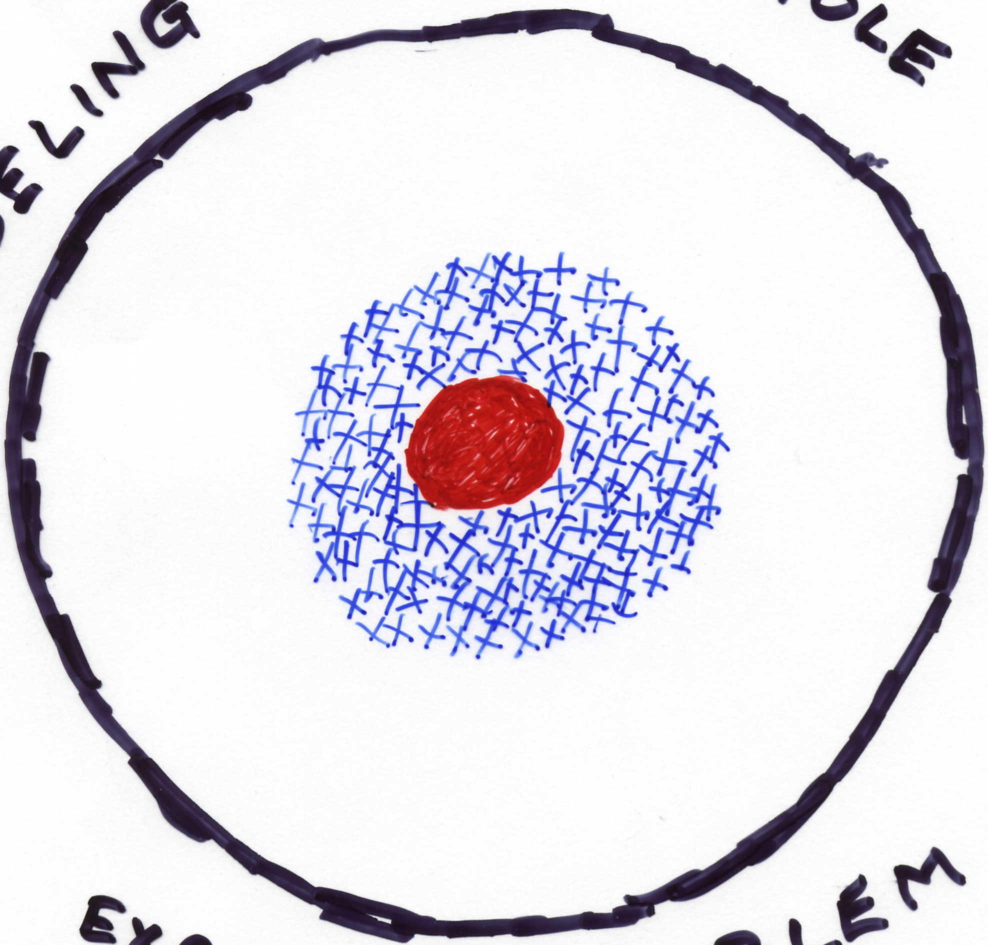
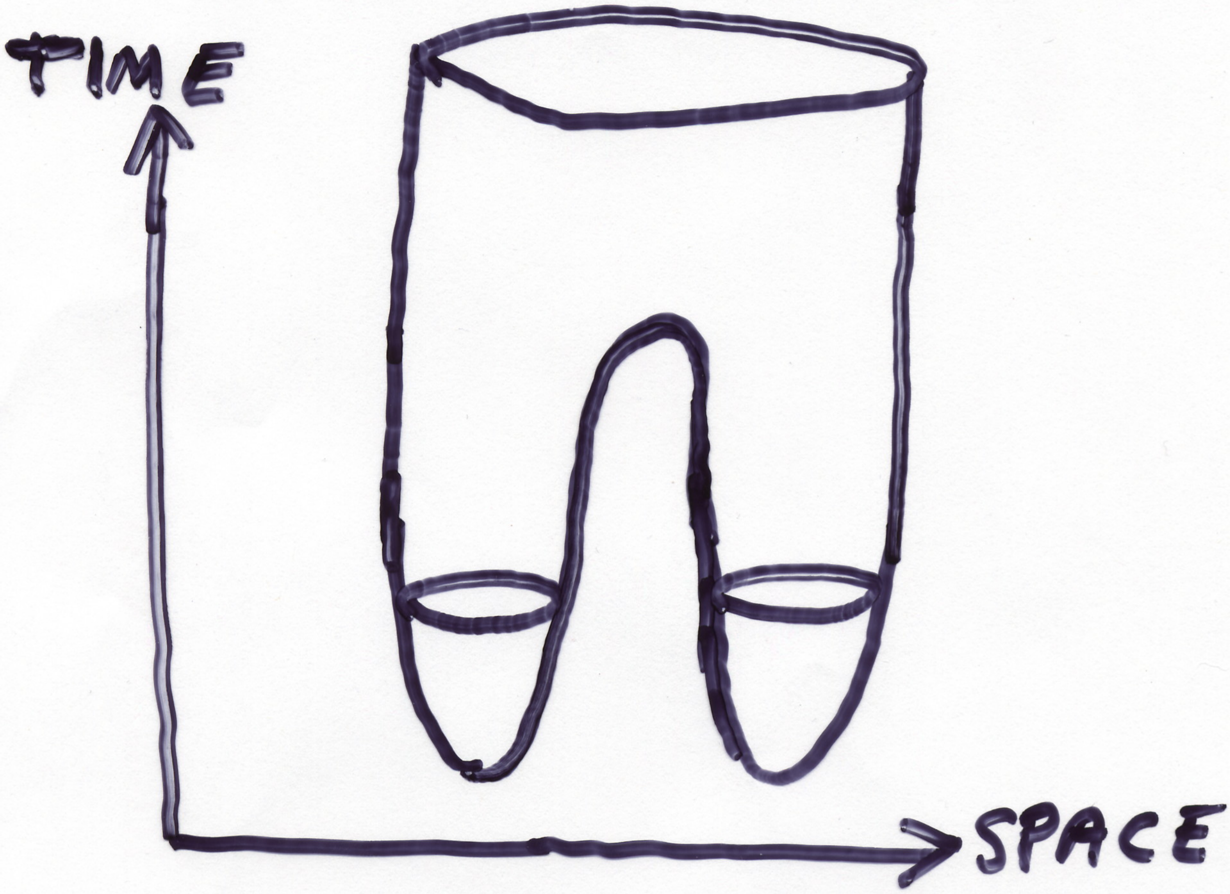
MODELING THE BLACK HOLE EXCISION PROBLEM



