Critical Behavior in Gravitational Collapse of a Yang-Mills Field

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We present results from a numerical study of spherically symmetric collapse of a self-gravitating, SU(2) gauge field. Two distinct critical solutions are observed at the threshold of black hole formation. In one case, the critical solution is discretely self-similar, and black holes of arbitrarily small mass can form. However, in the other instance, the critical solution is the $n = 1$ static Bartnik-Mckinnon sphaleron, and black hole formation turns on at finite mass. The transition between these two scenarios is characterized by the superposition of both types of critical behavior. [S0031-9007(96)00656-4]

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In a recent numerical study of gravitational collapse of a massless scalar field, a type of critical behavior was found at the threshold of black hole formation [1]. More precisely, in the analysis of spherically symmetric evolution of various one-parameter families of initial data describing imploding scalar waves, it was observed that there is generically a critical parameter value, $p = p^*$, which signals the onset of black hole formation. In the subcritical regime, $p < p^*$, all of the scalar field escapes to infinity leaving flat spacetime behind, while for supercritical evolutions, $p > p^*$, black holes form with masses well fit by a scaling law, $M_{\rm BH} \propto (p - p^*)^\gamma$. Here, the critical exponent, $\gamma \simeq 0.37$, is universal in the sense of being independent of the details of the initial data. Thus, the transition between no-black-hole and black-hole spacetimes may be viewed as a continuous phase transition with the black hole mass playing the role of order parameter. In the intermediate asymptotic regime (i.e., before a solution "decides" whether or not to form a black hole) near-critical evolutions approach a universal attractor, called the critical solution, which exhibits discrete self-similarity (echoing). Using the same basic technique of studying families which "interpolate" between no-black-hole and black-hole spacetimes, similar critical behavior has been observed in several other models of gravitational collapse [2–4].

In this Letter we summarize results from the numerical study of the evolution of a self-gravitating non-Abelian gauge field modeled by the SU(2) Einstein-Yang-Mills (EYM) equations. In addition to its intrinsic physical interest, we have chosen this model because, in contrast to all previously studied models, it contains static solutions which we suspected could affect the qualitative picture of critical behavior. Let us recall that these static solutions, discovered by Bartnik and Mckinnon (BK) [5,6], form a countable family X_n $(n \in N)$ of spherically symmetric, asymptotically flat, regular, but unstable, configurations.

Our main new result is the observation that for certain families of initial data the static BK solution, X_1, plays the role of a critical solution separating collapse from dispersal. Since in this case there is a finite gap in the spectrum of black hole masses, we call this "type I"

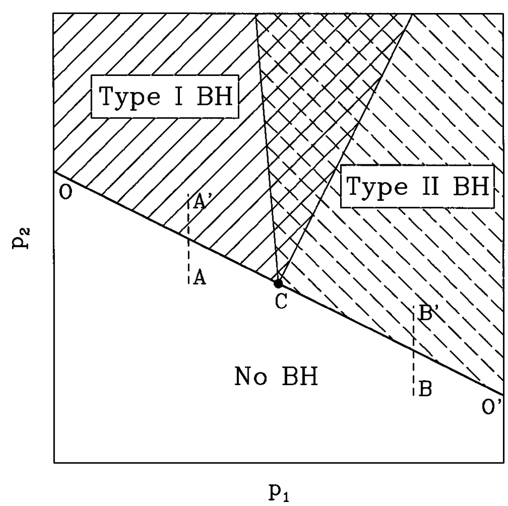

FIG. 1. Schematic representation of "phase space" for spherically symmetric Yang-Mills collapse, showing possible end states of evolutions from a sufficiently general two-parameter family of initial conditions. The critical line OO' demarks the threshold of black hole formation. An interpolating family such as AA' exhibits type I behavior: The critical solution is the static BK solution X_1, and the smallest black hole formed has finite mass. Families such as BB' exhibit type II behavior: The critical solution is discretely self-similar ($\Delta \approx 0.74$), black hole formation turns on at infinitesimal mass, and mass scaling with $\gamma \approx 0.20$ is observed. At C, the two types of critical behavior coexist.

