\cos\theta y_1 - \sin\theta x_1\right) + GM\frac{m_2}{{r_2}^3}d_2\left(\cos\theta y_2 - \sin\theta x_2\right) \\
&= -GM\frac{m_1}{{r_1}^3}d_1\left(\cos\theta y_c - \sin\theta x_c\right) + GM\frac{m_2}{{r_2}^3}d_2\left(\cos\theta y_c - \sin\theta x_c\right) \\
&= GM\left(\frac{m_2}{{r_2}^3}d_2 - \frac{m_1}{{r_1}^3}d_1\right)\left(\cos\theta y_c - \sin\theta x_c\right) \\
&= GM\frac{m_1 m_2}{m_1 + m_2}d\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)\left(\sin\theta x_c - \cos\theta y_c\right)
\end{aligned}
$$

so

$$
\ddot{\theta} = \frac{GM}{d}\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)\left(\sin\theta x_c - \cos\theta y_c\right)
$$

Summarizing, we have:

$$\begin{aligned}
\ddot{x}_c &= -GM\left(\frac{1-\mu}{{r_1}^3}x_1 + \frac{\mu}{{r_2}^3}x_2\right) \\
\ddot{y}_c &= -GM\left(\frac{1-\mu}{{r_1}^3}y_1 + \frac{\mu}{{r_2}^3}y_2\right) \\
\ddot{\theta} &= \frac{GM}{d}\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)(\sin\theta x_c - \cos\theta y_c)
\end{aligned}$$

where

$$\begin{aligned}
\mu &\equiv \frac{m_2}{m_1 + m_2} \\
d_1 &= \mu\, d \\
d_2 &= (1-\mu)d \\
x_i &= x_c \pm d_i \cos\theta \\
y_i &= y_c \pm d_i \sin\theta \\
{r_i}^3 &= \left({x_i}^2 + {y_i}^2\right)^{3/2}
\end{aligned}$$

- The total (conserved) energy of the system is

$$E_{\rm tot} = T_{\rm trans} + T_{\rm rot} + V_{\rm grav}$$

where

$$\begin{aligned} T_{\rm trans} &\equiv \frac{1}{2}\left(m_1 + m_2\right)\left(\dot{x}_c^2 + \dot{y}_c^2\right) \\ T_{\rm rot} &\equiv \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\, d^2\, \dot{\theta}^2 \\ V_{\rm grav} &\equiv -GM\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) \end{aligned}$$

- We can further simplify the system by adopting units in which

$$GM = 1$$

Example: Orbiting Dumbbell (Model for Hyperion)
Function: fcn_dumb

```
function dydt = fcn_dumb(t, y)
%  Function fcn_dumb evaluates derivatives for orbiting dumbbell
%  problem.
   global mu  d;

   dydt = zeros(6,1);

   xc = y(1);
   yc = y(3);
   th = y(5);
   om = y(6);

   d1 = mu * d;
   d2 = (1.0 - mu) * d;

   x1 = xc + d1 * cos(th);
   y1 = yc + d1 * sin(th);
   x2 = xc - d2 * cos(th);
   y2 = yc - d2 * sin(th);
```

```
    r1m3 = 1.0 / (x1^2 + y1^2)^1.5;
    r2m3 = 1.0 / (x2^2 + y2^2)^1.5;

    c1 = -1.0;
    c2 = 1.0 / d;

    dydt(1) = y(2);
    dydt(2) = c1 * ((1.0 - mu) * x1 * r1m3 + ...
                            mu  * x2 * r2m3);
    dydt(3) = y(4);
    dydt(4) = c1 * ((1.0 - mu) * y1 * r1m3 + ...
                            mu  * y2 * r2m3);
    dydt(5) = y(6);
    dydt(6) = c2 * (r1m3 - r2m3) * ...
                   (sin(th) * xc - cos(th) * yc);

end
```

Function: energy_dumb

```
function [t_trans, t_rot, v_grav, e_tot] = energy_dumb(y, m1, m2, d)
   % energy_dumb: Computes energy quantities from solution y.
   xc = y(:,1); vxc = y(:,2); yc = y(:,3);
   vyc = y(:,4); th = y(:,5); om = y(:,6);

   t_trans = 0.5 * (m1 + m2) * (vxc.^2 + vyc.^2);

   t_rot = 0.5 * (m1 * m2) / (m1 + m2) * d^2 * om.^2;

   mu = m2 / (m1 + m2);
   d1 = mu * d;
   d2 = (1 - mu) * d;
   x1 = xc + d1 * cos(th);
   y1 = yc + d1 * sin(th);
   x2 = xc - d2 * cos(th);
   y2 = yc - d2 * sin(th);
   r1 = sqrt(x1.^2 + y1.^2);
   r2 = sqrt(x2.^2 + y2.^2);
   v_grav = -(m1 ./ r1 + m2 ./ r2);

   e_tot = t_trans + t_rot + v_grav;
end
```

Script: ode45_dumb

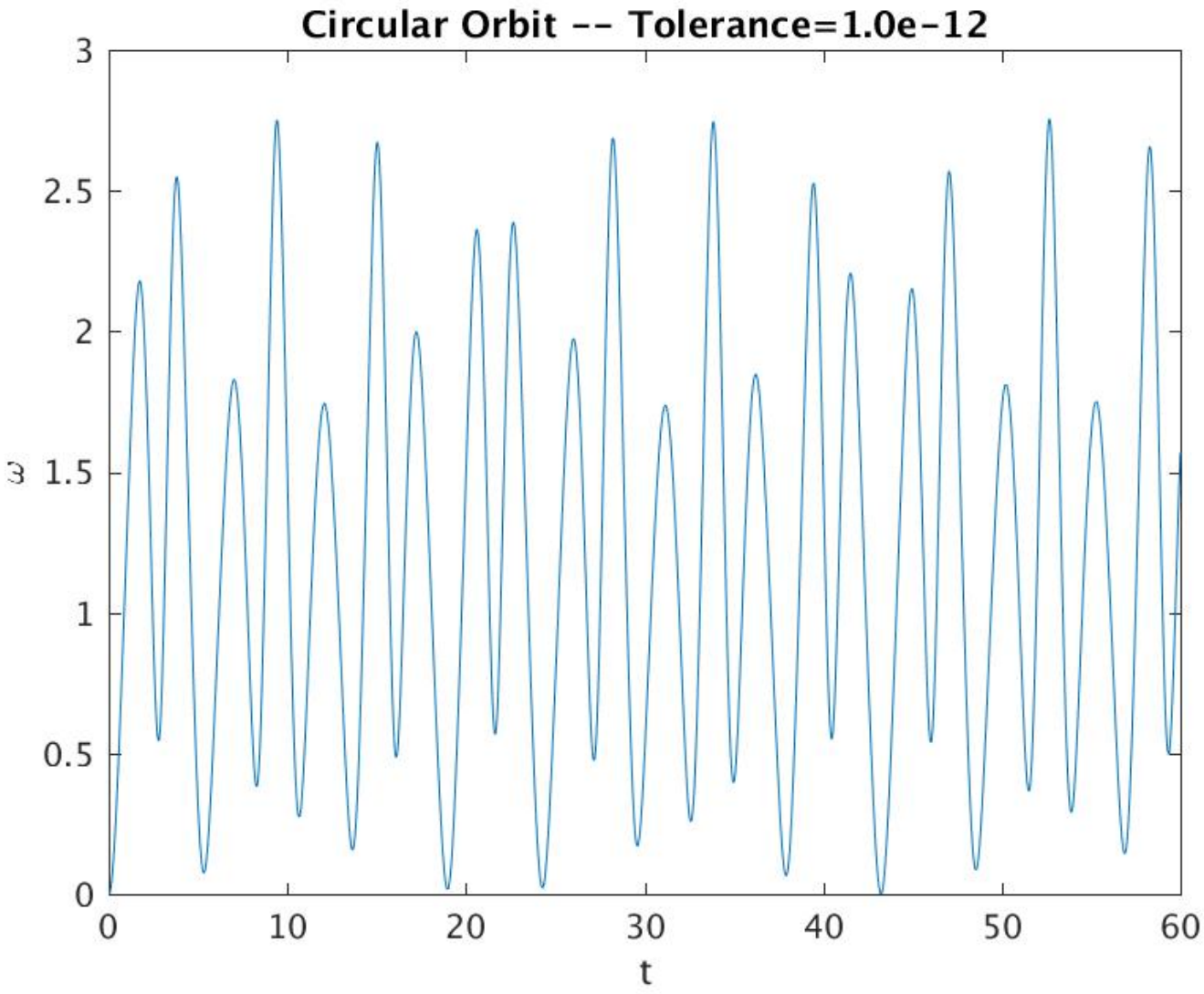
```
% ode45_dumb: Uses ODE45 to integrate equations for orbiting
% dumbbell problem
global mu  d;

m1bym2 = 2.0;
mu = 1.0 / (1.0 + m1bym2);
d = 0.3;

m1 = 1.0;
m2 = m1 / m1bym2;

tmax = 60;
tol = 1.0e-12;
options = odeset('AbsTol', tol * ones(1,6), 'RelTol', tol);

% Circular orbit ...
[tout yout] = ode45(@fcn_dumb, [0.0 tmax], ...
                    [1.0 0.0 0.0 1.0 0.0 0.0], options);

clf;
plot(tout, yout(:,6));
xlabel('t');
ylabel('\omega');
title('Circular Orbit -- Tolerance=1.0e-12');
```

```
print('dumb_omega_circular.jpg', '-djpeg');

[t_trans, t_rot, v_grav, e_tot] = energy_dumb(yout, m1, m2, d);

clf;
plot(tout, e_tot);
xlabel('t');
ylabel('E');
title('Circular Orbit -- Tolerance=1.0e-12');
print('dumb_etot_circular.jpg', '-djpeg');

% Elliptical orbit ...
tmax = 200;
[tout yout] = ode45(@fcn_dumb, [0.0 tmax], ...
                    [1.0 0.0 0.0 1.2 0.0 0.0], options);

clf;
plot(tout, yout(:,6));
xlabel('t');
ylabel('\omega');
title('Elliptical Orbit -- Tolerance=1.0e-12');
print('dumb_omega_elliptical.jpg', '-djpeg');

[t_trans, t_rot, v_grav, e_tot] = energy_dumb(yout, m1, m2, d);
```

