```
c===========================================================
c     Solves 1-d linear boundary value problem
c
c         u''(x) = f(x)  on  x = [0,1]; u(0) = u0, u(1) = u1
c
c     using mixed fourth-order and second order finite
c     difference technique and LAPACK banded solver DGBSV.
c===========================================================
      program         bvp1d4

      implicit        none

      integer         i4arg

c-----------------------------------------------------------
c     Domain extrema and maximum system size.
c-----------------------------------------------------------
      real*8          xmin,          xmax
      parameter     ( xmin = 0.0d0,  xmax = 1.0d0 )

      integer         maxn
      parameter     ( maxn = 2**19 + 1 )
c-----------------------------------------------------------
c     Storage for discrete x-values, unknowns, exact
c     solution and right hand side values.
c-----------------------------------------------------------
      real*8          x(maxn),       u(maxn),
     &                uexact(maxn),  f(maxn)
```

```
c-----------------------------------------------------------
c     Number of lower and upper bands.
c-----------------------------------------------------------
      integer         kl,            ku
      parameter     ( kl = 2,        ku = 2  )
c-----------------------------------------------------------
c     Storage for LAPACK-banded-form of linear system,
c     right-hand-side of system and pivot vector,
c     for use with DGBSV.
c
c     Note that for pivoting purposes (row interchanges)
c     DGBSV requires an additional 'kl' rows of workspace.
c     Leading dimension of 'ab' is thus
c
c        ku + kl + kl + 1 = 7
c-----------------------------------------------------------
      integer         ldab
      parameter     ( ldab  = 7 )
      real*8          ab(ldab,maxn), rhs(maxn)
      integer         ipiv(maxn)
c-----------------------------------------------------------
c     Other standard LAPACK parameters.
c-----------------------------------------------------------
      integer         nrhs,          info
c-----------------------------------------------------------
c     Discretization level, size of system (# of discrete
c     unknowns) and output option.
c-----------------------------------------------------------
      integer         level,         n,             option
c-----------------------------------------------------------
c     Storage for difference coefficients.  Note: these
c     arrays have elements -2, -1, 0, 1 and 2.
c-----------------------------------------------------------
      real*8          cdd2(-2:2),    cdd4(-2:2),    c0(-2:2)
```

```
c-----------------------------------------------------------
c     Mesh spacing and related constants.
c-----------------------------------------------------------
      real*8          h,              hm2,            hm2by12
c-----------------------------------------------------------
c     Other locals.
c-----------------------------------------------------------
      integer         i,              j,              k
      real*8          rmserr

c-----------------------------------------------------------
c     Argument parsing.
c-----------------------------------------------------------
      level = i4arg(1,-1)
      if( level .lt. 0 ) go to 900
      n = 2 ** level + 1
      if( n .gt. maxn ) then
         write(0,*) 'Insufficient internal storage'
         stop
      end if
      option = i4arg(2,0)
```

```
c-----------------------------------------------------------
c     Set up finite-difference 'mesh' (discrete x-values)
c     and difference coefficient arrays.
c-----------------------------------------------------------
      h       =  1.0d0 / (n - 1)
      do j = 1 , n
         x(j) = xmin + (j - 1) * h
      end do
      x(n) = xmax

      hm2     =  1.0d0 / (h * h)
      hm2by12 = hm2 / 12.0d0

      c0(-2)   =  0.0d0
      c0(-1)   =  0.0d0
      c0( 0)   =  1.0d0
      c0( 1)   =  0.0d0
      c0( 2)   =  0.0d0

      cdd2(-2) =  0.0d0
      cdd2(-1) =  hm2
      cdd2( 0) =  -2.0d0 * hm2
      cdd2( 1) =  hm2
      cdd2( 2) =  0.0d0

      cdd4(-2) = -hm2by12
      cdd4(-1) =  16.0d0 * hm2by12
      cdd4( 0) = -30.0d0 * hm2by12
      cdd4( 1) =  16.0d0 * hm2by12
      cdd4( 2) = -hm2by12
```

```
c-----------------------------------------------------------
c     Set up exact solution and right hand side vector.