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behavior (in analogy to a first order phase transition), to distinguish it from "type II" behavior (which we also observe), where black hole formation turns on at infinitesimal mass. As sketched in Fig. 1, suitable two-parameter families of initial data exhibit both types of behavior.

We consider spherically symmetric Einstein-Yang-Mills equations with the gauge group SU(2). We write the time-dependent, spherically symmetric metric as

$$ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2 d\Omega^2. \quad (1)$$

For the Yang-Mills (YM) field, we assume the purely magnetic ansatz which in the Abelian gauge has the form [7] $F = dW \wedge \Omega - (1 - W^2)\tau_3 d\vartheta \wedge \sin\vartheta d\varphi$, where $\Omega = \tau_1 d\vartheta + \tau_2 \sin\vartheta d\varphi$ and the τ_i are Pauli matrices. Thus, the matter content of the model is described by a single function, $W(r,t)$, which we call the Yang-Mills potential. As follows from this ansatz, the vacuua of the YM field are given by $W = \pm 1$. Using overdots and primes in the following to denote $\partial/\partial t$ and $\partial/\partial r$, respectively, we introduce auxiliary YM variables: $\Phi \equiv W'$ and $\Pi \equiv a\dot{W}/\alpha$. The dynamics of the EYM model can then be computed from (see [7])

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)', \qquad \dot{\Pi} = \left(\frac{\alpha}{a}\Phi\right)' + \frac{\alpha a}{r^2}W(1 - W^2), \quad (2)$$

$$\frac{a'}{a} = \frac{1 - a^2}{2r} + \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1 - W^2)^2\right), \quad (3)$$

$$\frac{\alpha'}{\alpha} = \frac{a^2 - 1}{2r} + \frac{1}{r}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}(1 - W^2)^2\right), \quad (4)$$

$$W(r,t) \equiv \pm 1 + \int_0^r \Phi(\tilde{r},t)d\tilde{r}. \quad (5)$$

We have solved the initial value problem for many one-parameter families of asymptotically flat, regular initial data, some of which are listed in Table I. To ensure regularity at the origin we require that $W(r,t) \to \pm 1 + O(r^2)$ as $r \to 0$. The numerical results described below were generated using a modified version of the adaptive-mesh algorithm used to perform the original scalar field calculations [1]. The sensitivity of the mesh-refinement algorithm was again very helpful in efficiently computing near-critical solutions, but was not nearly as crucial as it was for the scalar case. In fact, we have also reproduced most of the following results using a uniform-grid code much like the one described in [8].

TABLE I. Initial data families used and type(s) of critical behavior observed. Adjustable parameters are enclosed in $[\cdots]$. Initial data for F_α is time symmetric ($\dot{W}(r,0) = 0$); for all other families, the initial data are ingoing only. $C(r,s) = (1 + a(1 + br/s)\exp[-2(r/s)^2])$ with constants a and b chosen so that $W(0,0) = 1$ and $W'(0,0) = 0$.

Family	$W(r,0)$	Type
$F_a[x;s]$	$C(r,s)\tanh[(x-r)/s]$	I, II
$F_b[a;\delta;q]$	$1 + a\exp(-[(r-20)/\delta]^q)$	II
$F_c[\delta]$	$[1-(r/\delta)^2]/([1-(r/\delta)^2]^2 + 4r^2)^{1/2}$	I
$F_d[a;q]$	$-1 + 2a\exp(-[(r-17)/4]^q)$	I, II

As stated above, depending on the particular form of initial data used, we have found two types of critical behavior.