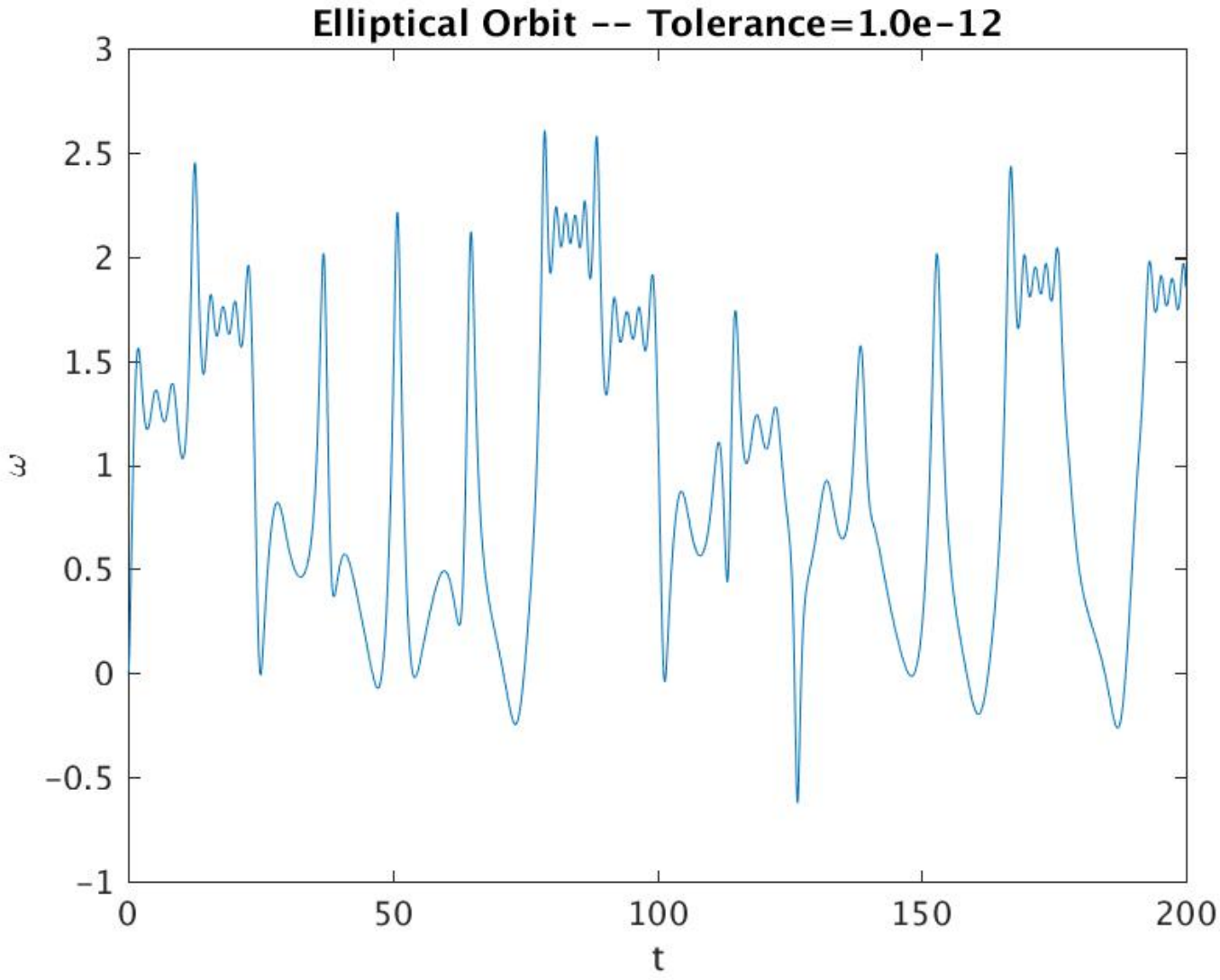
```
clf;
plot(tout, e_tot);
xlabel('t');
ylabel('E');
title('Elliptical Orbit -- Tolerance=1.0e-12');
print('dumb_etot_elliptical.jpg', '-djpeg');

clf;
hold on;
plot(tout,t_trans);
plot(tout,t_rot);
plot(tout,v_grav);
plot(tout,e_tot);
xlabel('t');
ylabel('Energy quantity');
legend('T_{\rm trans}', 'T_{\rm rot}', 'V_{\rm grav}', 'E_{\rm tot}');
title('Elliptical Orbit -- Tolerance=1.0e-12');
print('dumb_eall_elliptical.jpg', '-djpeg');
hold off;

% Elliptical orbit, longer time ...
tmax = 1000;
[tout yout] = ode45(@fcn_dumb, [0.0 tmax], ...
                    [1.0 0.0 0.0 1.2 0.0 0.0], options);
```

```
clf;
plot(tout, yout(:,6));
xlabel('t');
ylabel('\omega');
title('Elliptical Orbit -- Tolerance=1.0e-12');
print('dumb_omega_elliptical_long.jpg', '-djpeg');
```

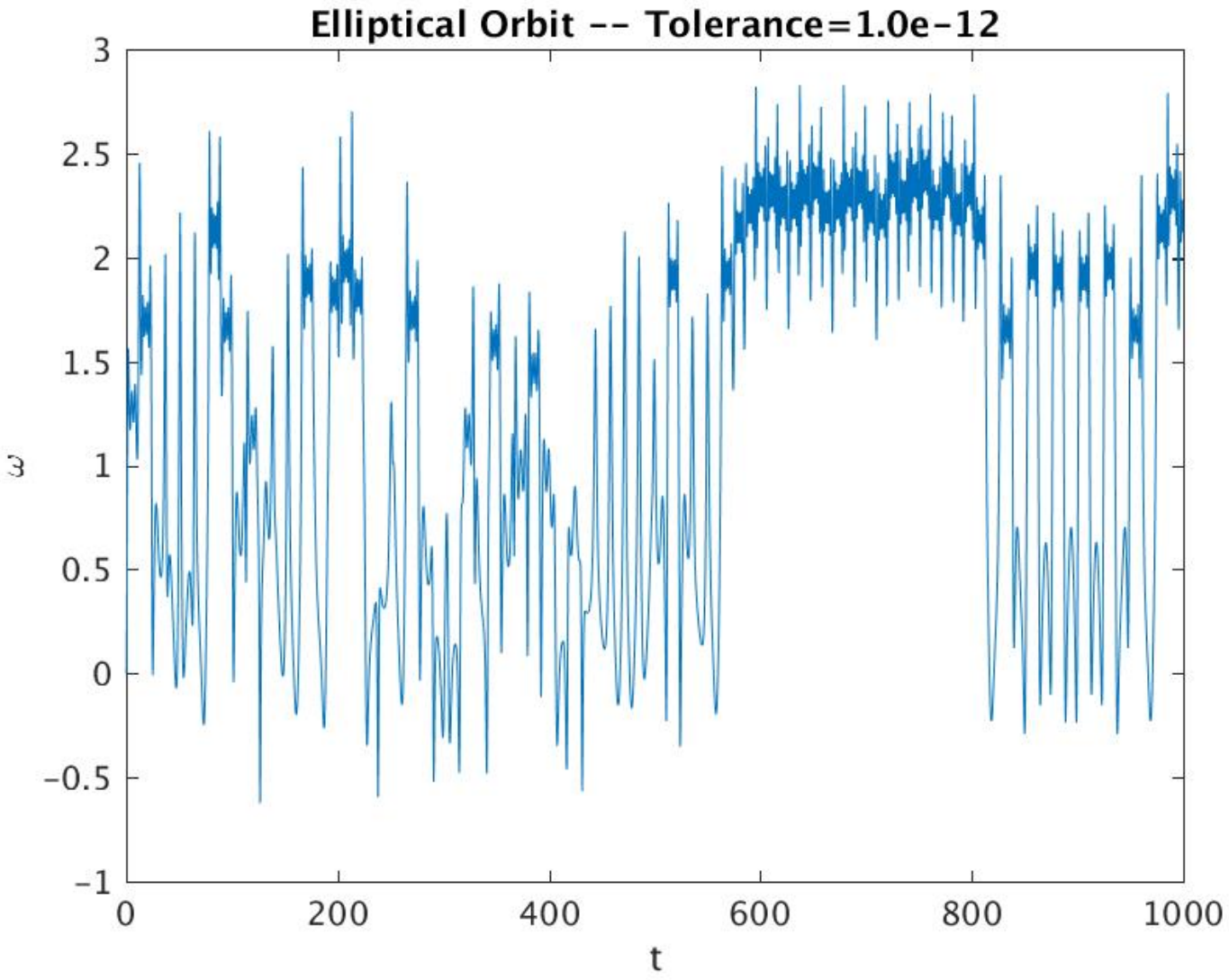
- Plot of $\omega(t)$ for circular orbit



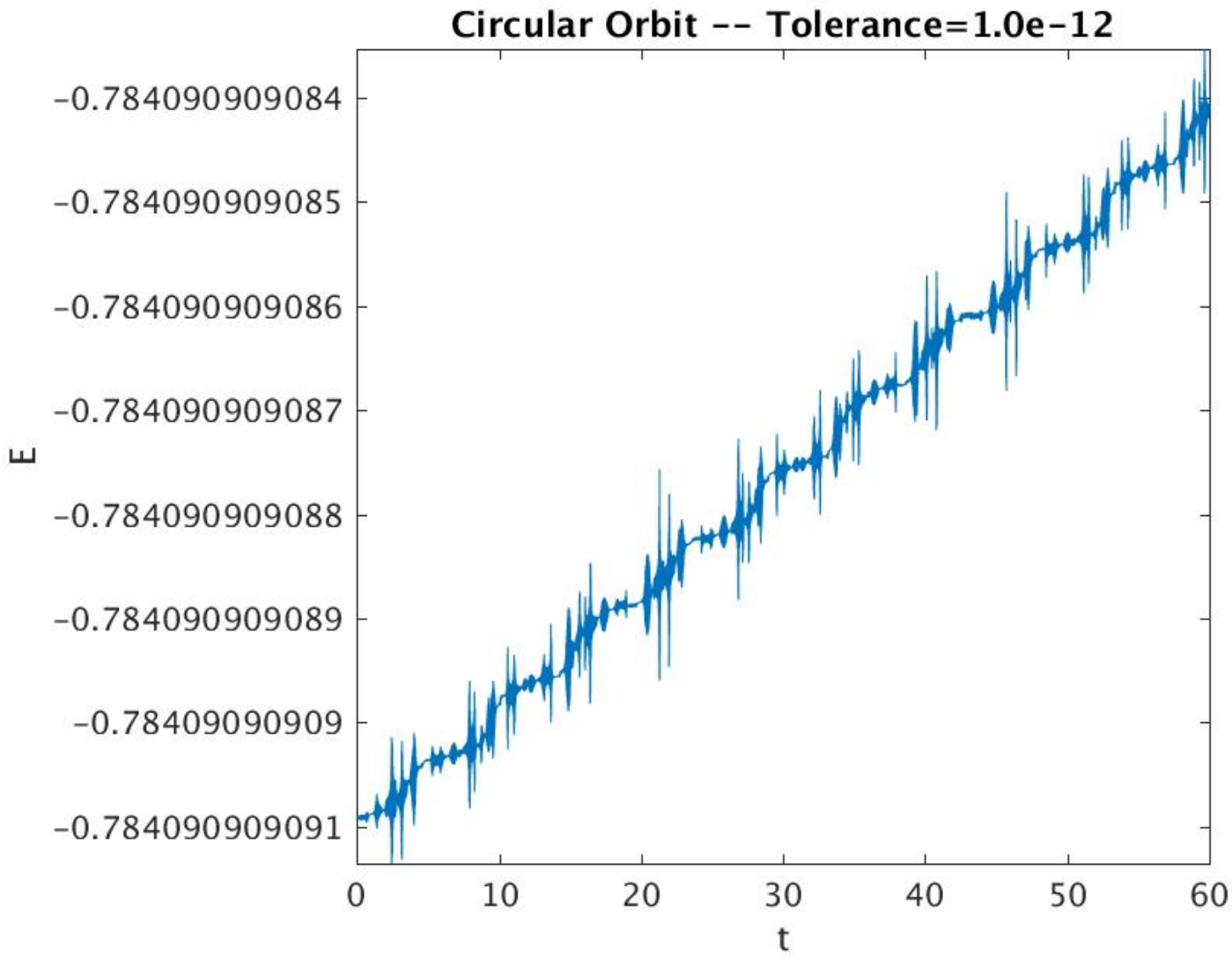
- Plot of $\omega(t)$ for elliptical orbit