c-----------------------------------------------------------
      call exact(uexact,f,x,n)

c===========================================================
c     Set up banded system.  Recall that for LAPACK
c     banded storage for LU decomposition
c
c     a( i , j ) -> ab( kl + ku + 1 + i - j , j )
c===========================================================

c-----------------------------------------------------------
c     i = 1:  (Left boundary) u(1) = u_0
c-----------------------------------------------------------
      i = 1

      do k = 0 , 2
         j = i + k
         ab(kl + ku + 1 + i - j,j) = c0(k)
      end do
      rhs(i) = uexact(i)
c-----------------------------------------------------------
c     i = 2:  O(h^2) approximation of u''(x) = f(x)
c-----------------------------------------------------------
      i = 2

      do k = -1 , 2
         j = i + k
         ab(kl + ku + 1 + i - j,j) = cdd2(k)
      end do
      rhs(i)  = f(i)
```

```
c-----------------------------------------------------------
c     i = 3, ..., n-2: O(h^4) approximation of u''(x) = f(x)
c-----------------------------------------------------------
      do i = 3 , n - 2
         do k = -2 , 2
            j = i + k
            ab(kl + ku + 1 + i - j,j) = cdd4(k)
         end do
         rhs(i) = f(i)
      end do
c-----------------------------------------------------------
c     i = n-1:  O(h^2) approximation of u''(x) = f(x)
c-----------------------------------------------------------
      i = n - 1

      do k = -2 , 1
         j = i + k
         ab(kl + ku + 1 + i - j,j) = cdd2(k)
      end do
      rhs(i)  = f(i)
c-----------------------------------------------------------
c     i = n:  (Right boundary) u(n) = u_1
c-----------------------------------------------------------
      i = n

      do k = -2 , 0
         j = i + k
         ab(kl + ku + 1 + i - j,j) = c0(k)
      end do
      rhs(i) = uexact(i)
```

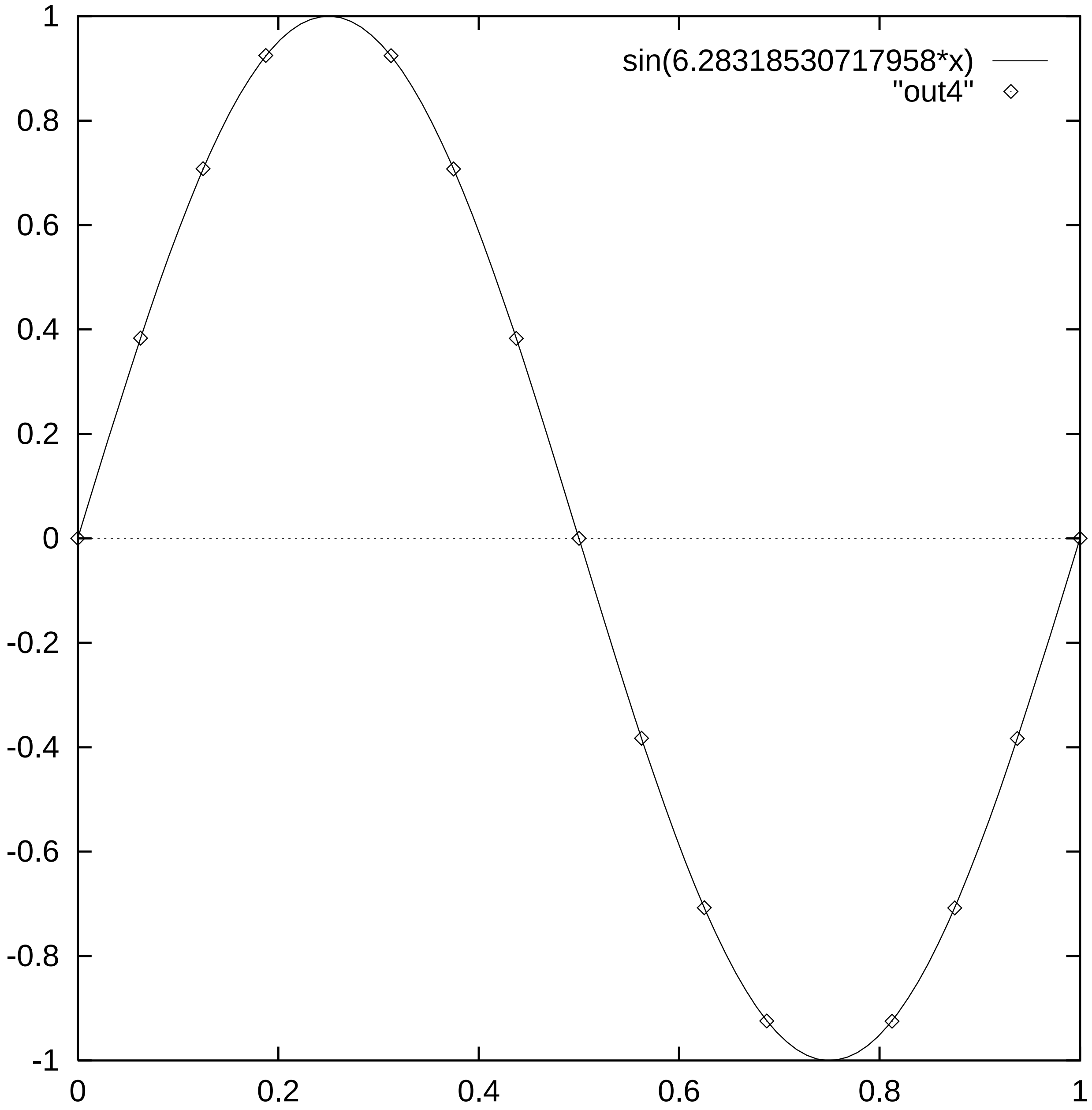
```
c===========================================================
c     Solve banded system.