Type II behavior.—In this case, we observe a continuous no-black-hole–black-hole phase transition, and the overall picture of criticality is very much analogous to that of scalar field collapse. For a generic type II family, and in the near-critical, nonlinear regime, we conjecture that the evolution asymptotes to a locally unique (up to $r, t \to \sigma r, \sigma t$, for arbitrary $\sigma > 0$) discretely self-similar solution, with an echoing exponent, $\Delta \approx 0.74$. As with the scalar field case, we expect that the precisely critical solution echoes an infinite number of times, exhibits unbounded growth of curvature near $r = 0$, and is singular at the origin at some finite value of central proper time, T^*. Figure 2 shows profiles of various echoing quantities at $T \approx T^*$ from a family F_a calculation (see Table I) with $|s - s^*|/s^* \approx 10^{-15}$. Typical evidence for scale periodicity is shown in Fig. 3.

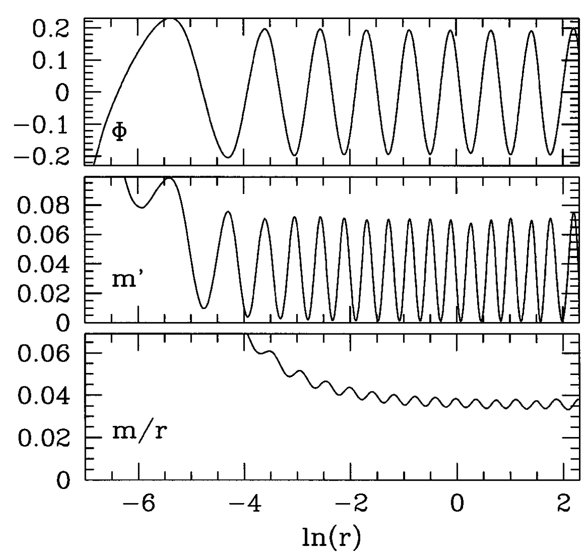

FIG. 2. Late time profiles of marginally subcritical type II collapse using family F_a (see Table I). Note that the mass aspect, $m(r,t)$, is defined via $a^2 = (1 - 2m/r)^{-1}$. The large number of echoes visible here (for fixed $|p - p^*|/p^*$), relative to the scalar case, is a reflection of the relatively small value of the echoing exponent ($\Delta_{\rm YM} \approx 0.74$ vs $\Delta_{\rm SF} \approx 3.44$). However, this is partly offset by the fact that the mass scaling exponents for the two models also differ significantly. In general, dimensional/scaling considerations suggest that $\Delta\delta n = -\gamma\delta\pi$, where δn and $\delta\pi$ are the changes in echo number, n, and $\pi \equiv \ln|p - p^*|$, respectively [9].

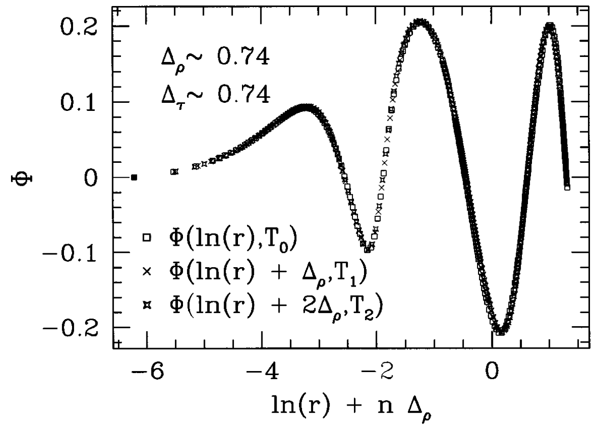

FIG. 3. Scale periodicity of the type II solution. This plot shows the superposition of a near-critical profile of Φ (at a particular time) with the first two echoes which subsequently develop. The central proper time, T_0, at which the earliest profile is monitored is arbitrary; times T_1 and T_2 and the rescaling exponent, Δ_ρ, are then chosen to minimize $\Phi[\ln(r) + n\Delta_\rho, T_n] - \Phi[\ln(r), T_0]$. An independent estimate of Δ is generated by first estimating the critical time, T^*, for the family, and then computing $\Delta_\tau \equiv \ln[(T^* - T_n)/(T^* - T_{n+1})]$.