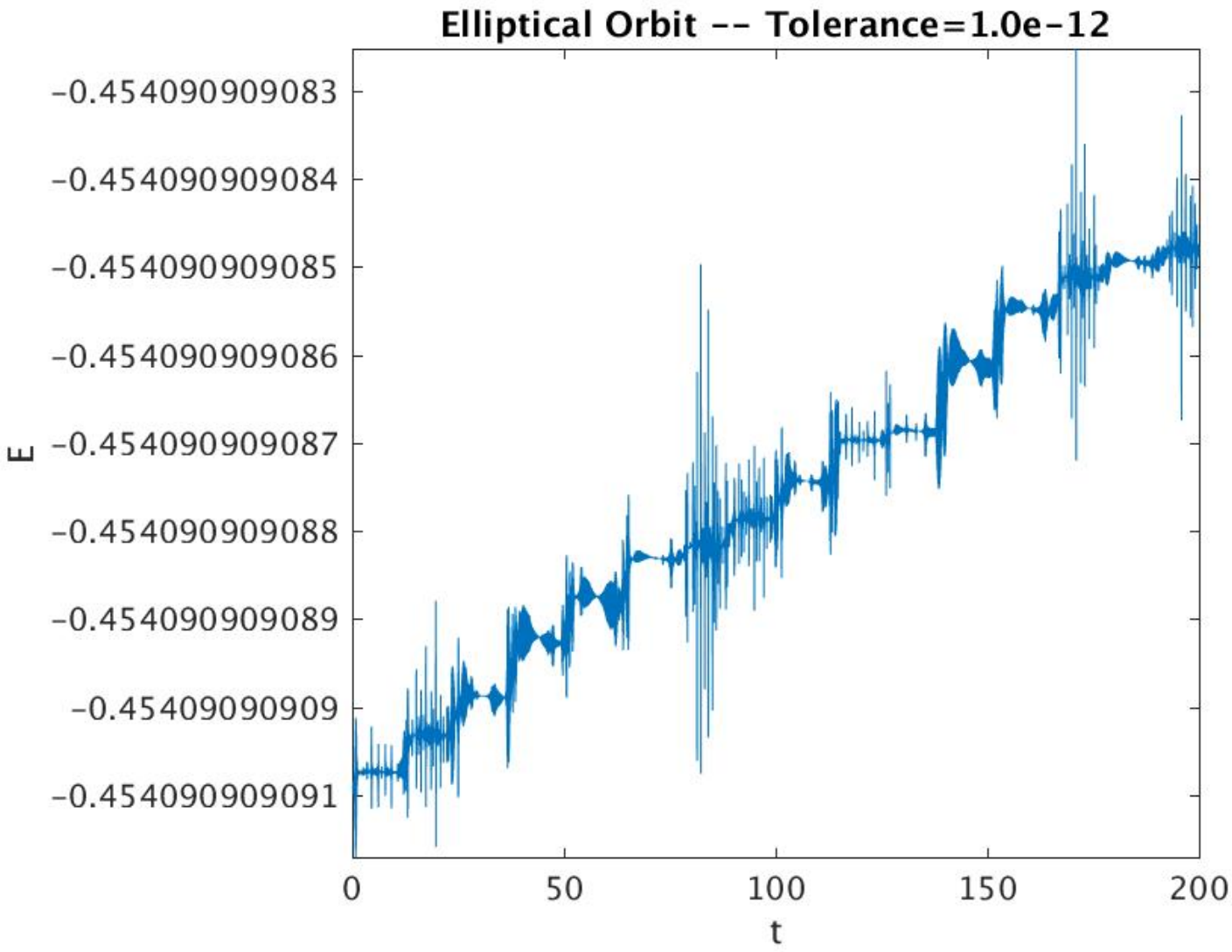
- Plot of $\omega(t)$ for elliptical orbit, longer integration time



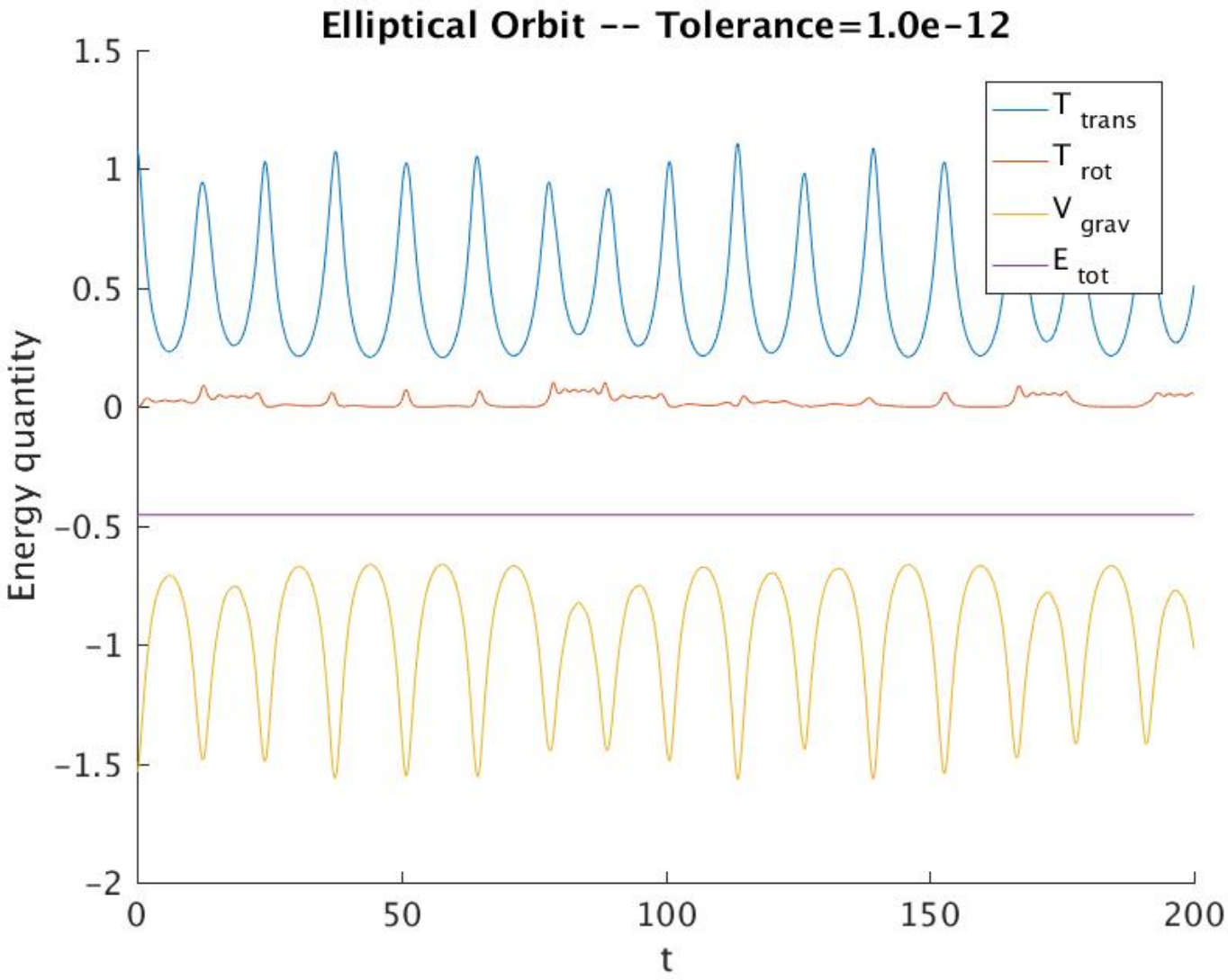
- Total energy, circular orbit



- Total energy, elliptical orbit



- Various energy quantities, elliptical orbit



Example: Toda Lattice
Function: `fcn_toda`

```
function dydt = fcn_toda(t, y)
%  Function dydt evaluates derivatives for 1D Toda lattice.
%
%  y is a vector of length 2*ns, where ns is the number of sites.
%  y(1:ns)      = generalized cooordinates
%  y(ns+1:2*ns) = generalized momenta
%
%  Static boundary conditions are imposed on the momenta.

   ns = length(y) / 2;
   dydt = zeros(length(y), 1);

   % Time derivatives of coordinates ...
   dydt(1:ns, 1) = y(ns+1:2*ns);
   % Time derivatives of momenta ...
   dydt(ns+2:2*ns-1, 1) = exp(-(y(2:ns-1) - y(1:ns-2))) - ...
                          exp(-(y(3:ns) - y(2:ns-1)));
   % Impose static boundary conditions on momenta ...
   dydt(ns+1, 1) = 0;
   dydt(2*ns, 1) = 0;
end
```

Script: ode45_toda1

```
% ode45_toda1: Uses ODE45 to integrate equations for 1D Toda lattice
% following
% https://www.mat.univie.ac.at/~gerald/ftp/book-jac/toda.html
%
% Single soliton setup.

% Set to 1 to enable .avi creation
avienable = 1;
avisecs = 20;
aviframerate = 25;
aviframes = avisecs * aviframerate;
avifilename = 'toda1.avi';

% Minimum and maximum time ...
tmin = 0.0;
tmax = 200.0;

% Sites and number of sites ...
n = 1:401;
ns = length(n);

% Tolerance parameters ...
abstol = 1.0e-6;
reltol = 1.0e-6;
```

```
options = odeset('AbsTol', abstol * ones(1,2*ns), 'RelTol', reltol);

% Set initial data ...

% yinit = solitons(n, n0, qp, gamma, kappa, sigma);
yinit = solitons(n, ...
   [50.0], ...
   [1.0], ...
   [1.0], ...
   [0.05], ...
   [1.0]);

% Integrate system using ode45 ...
[tout yout] = ode45(@fcn_toda, [tmin tmax], yinit, options);

% Open .avi file if video generation enabled ...
if avienable
   avifreq = round(length(tout) / aviframes);
   aviobj = VideoWriter(avifilename);
   open(aviobj);
end

% Loop over time steps ...
for it = 1 : length(tout)
   if avienable
```

```
   if (it == 1) | (it == length(tout))
      framecount = 5 * aviframerate;
   elseif rem(it,avifreq) == 0
      framecount = 1;
   else
      framecount = 0;
   end
end
% Plot displacements ...
if ~avienable | (avienable & framecount)
   clf;
   hold on;
   axis square;
   box on;
   xlim([0, ns]);
   ylim([min(yinit(1:ns)), max(yinit(1:ns))]);
   xlabel('n');
   ylabel('q(n)');
   plot(n, yout(it,1:ns), 'Marker', 'o', 'Markersize', 2.0, ...
      'MarkerEdgeColor', 'r', 'MarkerFaceColor', 'r');
   titlestr = sprintf('Step: %d    Time: %.1f', it, tout(it));
   title(titlestr, 'FontSize', 16, 'FontWeight', 'bold', ...
         'Color', [0.25, 0.42, 0.31]);
   drawnow;
end
```

```
   % Output JPEG of first timestep ...
   if it == 1
      print('toda1.jpg', '-djpeg');
   end
   % avi recording, if enabled ...
   if avienable
      for iframe = 1 : framecount
         writeVideo(aviobj, getframe(gcf));
      end
   end
end

% Close avi file ...
if avienable
   close(aviobj);
   fprintf('Created video file: %s\n', avifilename);
end
```

