PHY329 WAVE EQUATION (note correction) OCT 23, 1996

GENERATING INITIAL DATA: $\hat{u}_j^0$ and $\hat{u}_j^1$

First note a slight change in notation from the original lecture notes. Here, for ease of exposition, $\hat{u}_j^n$ is the quantity which satisfies the difference equation (22.4), while u_j^n satisfies the differential equation (22.2).

We begin by assuming that our difference solution will admit a Richardson expansion

$$\hat{u}(x,t) = u(x,t) + h^2 e_2(x,t) + h^4 e_4(x,t) + \cdots \tag{22.7}$$

and ask what this implies about how accurately we must specify $\hat{u}_j^0$ and $\hat{u}_j^1$.

First recall that the problem prescription includes the initial conditions:

$$u\,(x,0) = f\,(x) \qquad\qquad u_t\,(x,0) = g\,(x)$$

Clearly then, we can (and might as well) specify $\hat{u}_j^0$ exactly

$$\hat{u}_j^0 = f_j$$

which implies that $e_2(x,0) = 0$. Then in order for (22.7) to hold, we must have

$$\begin{aligned}\hat{u}_j^1 &\equiv \hat{u}(x_j,\, \triangle t\,) \\ &= u(x_j,\, \triangle t\,) + h^2 e_2(x_j,\, \triangle t\,) + O(h^4)\end{aligned}$$

which implies that

$$\hat{u}(x_j,\, \triangle t\,) - u(x_j,\, \triangle t\,) = h^2 e_2(x_j,\, \triangle t\,) + O(h^4)$$

But

$$\begin{aligned} e_2(x_j,\, \triangle t\,) &= e_2(x_j, 0) + \,\triangle t\, \frac{\partial e_2}{\partial t}(x_j, 0) \, + \, O(\,\triangle t^{\,2}) \\ &= \,\triangle t\, \frac{\partial e_2}{\partial t}(x_j, 0) + O(\,\triangle t^{\,2}) = \lambda h \frac{\partial e_2}{\partial t}(x_j, 0) + O(\,\triangle t^{\,2}) \end{aligned}$$

Thus,

$$\hat{u}(x_j,\, \triangle t\,) - u(x_j,\, \triangle t\,) = h^2 \left(\lambda h \frac{\partial e_2}{\partial t}(x_j, 0) \right) \, + \, \cdots = O(h^3)$$

So we see that we must specify $\hat{u}^{\,1}_{j}$ to $O(h^3)$ accuracy, *not* $O(h^2)$ as was erroneously claimed in the lecture. However, this is readily done by simply Taylor-expanding to higher order:

$$u(x_j,\, \triangle t\,) = u(x_j, 0) + \,\triangle t\; u_t(x_j, 0) + \frac{1}{2}\,\triangle t^{\,2}\, u_{tt}(x_j, 0) + O(\,\triangle t^{\,3})$$

Recalling (22.6) where we specified the initial conditions in terms of initially left-moving and right-moving profiles, $l(x)$ and $r(x)$, and their derivatives, $l'(x)$ and $r'(x)$:

$$\begin{aligned} u\,(x, 0) &= l\,(x) + r\,(x) \\ u_t\,(x, 0) &= l'\,(x) - r'\,(x) \end{aligned}$$

and noting that

$$u_{tt}\left(x,0\right)=l''\left(x\right)+r''\left(x\right)$$

we have for our initialization:

$$\hat{u}_{j}^{0}=l_{j}+r_{j} \tag{22.7}$$

$$\hat{u}_{j}^{1}=l_{j}+r_{j}+\triangle t\left(l'_{j}-r'_{j}\right)+\frac{1}{2}\triangle t^{2}\left(l''_{j}+r''_{j}\right) \tag{22.8}$$