c===========================================================

      nrhs = 1
      call dgbsv( n, kl, ku, nrhs, ab, ldab, ipiv, rhs, n,
     &            info )

      if( info .eq. 0 ) then
c-----------------------------------------------------------
c        Solver successful, output either (x_j, u_j) or
c        (x_j, error_j) to stdout.  Also compute rms error
c        and output to standard error.
c-----------------------------------------------------------
         rmserr = 0.0d0
         do j = 1 , n
            if( option .eq. 0 ) then
               write(*,*) x(j), rhs(j)
            else
               write(*,*) x(j), (uexact(j) - rhs(j))
            end if
            rmserr = rmserr + (uexact(j) - rhs(j)) ** 2
         end do
         rmserr = sqrt(rmserr / n)
         write(0,*) 'rmserr = ', rmserr
      else
c-----------------------------------------------------------
c        Solver failed.
c-----------------------------------------------------------
         write(0,*) 'bvp1d4: dgbsv() failed, info = ', info
      end if

      stop
```

```
 900  continue
         write(0,*) 'usage: bvp1d4 <level> [<option>]'
         write(0,*)
         write(0,*) '       Specify option .ne. 0 for output'
         write(0,*) '       of error instead of solution'
      stop

      end

c===========================================================
c     Computes exact values for u(x) (unknown function)
c     and f(x) (right hand side function).  x array must
c     have been previously defined.
c===========================================================
      subroutine exact(u,f,x,n)

         implicit        none
         integer         n
         real*8          u(n),     f(n),    x(n)

         real*8          pi2
         integer         j

         pi2 = 8.0d0 * atan(1.0d0)
         do j = 1 , n
            u(j) = sin(pi2 * x(j))
            f(j) = -pi2 * pi2 * u(j)
         end do

         return

      end
```

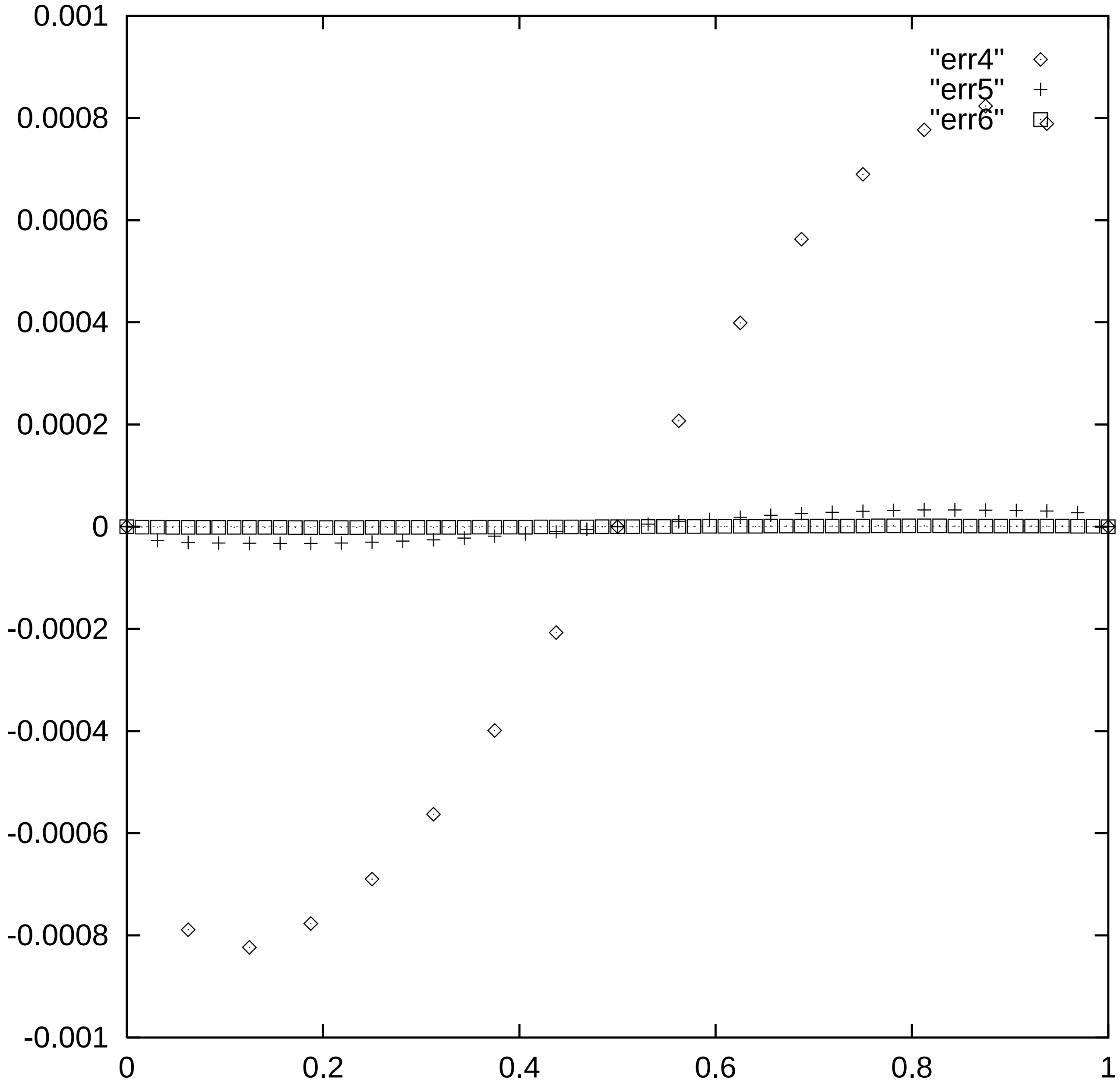
```
##############################################################
# Building 'bvp1d4' and sample output on the SGIs
###############################################################
sgi1% pwd; ls
/usr/people/phys410/linsys/ex3
Makefile   bvp1d4.f

sgi1% make
f77 -g -64 -c bvp1d4.f
f77 -g -64 -L/usr/local/lib bvp1d4.o \
    -lp410f -llapack -lblas -o bvp1d4

sgi1% bvp1d4
 usage: bvp1d4 <level> [<option>]

        Specify option .ne. 0 for output
        of error instead of solution

###############################################################