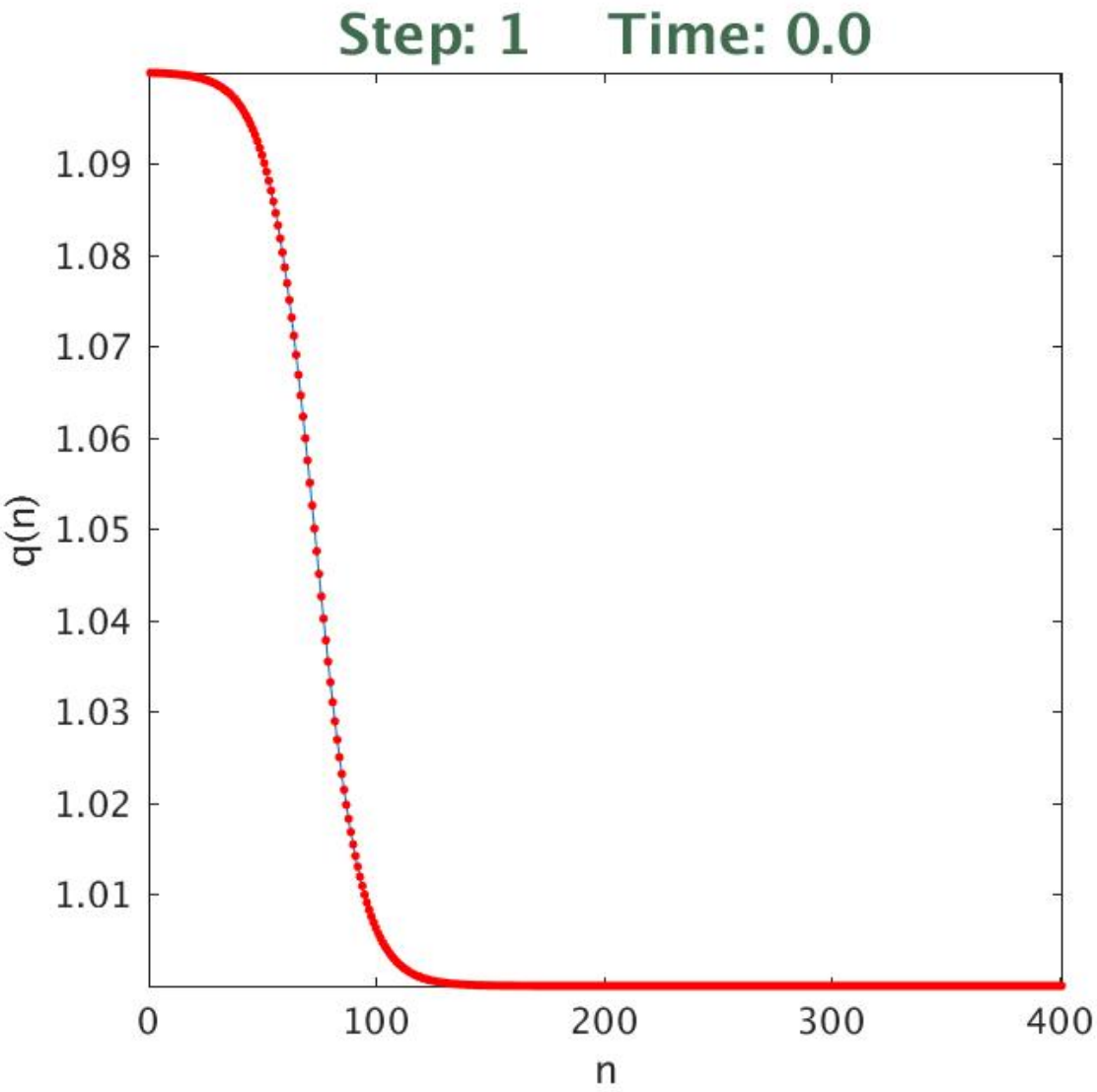
Function: solitons

```
function yinit = solitons(n, n0, qp, gamma, kappa, sigma)
%  solitons defines initial values corresponding to multi-soliton
%  Toda lattice.

   yinit = zeros(2 * length(n),1);
   for is = 1 : length(n0)
      t0 = n0(is) * kappa(is) / sinh(kappa(is));
      for ii = 1 : length(n)
         yinit(ii) = yinit(ii) + ...
            q1(n(ii), t0, qp(is), gamma(is), kappa(is), sigma(is));
         yinit(length(n) + ii) = yinit(length(n) + ii) + ...
            dq1dt(n(ii), t0, qp(is), gamma(is), kappa(is), sigma(is));
      end
   end
end
```

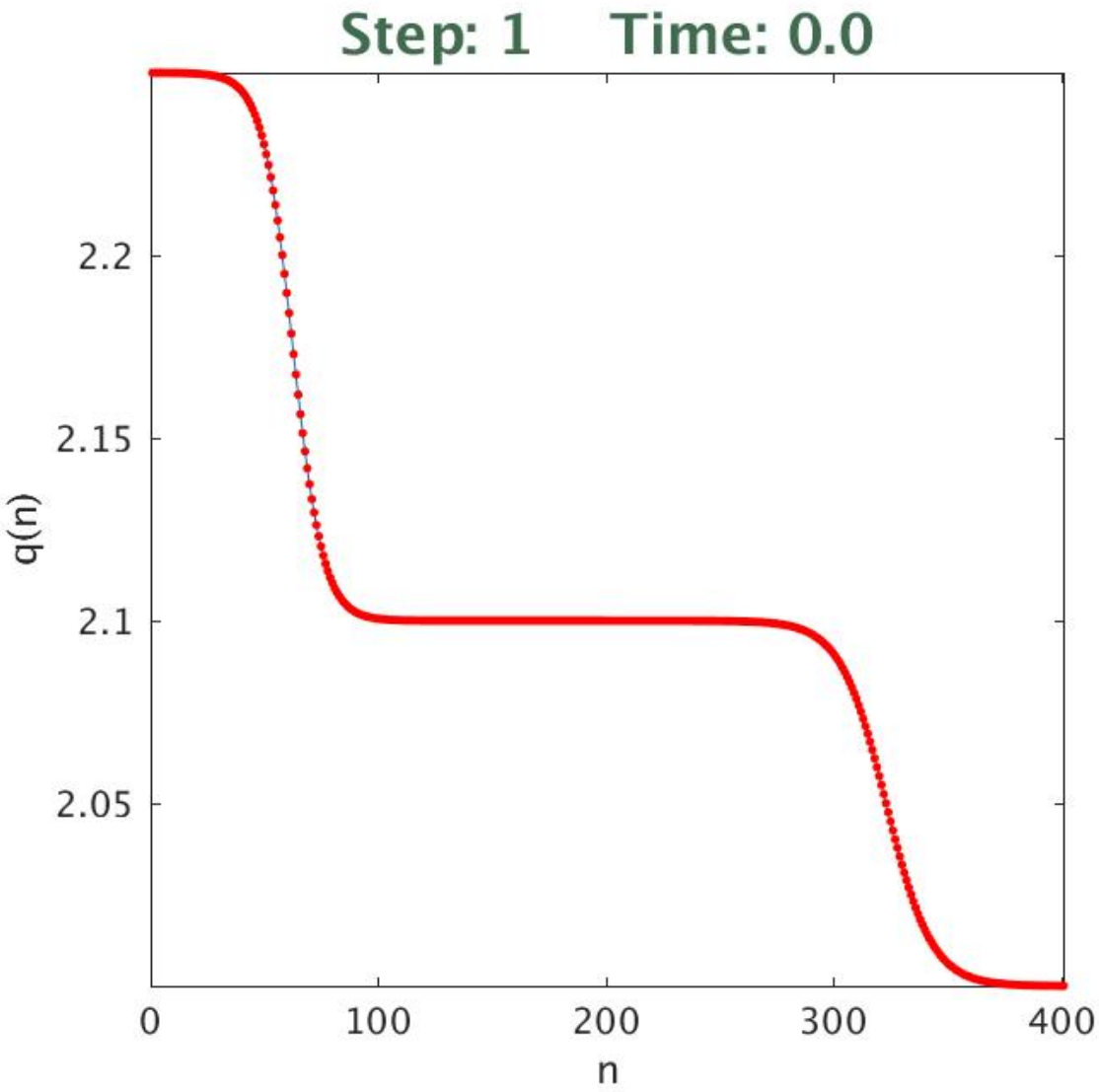
Functions: q1 and dq1dt

```
function q1nt = q1(n,t,qplus,gamma,kappa,sigma)
%  q1: Defines single soliton solution for Toda lattice following
%  equation (3) of
%  https://www.mat.univie.ac.at/~gerald/ftp/book-jac/toda.html
   q1nt = qplus + log( ...
          (1 + (gamma / (1 - exp(-2*kappa))) * ...
             exp(-2*kappa*n + 2*sigma*sinh(kappa)*t)) / ...
          (1 + (gamma / (1 - exp(-2*kappa))) * ...
             exp(-2*kappa*(n + 1) + 2*sigma*sinh(kappa)*t)) ...
       );
end

function dq1dtnt = dq1dt(n,t,qplus,gamma,kappa,sigma)
%  dq1dt: Defines time derivative of single soliton solution for Toda
%  lattice following equation (3) of
%  https://www.mat.univie.ac.at/~gerald/ftp/book-jac/toda.html
   t1 = exp(-2*kappa*n  + 2*sigma*sinh(kappa)*t);
   t2 = exp(-2*kappa*(n + 1) + 2*sigma*sinh(kappa)*t);
   t3 = 1 - exp(-2*kappa);
   t4 = (1 + gamma*t2 / t3);
   dq1dtnt = (2*gamma*sigma*sinh(kappa)*t1 / (t3 * t4) - ...
      2*(1 + gamma*t1/t3)*gamma*sigma*sinh(kappa) * t2 / ...
      (t4^2 * t3)) * t4 / (1 + gamma*t1 / t3);
end
```