As expected, type II families also exhibit mass scaling in the supercritical regime: $M_{\rm BH} \propto (p - p^*)^\gamma$. Typical results are shown in Fig. 4, and we estimate that the value $\gamma \approx 0.20$ is accurate to a few percent. We note in passing that this result is another piece of the growing body of evidence which has shown that γ is *not* constant across all collapse models [4,10,11]. As argued in [12], mass scaling and universality (initial-data independence), strongly suggest that the stable manifold of a type II solution, such as the one described here, is of codimension one. This picture, which predicts that γ is the reciprocal of the Lyapunov exponent of a single growing mode associated with a critical solution, has now been validated for at least two distinct matter models with *continuously* self-similar critical solutions [4,12–15]. Perturbative treatment of critical solutions with discrete symmetry is more involved, although considerable progress has been made for the scalar case [16]. Here we remark only that any techniques which work for the scalar field should be largely applicable to this solution.

Type I behavior.—As stated previously, solutions in a type I interpolating family asymptote to the *static* BK solution, X_1, as $p \to p^*$. This behavior appears generically for kink-type initial profiles of the YM potential W, such as F_c in Table I. In this case it has already been established that each of the BK solutions, X_n, has exactly n unstable modes (within the purely magnetic ansatz); hence we are certain that the stable manifold of the critical solution has codimension one. Initial data with small $|p - p^*|$ results in an evolution which approaches X_1 and stays in its vicinity for central proper time, $T \approx -\lambda \ln|p - p^*|$. The configuration then either disperses to infinity ($p < p^*$) or collapses to a black hole with finite mass ($p > p^*$). Typical results from a marginally subcritical type I evolution are shown in Fig. 5.

Our calculations are somewhat complementary to those performed by Zhou and Straumann [8]. Those authors

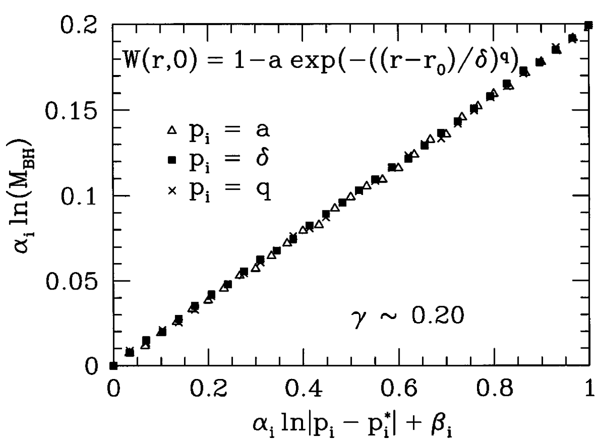

FIG. 4. Mass scaling of type II solutions. Each marker type corresponds to a different family of supercritical computations. For each family, constants α_i and β_i are chosen to unit normalize the x range and place the first data point (smallest black hole) at the origin. For all data sets, the least squares fit for the slope, γ, is 0.20, with an estimated uncertainty of a few percent. In addition, for all families, the unnormalized π range was 18; thus, in each case, the black hole mass spans a factor of $e^{18\gamma} \approx 37$.

