# PHYS 410/555 Computational Physics

## *The Method of Lines for the Wave Equation*

One approach to the numerical solution of time-dependent *partial differential equations* (PDEs) is to use a discretization technique, such as finite-differencing, but only apply it explicitly to the *spatial* part(s) of the PDE operator(s) under consideration. Following the spatial discretization, one is left with a set of coupled *ordinary differential equations* in $t$, which can then often be solved by a "standard" ODE integrator such as `LSODA`.

As an example of this technique, consider the *wave equation* in one space dimension (often called the "1D wave equation"):

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t) \tag{1}$$

Introducing the notation that a subscript denotes partial differentiation, and suppressing the explicit $x$ and $t$ dependence, (1) can also be written as

$$u_{tt} = c^2 u_{xx} \tag{2}$$

As you probably know, the wave equation describes propagation of disturbances, or waves, at a speed $c$: waves can either travel to the right (velocity $+c$), or to the left (velocity $-c$). Without loss of generality, we can always choose units such that $c = 1$, and, for convenience, we will do so. Our wave equation then becomes:

$$u_{tt} = u_{xx} \tag{3}$$

As with any differential equation, boundary conditions play a crucial role in fixing a solution of (3). Here, we will solve the wave equation on the domain

$$0 \le x \le 1 \qquad t \ge 0 \tag{4}$$

and will thus have to provide boundary conditions at $x = 0$ and $x = 1$, as well as initial conditions at $t = 0$.

For concreteness, we will prescribe *Dirichlet boundary conditions*:

$$u(0,t) = u(1,t) = 0 \tag{5}$$

as well as the following *initial conditions*:

$$u(x,0) = u_0(x) = \exp\left(-\left(\frac{x - x_0}{\Delta}\right)^2\right) \tag{6}$$

$$u_t(x,0) = 0 \tag{7}$$

where $x_0$ $(0 < x_0 < 1)$ and $\Delta$ are specified constants.

If we think in terms of small-amplitude waves propagating on a string, then the Dirichlet conditions correspond to keeping the ends of the string fixed. The interpretation of the initial conditions is as follows: In solving (3) we have the freedom to specify the amplitude of the disturbance for all values of $x$, as well as the time-rate of change of that amplitude, again for all values of $x$.

We thus have two functions worth of freedom in specifying our initial conditions. We set the initial amplitude to some functional form given by $u_0(x)$; here we use a "gaussian pulse" that is centred at $x_0$, and that has an overall effective width of a few $\times\ \Delta$. We also set the initial time rate of change of the amplitude to be 0 for all $x$.

Such data is known as *time symmetric*, since it defines an instant in the evolution of the wave equation where there is a $t \to -t$ symmetry. In other words, with time symmetric initial data, if we integrate *backward* in time, we will see exactly the same solution as a function of $-t$ as we see integrating forward in time.

Since the wave equation describes propagating disturbances, and given that the initial conditions *are* time symmetric, a little reflection might convince you that the initial conditions (6) and (7) must represent a superposition of equal amplitude right-moving and left-moving pulses. Thus, we should expect the solution of (3), subject to (5), (6) and (7) to describe the propagation of two equal-amplitude pulses that are initially coincident, but that subsequently move apart reflect off $x = 0$ and $x = 1$ respectively, move together, through each other, then apart, etc. etc. Indeed, this is precisely the behaviour we will observe in our subsequent numerical solution.

As mentioned above, the *method of lines*, involves an explicit discretization *only of the spatial part of the PDE operator*. Here we will use the familiar $O(h^2)$ finite-difference approach to the treatment of $u_{xx} \equiv \partial^2 u / \partial x^2$.

However, before proceeding to the spatial discretization, we first note that (3) is a second-order-in-time equation. In order that our approach eventually produce a set of *first order* ODEs in $t$, we introduce an auxiliary variable, $v(x,t)$,

$$v(x,t) \equiv u_t(x,t) \equiv \frac{\partial u}{\partial t}(x,t) \tag{8}$$

and then rewrite (3) as the *system:*

$$u_t = v \tag{9}$$

$$v_t = u_{xx} \tag{10}$$

The boundary conditions become

$$u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0 \tag{11}$$

while the initial conditions are now

$$u(x,0) = u_0(x) = \exp\left(-\left(\frac{x-x_0}{\Delta}\right)^2\right) \tag{12}$$

$$v(x,0) = 0 \tag{13}$$

We can now proceed with the spatial discretization. To that end, we replace the continuum spatial domain $0 \leq x \leq 1$ by a uniform finite difference mesh, $x_j$:

$$x_j \equiv (j-1)h \qquad j = 1, 2, \cdots N \qquad h \equiv (N-1)^{-1} \tag{14}$$

and introduce the discrete unknowns, $u_j$ and $v_j$:

$$u_j \equiv u_j(t) \equiv u(x_j, t) \tag{15}$$

$$v_j \equiv v_j(t) \equiv v(x_j, t) \tag{16}$$

Using the usual centred, $O(h^2)$ approximation for the second spatial derivative,

$$u_{xx}(x_j) = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + O(h^2) \tag{17}$$

eqs. (9) and (10) become a set of $2(N-2)$ coupled ODEs for the $2(N-2)$ unknowns $u_j(t)$ and $v_j(t)$, $j = 2, \cdots N-1$:

$$\frac{du_j}{dt} = v_j \qquad j = 2, \cdots N-1 \tag{18}$$

$$\frac{dv_j}{dt} = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \qquad j = 2, \cdots N-1 \tag{19}$$

We can implement the Dirichlet boundary conditions as follows: if the boundary conditions are satisfied at the initial time, $t = 0$, then they will be satisfied at all future times provided that the time derivatives of $u$ and $v$ vanish at the boundaries. Using this observation, we can now write down a complete set of $2N$ coupled ODEs in the $2N$ unknowns $u_j(t)$ and $v_j(t)$ which can then be solved using `LSODA`:

$$\frac{du_1}{dt} = 0 \tag{20}$$

$$\frac{du_j}{dt} = v_j \qquad j = 2, \cdots N-1 \tag{21}$$

$$\frac{du_N}{dt} = 0 \tag{22}$$

$$\frac{dv_1}{dt} = 0 \tag{23}$$

$$\frac{dv_j}{dt} = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \qquad j = 2, \cdots N-1 \tag{24}$$

$$\frac{dv_N}{dt} = 0 \tag{25}$$

This solution procedure is implemented by the the program `wave` (See `~phys410/ode/wave`). You will follow an analogous approach to solve the *diffusion equation* in the final homework.