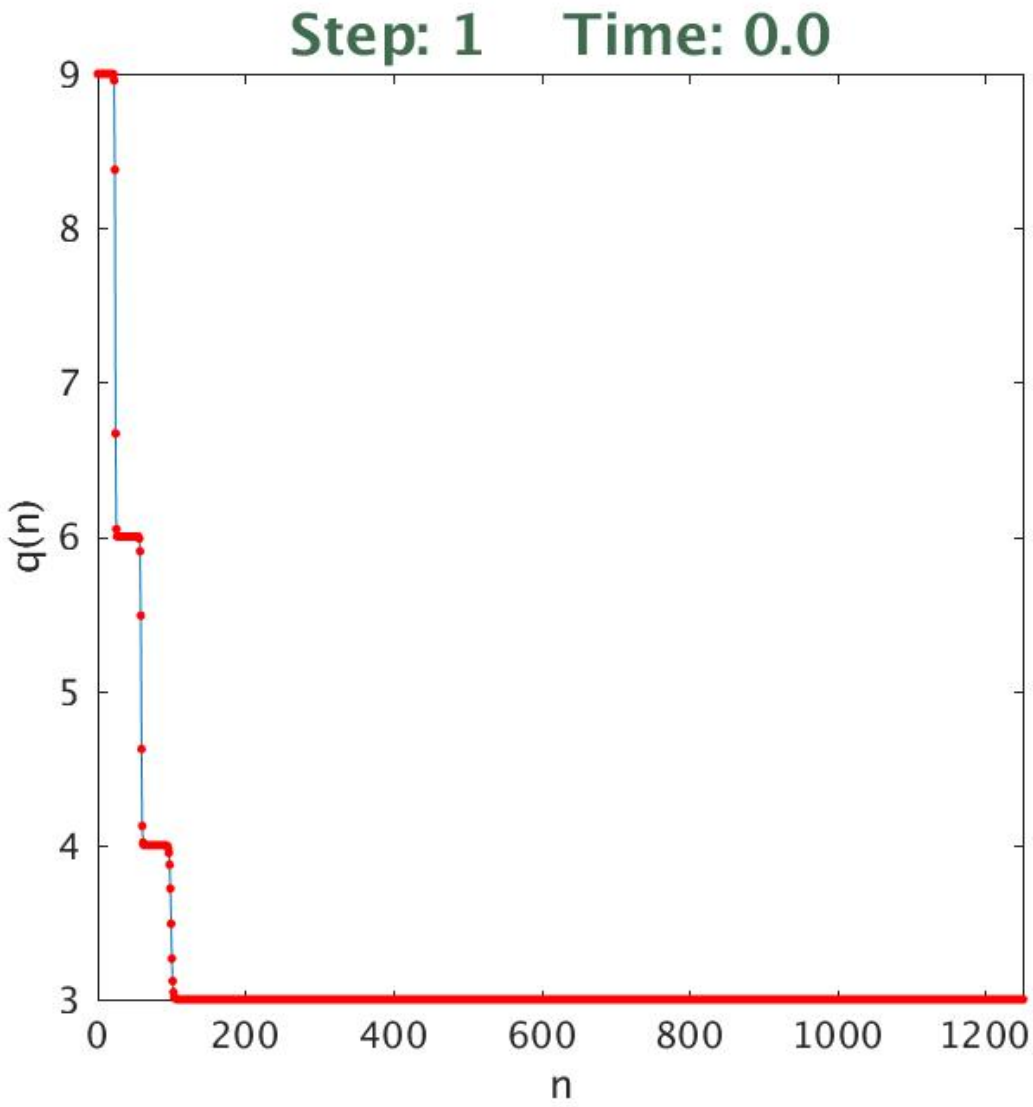
- Initial conditions: one soliton



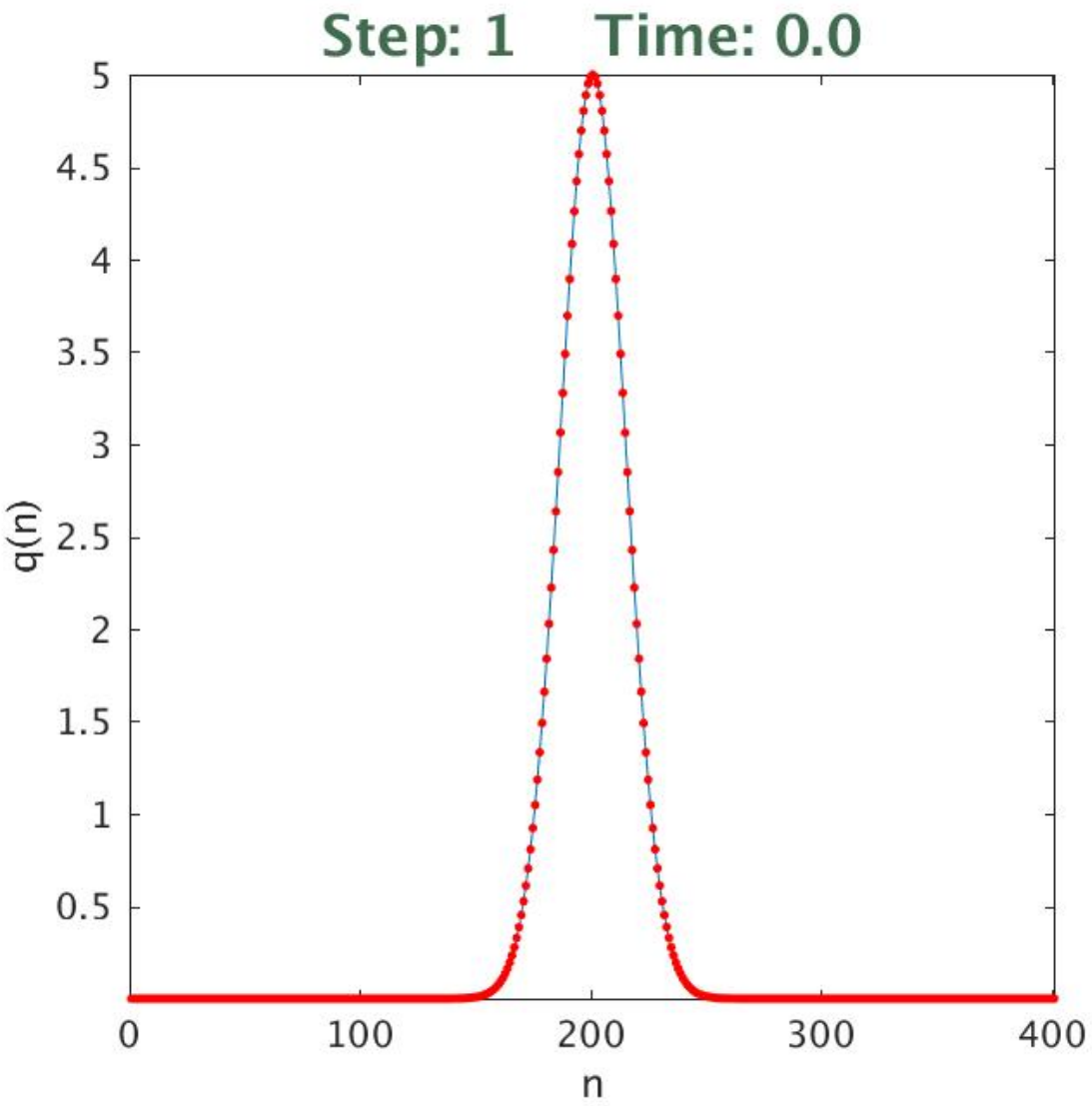
- Initial conditions: two solitons



- Initial conditions: three solitons



- Initial conditions: Gaussian pulse, $p(n,0) = 0$



Example: van der Pol Oscillator
Function: `fcn_vdp`

```
function dydt = fcn_vdp(t, y)
% Function dydt evaluates derivatives for van der Pol oscillator
% following Tsatsos:
% https://arxiv.org/pdf/0803.1658.pdf
   global a b omega;

   dydt = ones(2,1);
   dydt(1) = y(2);
   dydt(2) = -y(1) - a * (y(1)^2 - 1) * y(2) + b * sin(omega * t);
end
```

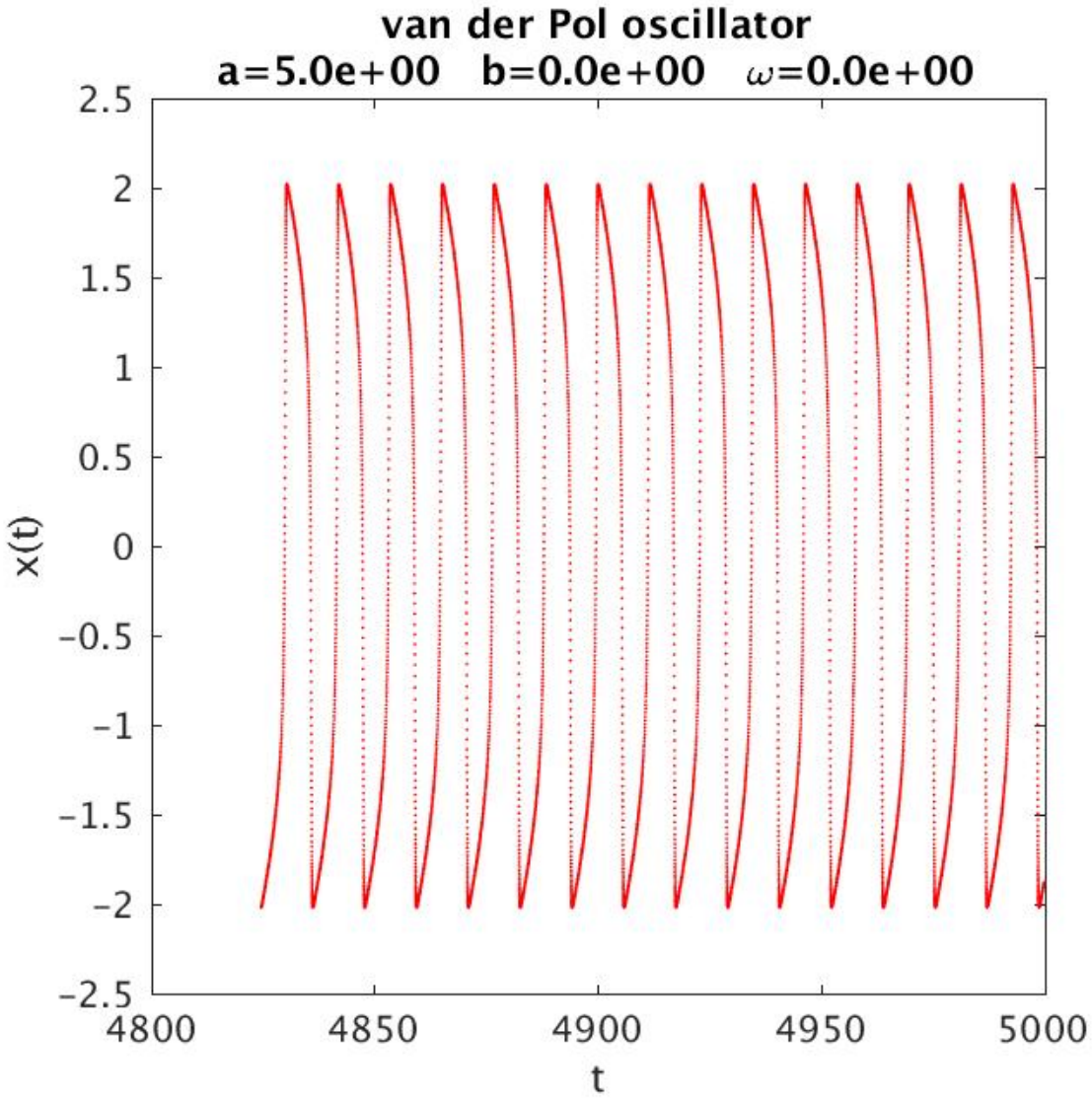
Function: ode45_vdp_b

```
% ode45_vdp: Uses ODE45 to integrate equations for Van de Pol
% oscillator following Tsatsos:
% https://arxiv.org/pdf/0803.1658.pdf
%
% This script generates bifurcation diagrams.
global a b omega;

% Tolerance parameters ...
abstol = 1.0e-6;
reltol = 1.0e-6;
options = odeset('AbsTol', abstol * [1 1]', 'RelTol', reltol);

% Control parameters ...
a = 5.0;
b = 5.0;
vomega = 3.3695:0.000005:3.370;
nomega = length(vomega);

% Number of points in Poincare section ...
np = 500;

% Define space for bifurcation plot variables ...
xb = zeros(np,nomega);
```

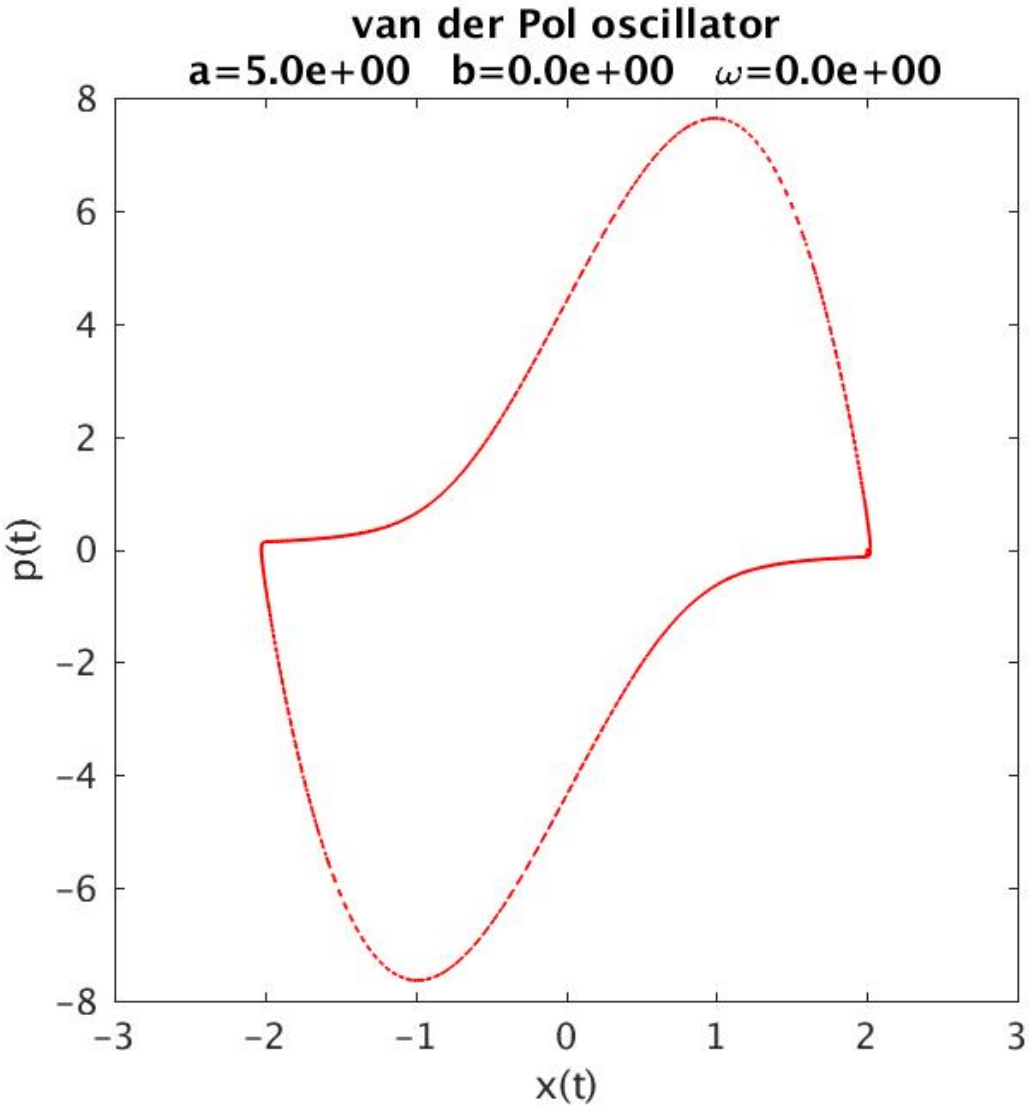
```
% Initial data ...
yinit = [2.0, 0.0];

for iomega = 1 : nomega
   iomega
   omega = vomega(iomega);
   % Output times ...
   tspan = [0:2000] * (2.0 * pi / omega);
   % Integrate system using ode45 ...
   [tout yout] = ode45(@fcn_vdp, tspan, yinit, options);

   % Save Poincare points ...
   xb(:,iomega) = yout(end-np+1:end,1);

   % Make bifurcation plot ...
   figure(1);
   clf;
   hold on;
   axis square;
   box on;
   xlabel('\omega');
   ylabel('x_p(t)');
   titlestr = ...
      sprintf('van der Pol oscillator\na=%.1f   b=%.1f', a, b);
   title(titlestr);
```

```
   for i = 1 : iomega
      plot(repmat(vomega(i),1,np), xb(:,i), ...
         'ro', 'Markersize', 0.5);
   end
   drawnow
end
print('vdp_bifurcation.jpg', '-djpeg');
```