JEFF WINICOUR

THE BINARY BLACK HOLE PROBLEM



THE TWO SIDES OF THE PROBLEM:

- Mathematical - PDE's
- Geometrical - Spacetime picture

BOTH ARE NECESSARY.

BOTH ARE COMPLICATED.

MINKOWSKI SPACE

Spacetime coordinates $x^\alpha = (t, x^i) = (t, x, y, z)$
Proper distance - Minkowski metric

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -c^2 dt^2 + d\ell^2$$

$$d\ell^2 = \delta_{ij} dx^i dx^j$$

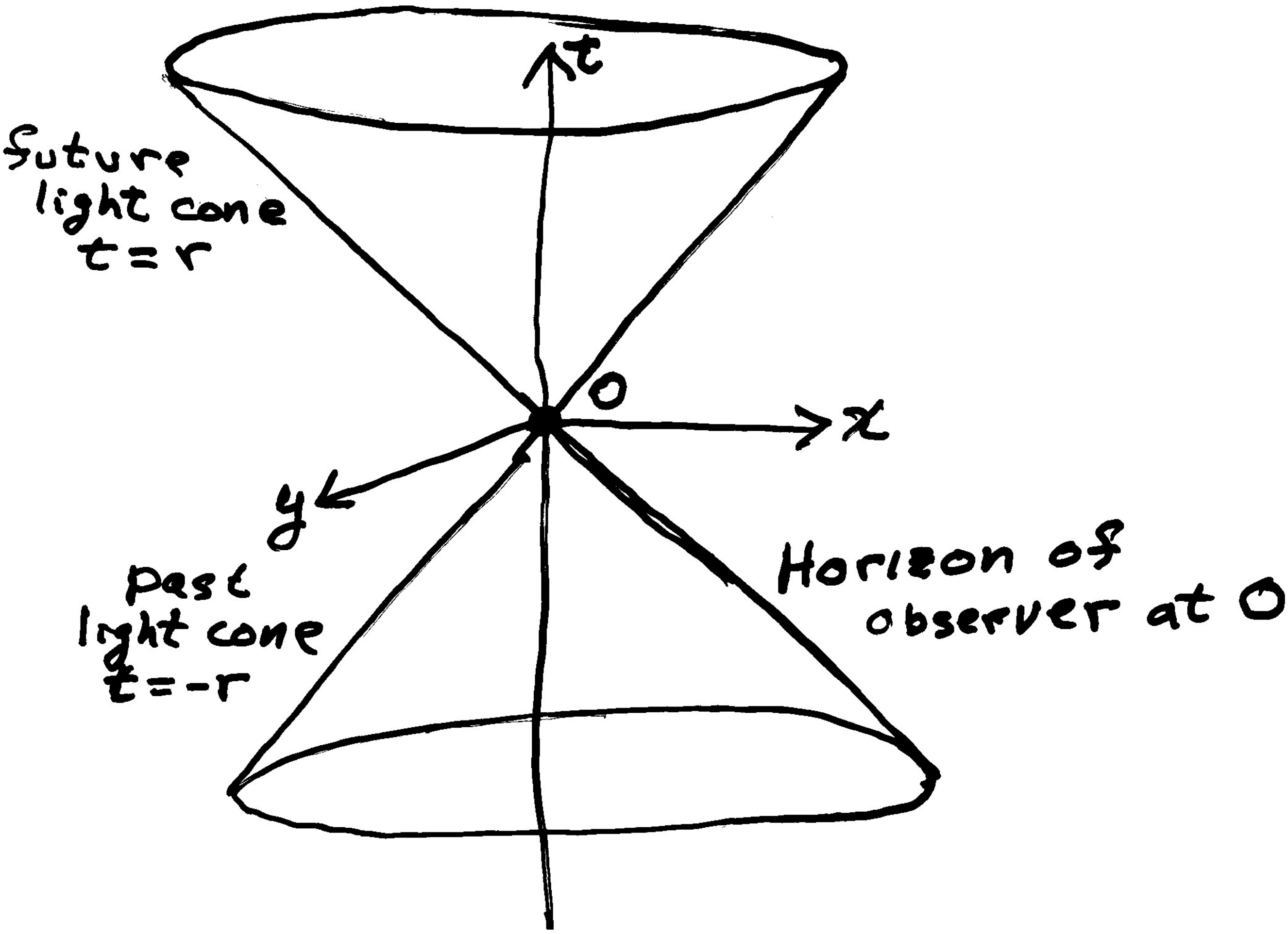
PDE side: c is speed of the characteristics of Maxwell's equations. The electromagnetic field components satisfy

$$\Box_\eta \Phi = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \Phi = \left(-\frac{1}{c^2}\partial_t^2 + \nabla^2 \right) \Phi = 0$$

Light cone: $\eta_{\alpha\beta} x^\alpha x^\beta = -c^2 t^2 + r^2 = 0$

$$r^2 = x^2 + y^2 + z^2$$

Standard convention: $c = 1$
Light cone is chief invariant feature of special relativity - event horizons



SPATIAL DISCRETIZATION

Theorems for the finite difference algorithms for the linearized problem $\phi \approx 1 + u$. We consider

$$W := u_{tt} - 2aD_0u_t - (b - a^2)D_+D_-u = 0$$

$$V := u_{tt} - 2aD_0u_t - bD_+D_-u + a^2D_0^2u = 0$$

W-algorithm: Stable for $b > a^2$ (energy conserving), admits stable Dirichlet or Neumann timelke boundary. Unstable for $b < a^2$

V-algorithm: Stable, admits stable spacelike excision boundary. Problematic at the outer timelike boundary.

