Finite Difference Solution of the Korteweg & de Vries (KdV) Equation

PHYS 210 Term Project Proposal

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- **Overview**
  - The KdV equation is a nonlinear wave equation in one space variable and time which admits interesting "particle-like" solutions known as solitons
  - The KdV solitons travel with velocities that are dependent on their amplitudes, and solitons with different amplitudes can undergo interactions (collisions) which, when done, leave each soliton unchanged
- **Project Goals**
  - To write an MATLAB (octave) code which solves the KdV equation numerically, using second-order finite difference techniques
  - To establish correctness of the implementation of the code through convergence tests, independent residual evaluation, and comparison with known solutions
  - To investigate a variety of initial conditions for the equation, including those describing single and multi-soliton solutions

- **Mathematical Formulation (Equations of Motion)**
  - The KdV equation can be written in the form

$$\frac{\partial u}{\partial t} + 12u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

where $u \equiv u(t, x)$

  - The equation will be solved as an initial boundary problem on the domain

$$-x_{\max} \le x \le x_{\max} \qquad 0 \le t \le t_{\max} \tag{2}$$

with initial conditions

$$u(0, x) = u_0(x) \tag{3}$$

where $u_0(x)$ is some specified function, and boundary conditions

$$u(t, -x_{\max}) = u(t, x_{\max}) = 0 \tag{4}$$

- **Numerical Approach**
  - The KdV equation will be discretized using a second-order finite difference technique, wherein the continuum domain (t, x) will be replaced with a discrete grid (lattice) of points (t^n, x_j) such that

$$\begin{aligned} x_j &= -x_{\max} + jh, \quad j = 0, 2, \cdots n_x &(5)\\ t^n &= n\lambda h, \quad n = 0, 1, 2, \cdots n_t &(6)\end{aligned}$$

  - I will then approximate (discretize) the KdV equation as follows

$$\frac{u_j^{n+1} - u_j^n}{\lambda h} + \mu_t \left(D_x u_j^n\right) + 12\mu_t \left(u_j^n\right) \mu_t \left(D_x u_j^n\right) + \mu_t \left(D_{xxx} u_j^n\right) = 0 \quad (7)$$

where the operators μ_t, D_x and D_{xxx} are defined by:

$$\begin{aligned} \mu_t \left(u_j^n\right) &= \frac{1}{2}\left(u_j^{n+1} + u_j^n\right) &(8)\\ D_x \left(u_j^n\right) &= \frac{u_{j+1}^n - u_{j-1}^n}{2h} &(9)\\ D_{xxx} \left(u_j^n\right) &= \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{h^3} &(10)\end{aligned}$$

- **Numerical Approach (continued)**
  - When supplemented with discrete versions of the boundary conditions, and assuming that the values of $u_j^n, j = 0, 2, \cdots n_x$ are known (for $n = 0$, these will be determined from the initial conditions), the approximation to the KdV equation becomes a set or nonlinear algebraic equations in the unknowns $u_j^{n+1}, j = 0, 2, \cdots n_x$
  - I will solve this system of equations using a multi-dimensional Newton-Raphson method
  - Each iteration of the Newton method will require the solution of a banded linear system which will be solved using MATLAB (octave) functions that are based on LAPACK (Linear Algebra PACKage) routines
  - When the Newton method has converged, I will have advanced the discrete solution in the time, and can then repeat the process for the desired number of time steps
- **Visualization and Plotting Tools**
  - I will use **xvs** for interactive analysis and generation of mpeg animations, and **sm** for plots to be included in my report

- **Testing & Numerical Experiments**
  - **Testing**
    - Convergence testing: Fix initial data, compute solutions using discretization scales $h, h/2, h/4, h/8 \cdots$, and ensure that $O(h^2)$ convergence behaviour is obtained
    - Independent residual evaluation: Compute solutions as above with varying h, and apply a finite difference approximation of the PDE that is distinct from the one used to generate the approximate solution. Check that the residuals are $O(h^2)$
    - Check the numerical results against the known closed-form solutions for single solitons
  - **Numerical Experiments**
    - Investigate the interaction of two or more solitons with different amplitudes, and measure the expected "phase-shift" effect which results from the interaction (collision)
    - Investigate evolution of generic types of initial data (such as Gaussian profiles) to see if solitons generically emerge

- **Project Timeline**

Dates	Activities
10/23–10/29	Do basic research, derive equations & design code
10/30–11/05	Implement code
11/06–11/12	Test code
11/13–11/19	Run numerical experiments, begin presentation & report
11/20–11/26	Analyze data, continue work on presentation & report
11/27–11/30	Polish presentation and work on final draft of report
12/01	Give presentation!
12/01–12/03	Finish final draft of report
12/04	Submit report!

- **References**

- R.K.Dodd et al, *Solitons and Nonlinear Equations*, Academic Press, London, (1982)
- http://bh0.phas.ubc.ca/~matt/Teaching/05Fall/PHYS410/Projects/kdv.pdf

QUESTIONS?

COMMENTS?

SUGGESTIONS?

THUNDEROUS APPLAUSE!!