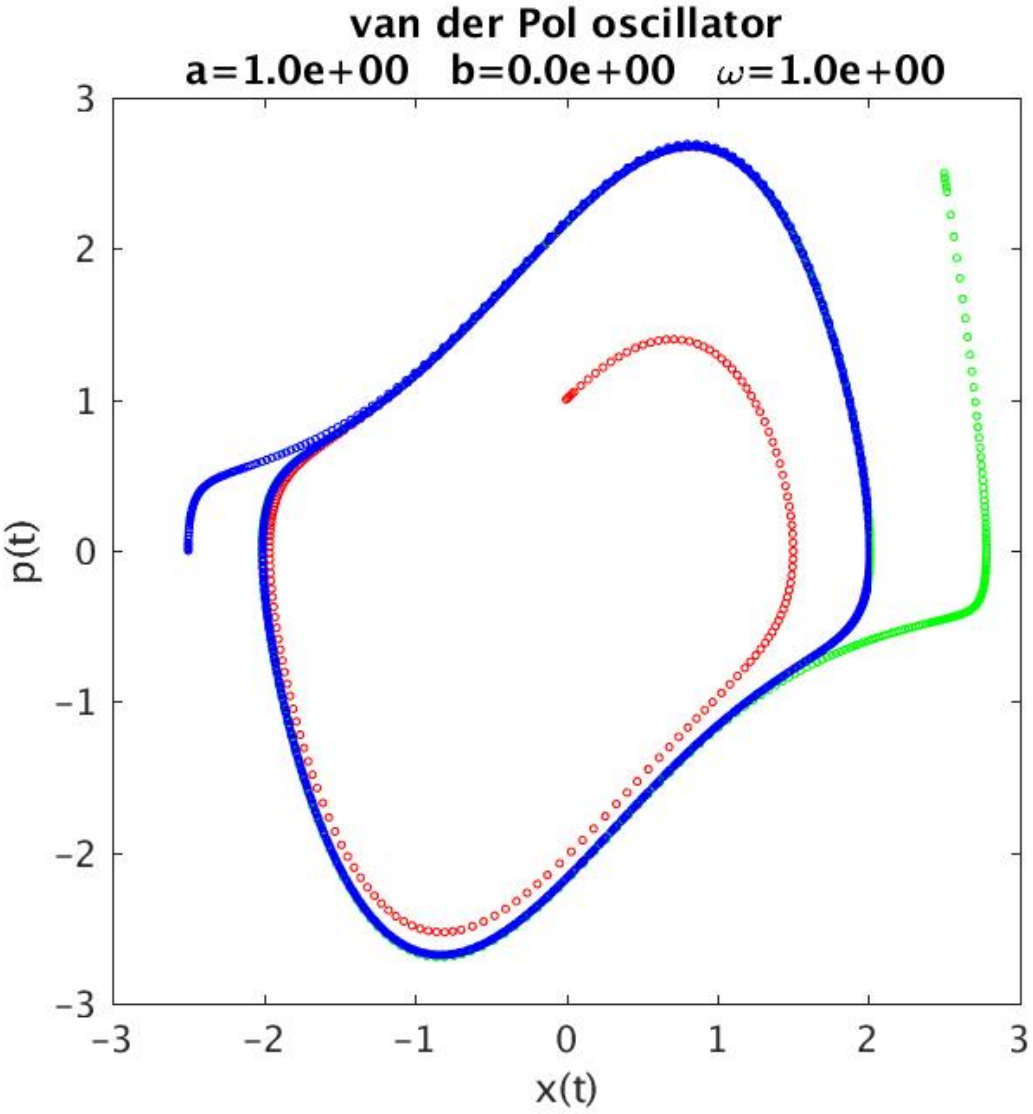
- Oscillator displacement for typical evolution with no driving ($b = 0$)



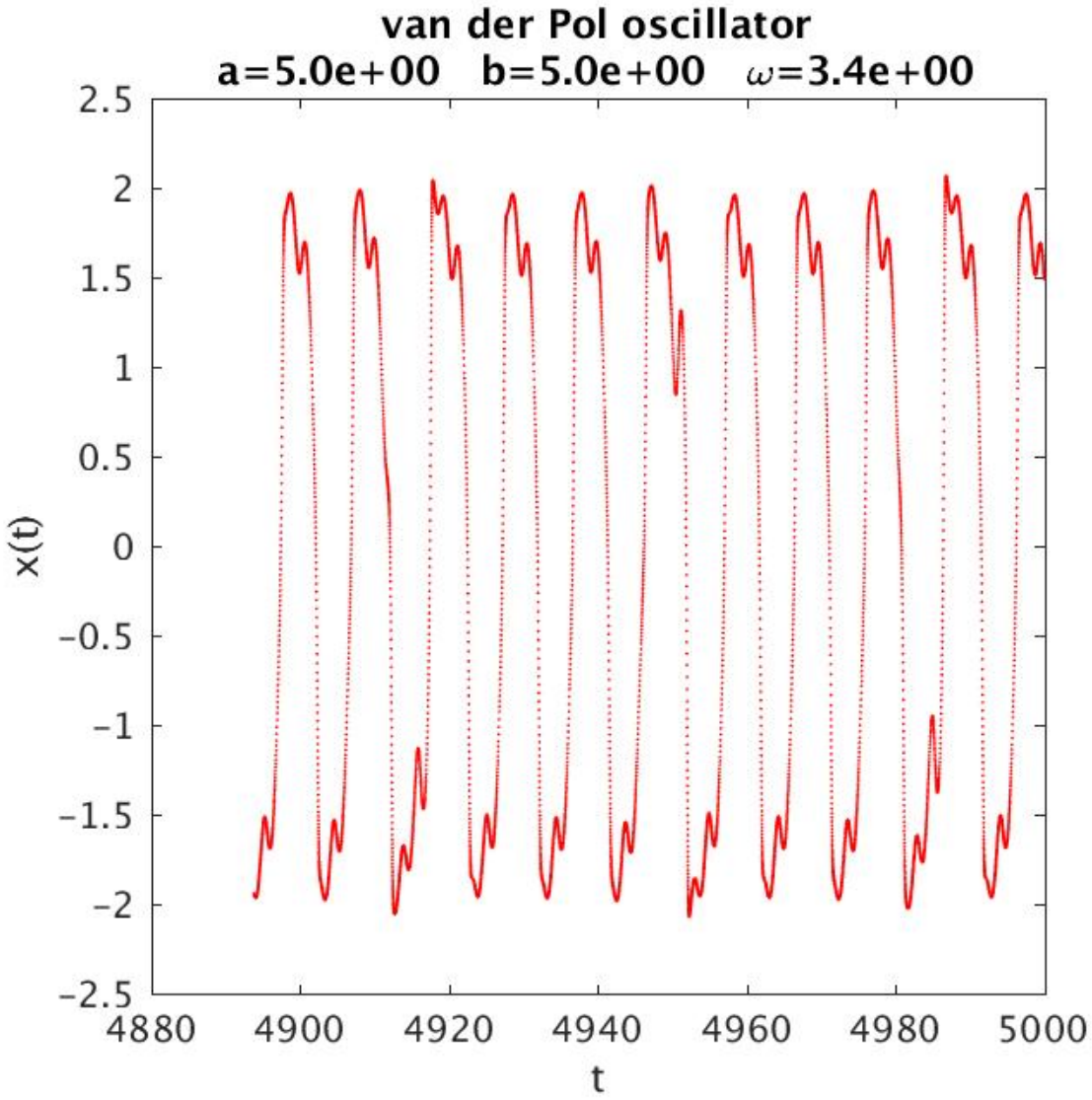
- Phase-space trajectory for typical evolution with no driving ($b = 0$)



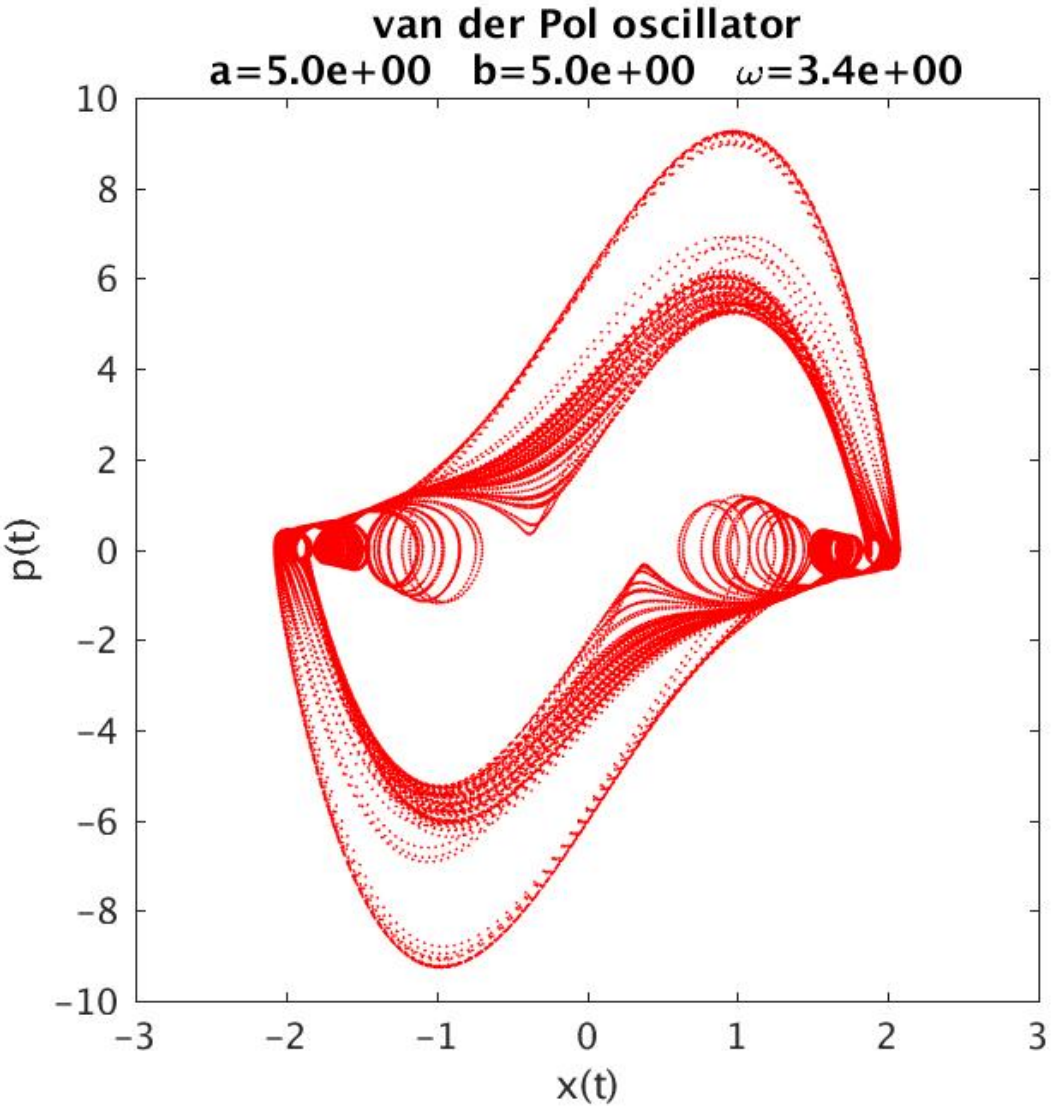
- Phase-space evolution of 3 distinct initial conditions with $b = 0$ (no driving force). Illustrates how dynamics tend to limit cycle (depends only on a).