# Note: compare with completely second-order 'bvp1d 4'
# which results in rms error of approximately 9.0E-03.
# These results are about 15 times better at this resolution
# (h = 1/16).
###############################################################
sgi1% bvp1d4 4
   0.0000000000000000E+00 -1.0547118733938987E-14
   6.2500000000000000E-02  0.3834724412118576
   0.1250000000000000      0.7079302872941245
   0.1875000000000000      0.9246563908935262
   0.2500000000000000       1.000689732294703
   0.3125000000000000      0.9244421766816860
   0.3750000000000000      0.7075056502724236
   0.4375000000000000      0.3828904610080090
   0.5000000000000000     -3.1565329029368671E-15
```

```
   0.5625000000000000     -0.3828904610080154
   0.6250000000000000     -0.7075056502724303
   0.6875000000000000     -0.9244421766816929
   0.7500000000000000      -1.000689732294711
   0.8125000000000000     -0.9246563908935345
   0.8750000000000000     -0.7079302872941338
   0.9375000000000000     -0.3834724412118676
    1.000000000000000     -2.4492935982947064E-16
 rmserr =   5.8394829778013078E-04

##############################################################
# Convergence test: Solve BVP on a sequence of levels,
# redirect stdout so that only overall RMS error appears
# on terminal.  Rate of convergence is not as definitive
# as it was for the second order calculation, but clearly
# this method converges much more rapidly than the second
# order method.
##############################################################
sgi1% foreach level (4 5 6 7 8 9 10)
foreach? bvp1d4 $level > /dev/null
foreach? end
 rmserr =   5.8394829778013078E-04
 rmserr =   2.5181486533000874E-05
 rmserr =   1.1531108217752065E-06
 rmserr =   5.8557438669720841E-08
 rmserr =   3.2465293440184777E-09
 rmserr =   1.8966747918271887E-10
 rmserr =   9.4215663095393918E-12
```

```
#################################################################
# Making output files for subsequent plotting via gnuplot.
# See previous handout for 'bvp1d' for typical 'gnuplot'
# "script" files.
#################################################################
sgi1% bvp1d4 4 > out4
 rmserr =   5.8394829778013078E-04
sgi1% bvp1d4 4 1 > err4
 rmserr =   5.8394829778013078E-04
sgi1% bvp1d4 5 1 > err5
 rmserr =   2.5181486533000874E-05
sgi1% bvp1d4 6 1 > err6
 rmserr =   1.1531108217752065E-06
```



sin(6.28318530717958*x)
"out4"
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1



0.001
0.0008
0.0006
0.0004
0.0002
0
-0.0002
-0.0004
-0.0006
-0.0008
-0.001
0
0.2
0.4
0.6
0.8
1
"err4"
"err5"
"err6"