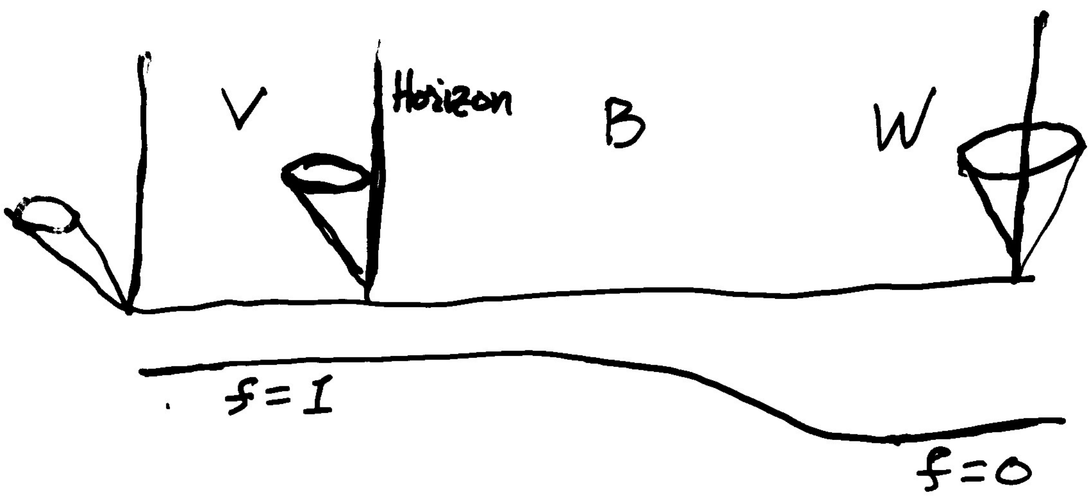
Stable blended algorithm for the global excision problem

$$B = W - D_-\Big((A_+(fa^2))D_+u\Big) + D_0\Big(fa^2D_0u\Big) = 0$$

reduces to W near the outer boundary and to V inside the horizon

Implementation as non-linear code: Discrete version of monopole conservation

$$\partial_t \int \frac{1}{\Phi} g^{t\alpha} \partial_\alpha \Phi dv = 0$$

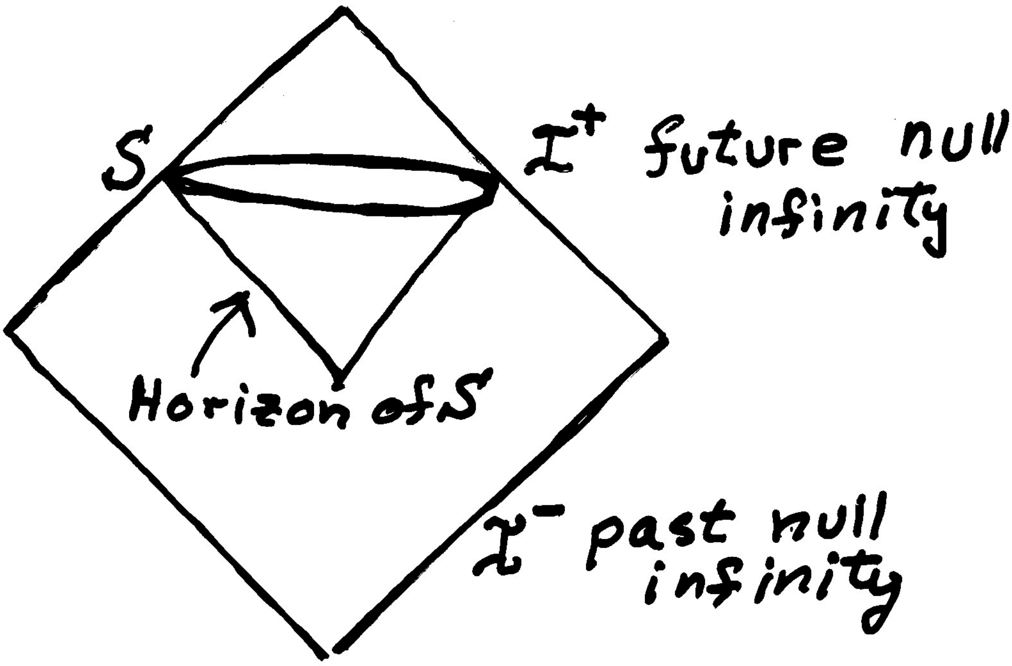


COMPACTIFIED MINKOWSKI SPACE

Conformally rescaled Minkowski metric

$$d\hat{s}^2 = \frac{1}{r^2} ds^2$$

PENROSE PICTURE: Extension to boundary points at $r = \infty$ along the light rays



Outgoing radiation fields propagate to $\mathcal{I}^+$

SPACETIME OF GENERAL RELATIVITY

Spacetime of general relativity is locally a curvy version of Minkowski space.