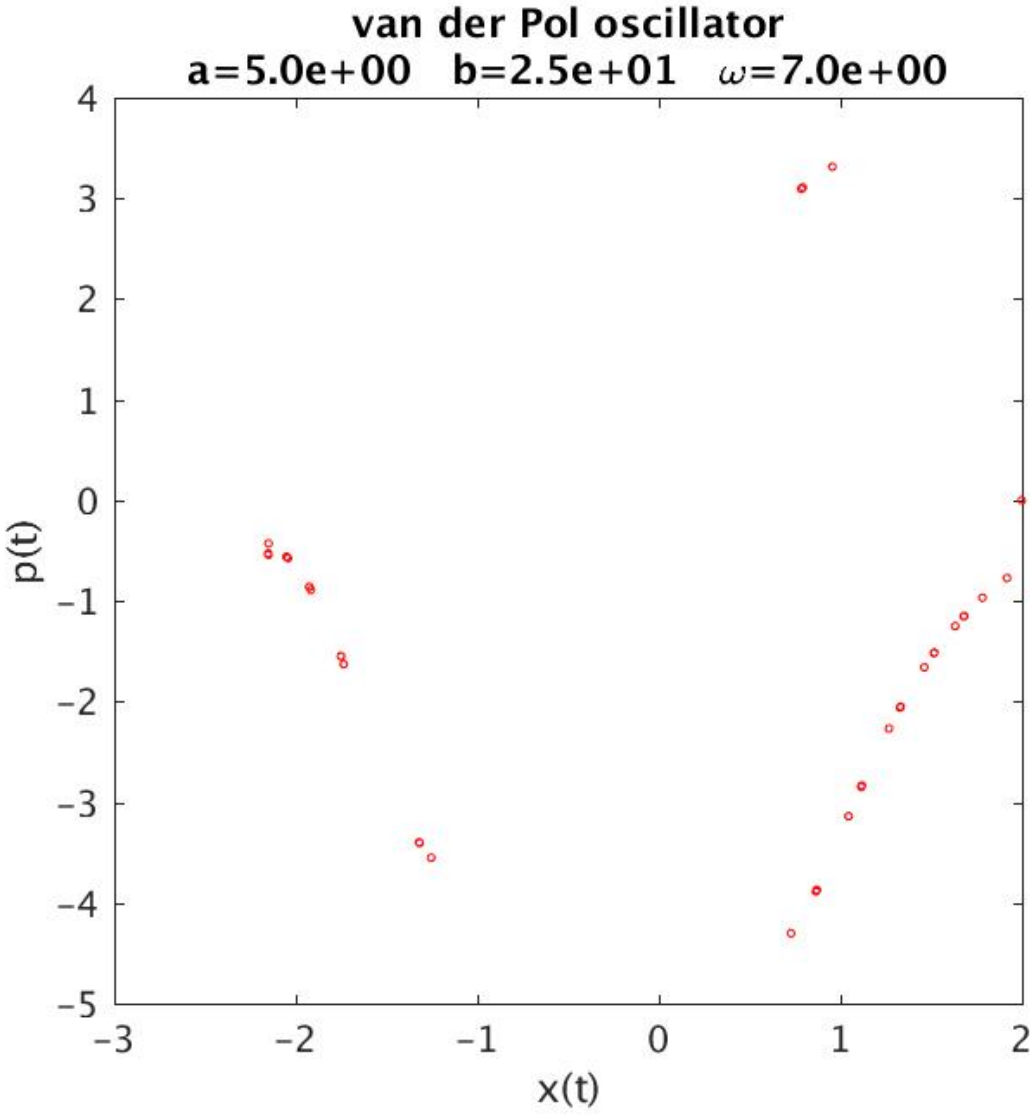
- (Driven) oscillator displacement in a chaotic regime.



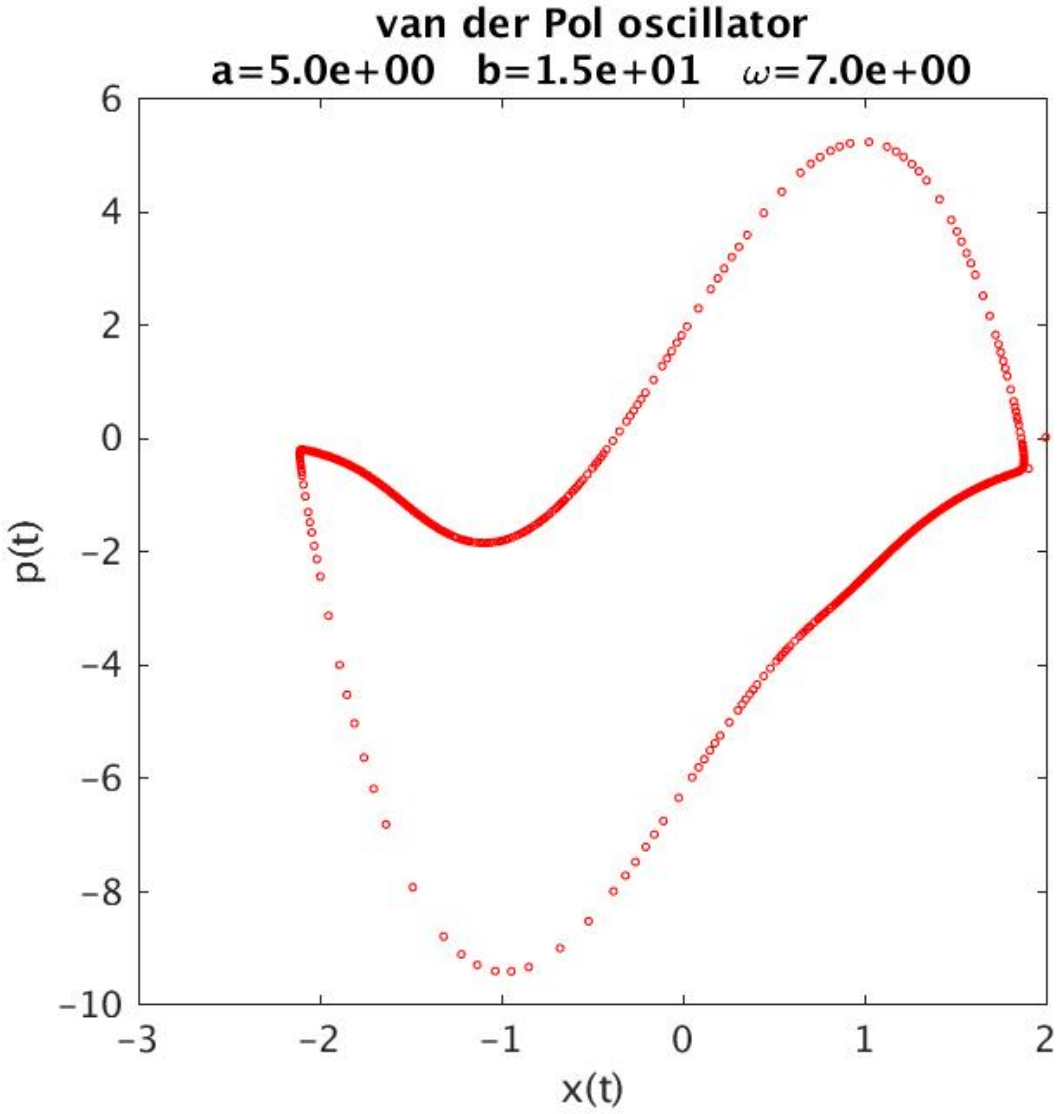
- Phase-space evolution for driven oscillator in a chaotic regime.



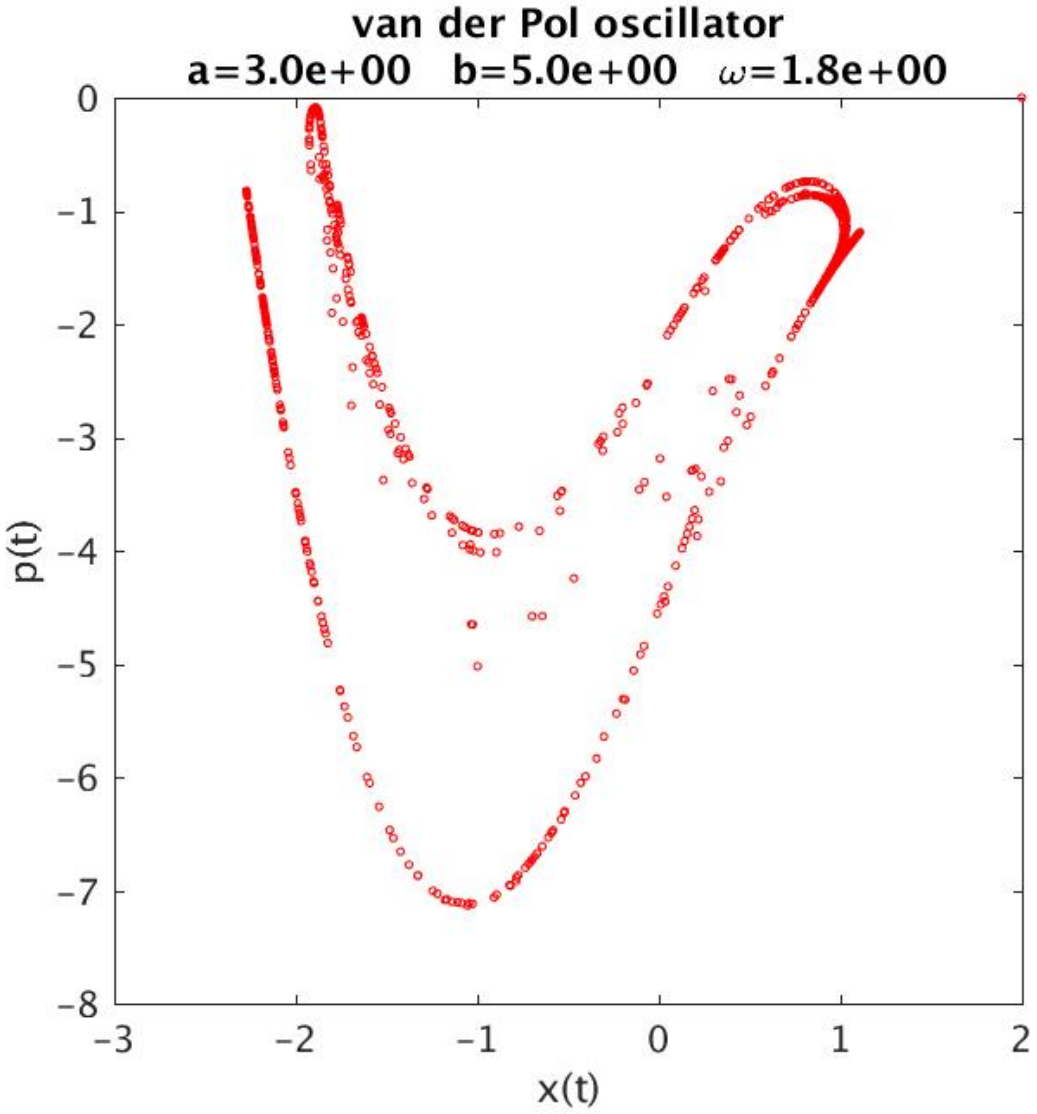
- Poincaré section for driven oscillator showing periodic trajectory. Output times are $2\pi n/\omega,\ n = 0, \ldots, 1000$.