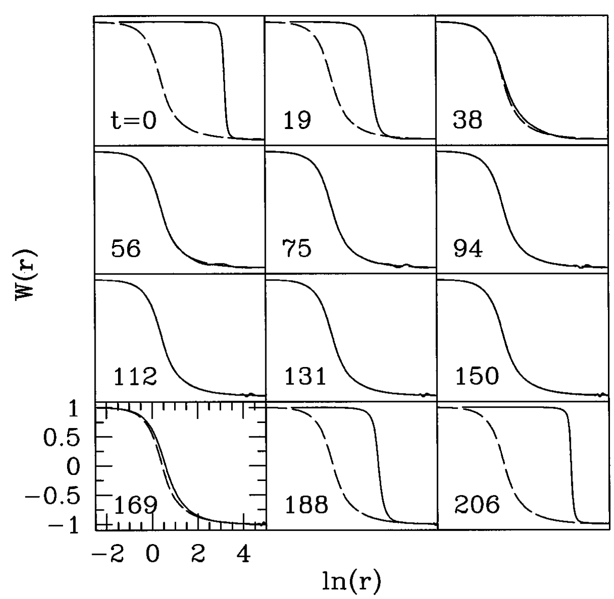

FIG. 5. Marginally subcritical type I evolution. Here we plot the dynamical evolution of $W(r,t)$ (solid line) and superimpose the static BK configuration X_1 (dashed line). Initially, the evolution (family F_c, $|\delta - \delta^*|/\delta^* \approx 10^{-15}$) is nearly linear and almost purely ingoing. When the pulse arrives at the center, it sheds off YM radiation, approaches X_1 and stays near it for some time, then disperses to infinity.

generally used initial conditions describing small deviations from the static solution, X_1, and studied the subsequent evolution to verify the prediction of perturbative instability [17]. Here, by construction, we *generate* X_1 as the boundary between collapse and dispersal, and thus immediately verify its instability. In this case, the reciprocal Lyapunov exponent of the single unstable mode yields the characteristic time scale, λ, for X_1's decay. Using central-proper-time normalization, the value computed from perturbation theory [17] is $\lambda = 0.5519\ldots$. We can measure this exponent quite directly from our simulations by computing the variation of the lifetime of near-critical solutions with respect to variations in $\pi \equiv \ln|p - p^*|$. Specifically, defining $T_r(\pi)$ to be the central proper time at which the zero crossing of $W(r,t)$ reaches radius r as it propagates outward, we expect $-dT_r/d\pi \rightarrow \lambda$ as $\pi \rightarrow -\infty$ and for sufficiently large r. Some numerical regularization is provided by monitoring T_r at several discrete radii r_i, $i = 1,\ldots,n$, and computing the averaged quantity $\bar{T}_r(\pi) \equiv n^{-1}\sum T_{r_i}(\pi)$. When this is done for F_c data with $r_1 = 400$, $r_n = 475$, $n = 16$, we find $0.5520 < -d\bar{T}_r/d\pi < 0.5525$ for $-19 < \pi < -10$. In addition, we can get a good estimate of the unstable *eigenmode* by studying near-critical departures from the static solution.

As noted in the introduction, type I behavior is clearly characterized by a gap in the black hole mass spectrum at threshold. We conjecture that the mass gap is universal, and observe that our calculations suggest that it is very close (1% or less) to the total mass, $m_1 = 0.828640\ldots$, of X_1.

Type I and II coexistence.—We can only briefly describe what is one of the more interesting features of critical behavior in the EYM model: the fact that for certain two-parameter families—such as F_a and F_d in Table I—there exists a critical line in parameter space (see Fig. 1) which interpolates between type I and type II behavior. The results we have obtained lead us to conjecture that the transition point (C in Fig. 1) represents the coexistence of the two distinct critical solutions described above. In other words, near C, we see echoing occurring in the context of the static X_1 background. Thus, we can have arbitrarily small black hole formation within a configuration which itself is arbitrarily close to forming a finite-mass black hole. We end by noting that although the supercritical regime in this model—especially in the cross-hatched overlap region sketched in Fig. 1—is still very much *terra incognita*, we have preliminary evidence that further phenomenological richness lurks there. In particular, we have observed an intriguing discontinuity in the spectrum of black hole masses which suggests the existence of another type of critical behavior in the formation of black holes [18].

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