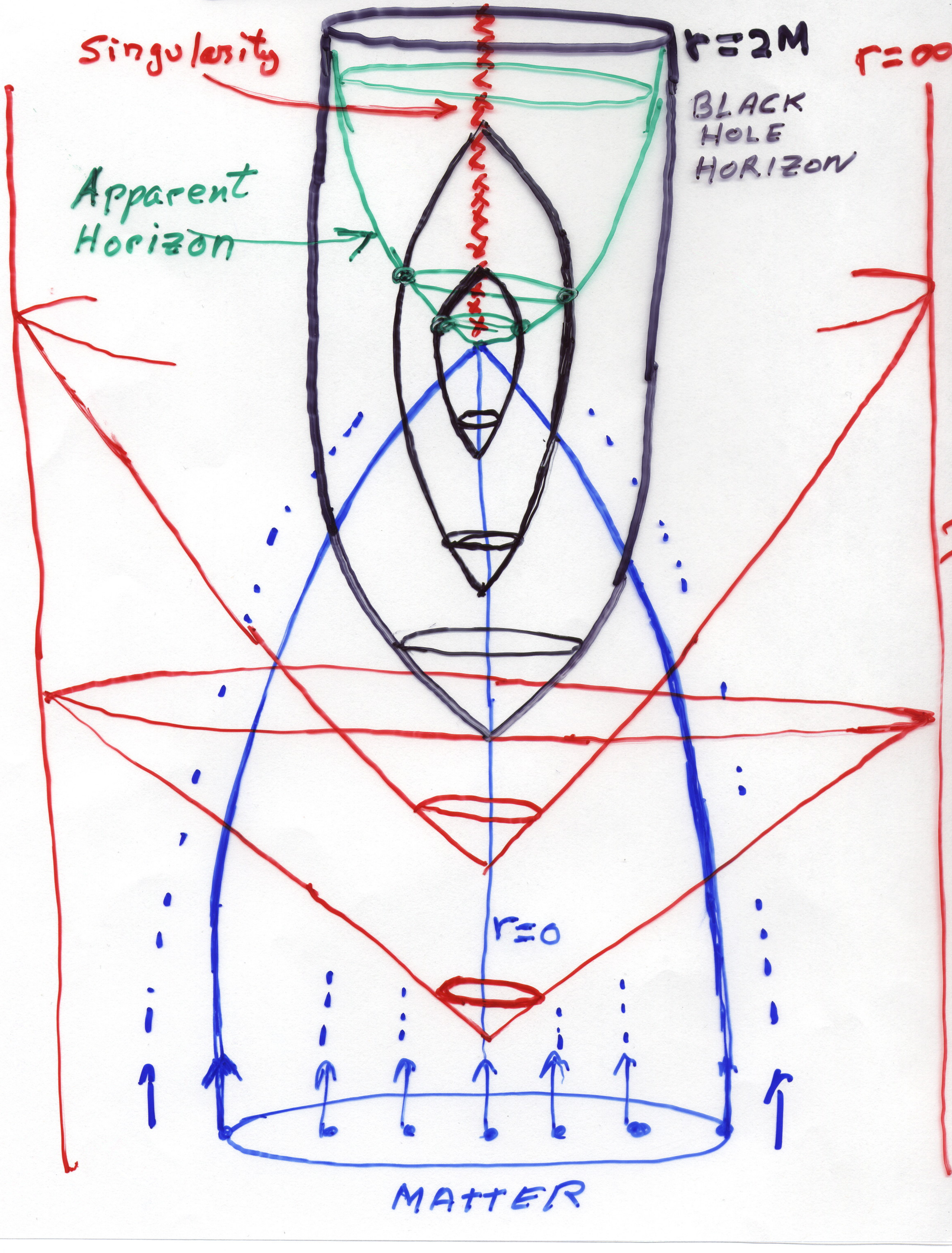
Spacetime metric

$$ds^2 = g_{\alpha\beta}(x^\mu) dx^\alpha dx^\beta$$