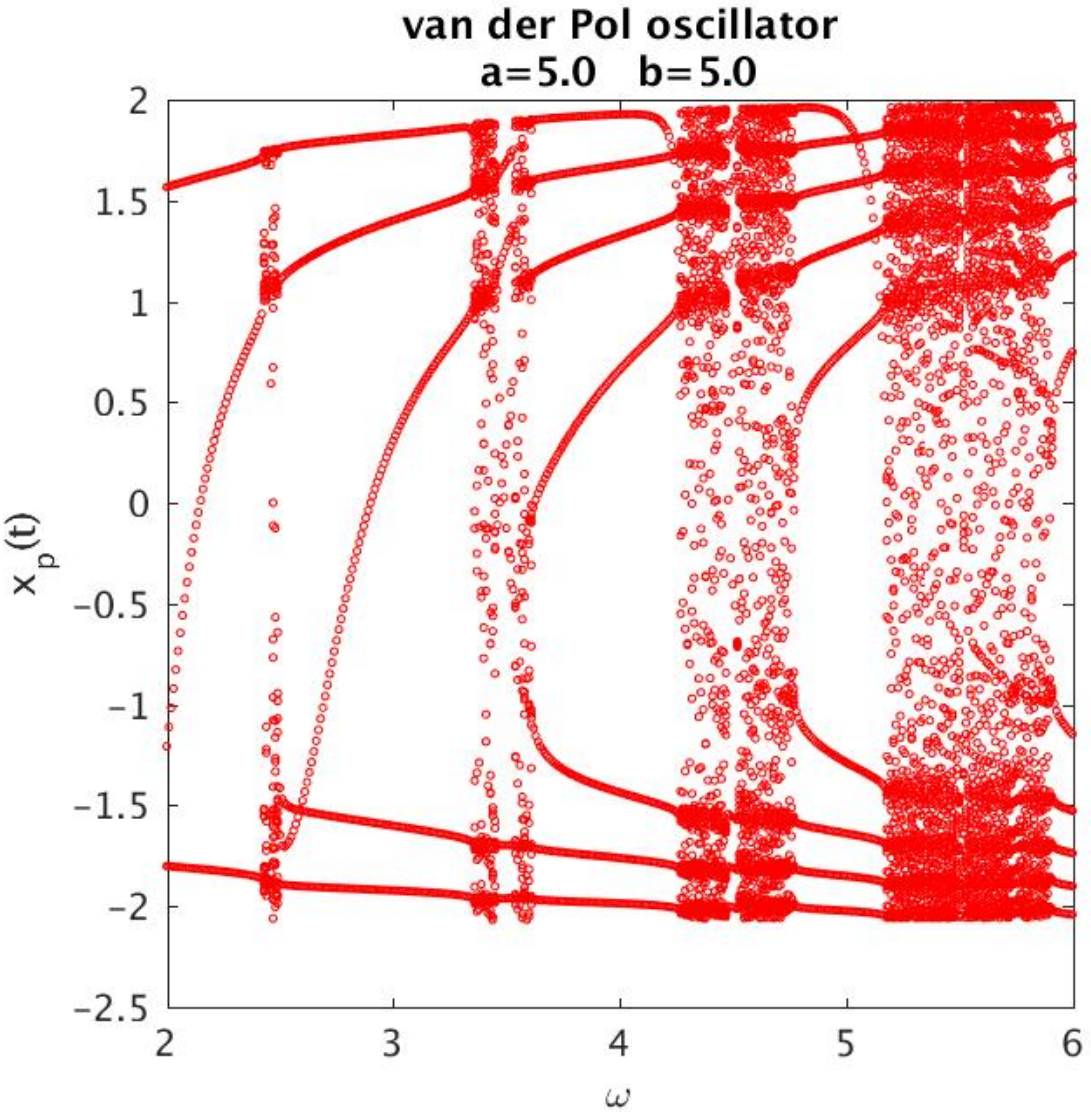
- Poincaré section for driven oscillator showing almost periodic trajectory. Output times are $2\pi n/\omega,\ n = 0, \ldots, 1000$.



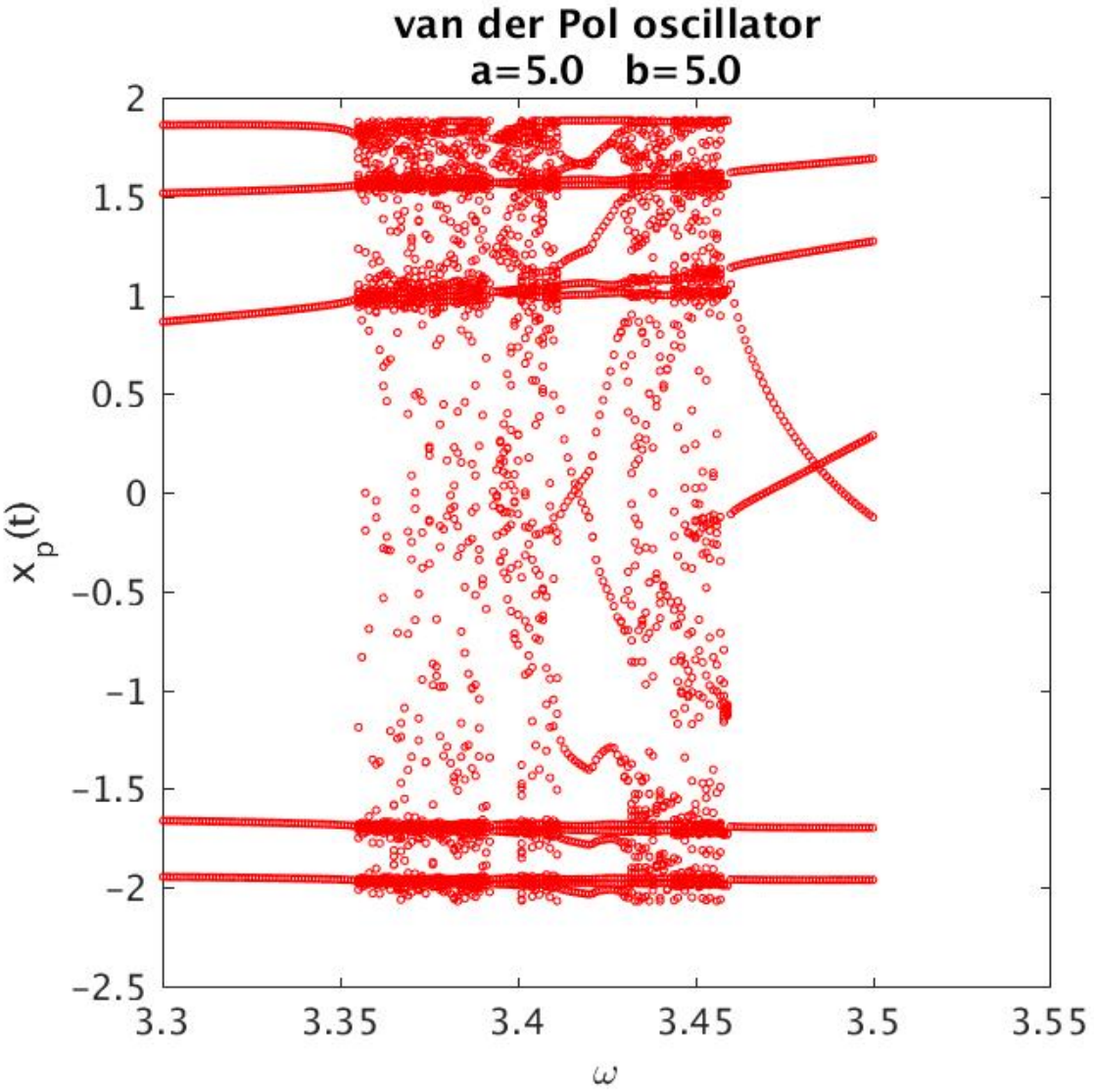
- Poincaré section for driven oscillator showing chaotic trajectory. Output times are $2\pi n/\omega,\ n = 0, \ldots, 1000$.



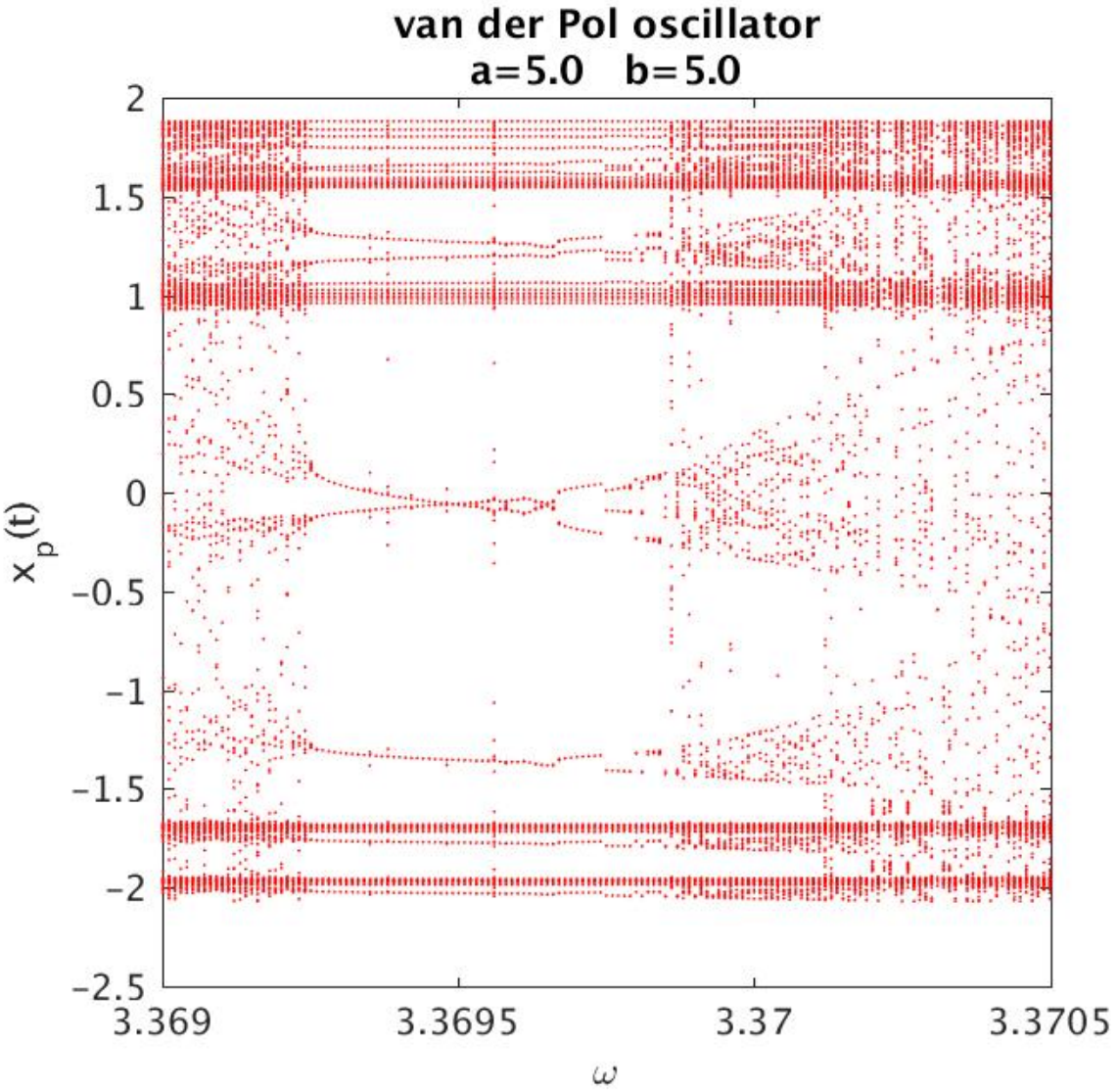
- **Bifurcation diagram (1 of 4).** Driving frequency ω is the control parameter. Each vertical slice in the plot represents a single evolution with output times $t_p = 2\pi n/\omega,\ n = 1500, \ldots, 2000$, and the $x_p(t)$ are the oscillator displacements at those times. Regions where the motion is periodic as well as chaotic are clearly seen.



- Bifurcation diagram (2 of 4). Zoomed-in view of previous diagram.



- Bifurcation diagram (3 of 4). Zoomed-in view of previous diagram.



- Bifurcation diagram (4 of 4). Zoomed-in view of previous diagram. Central region (vertically) shows "classic" period doubling to complete chaotic behaviour