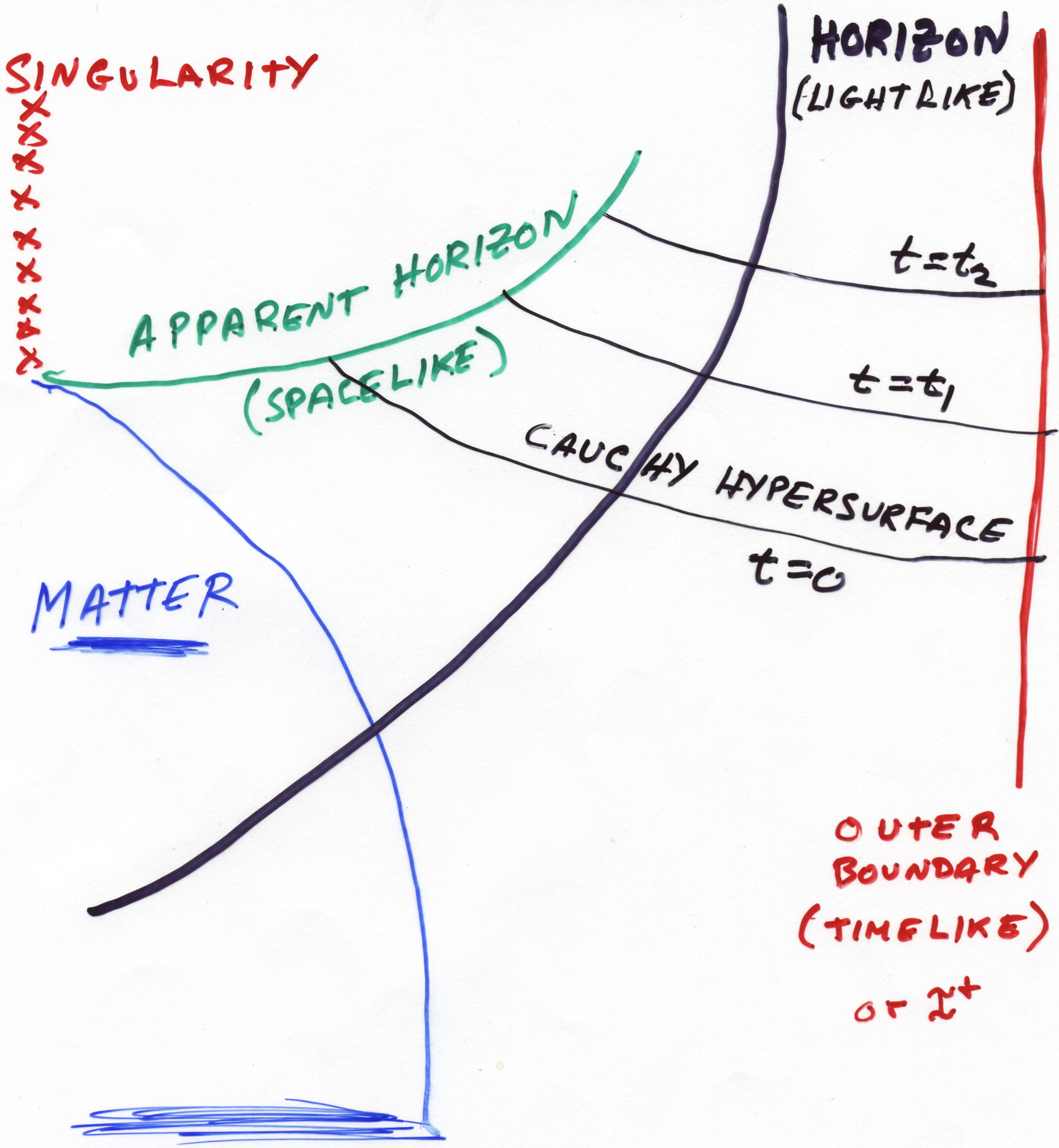
Weakly curved spacetimes have Penrose picture similar to Minkowski space

But curvature can lead to drastic global effects

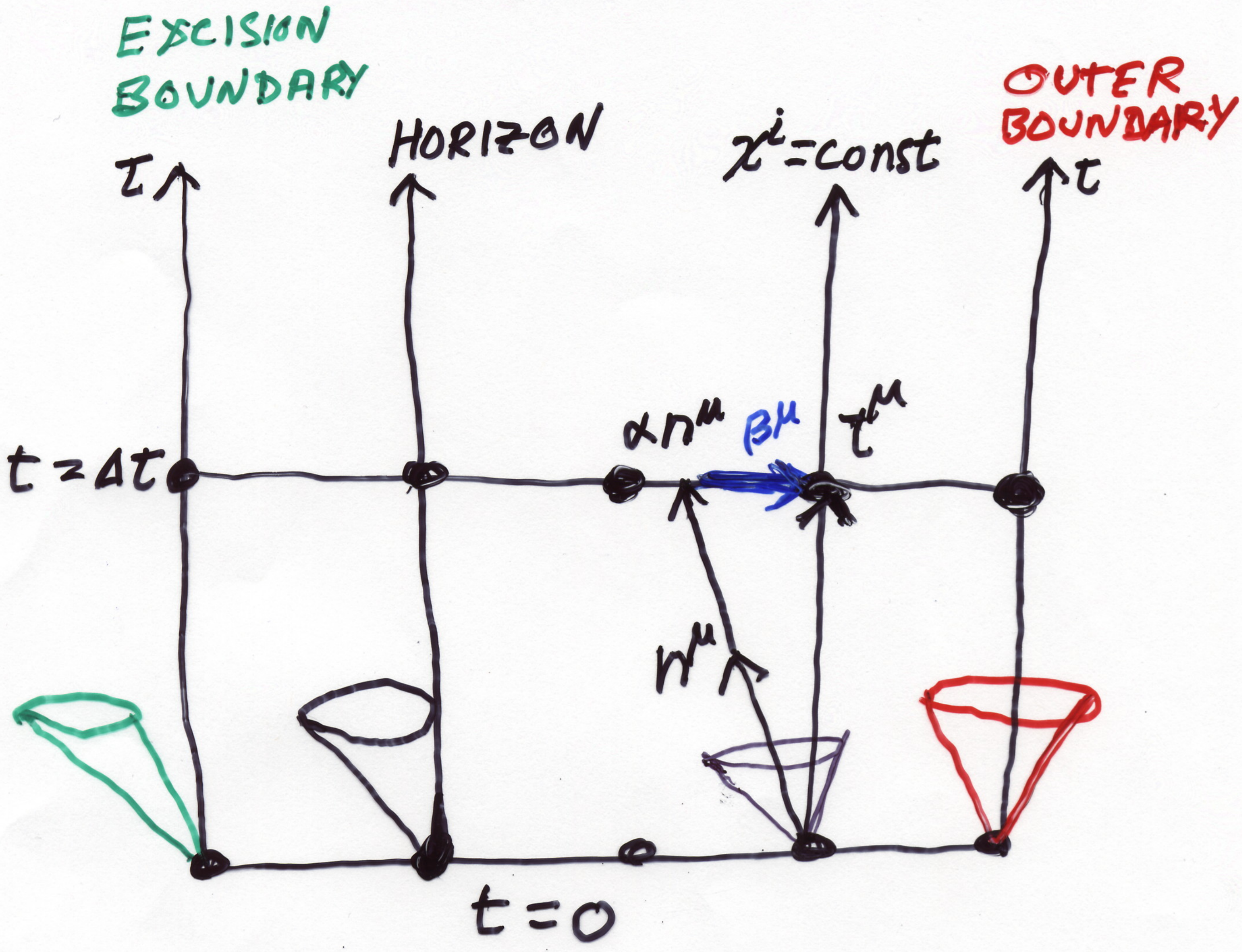
SPHERICAL COLLAPSE



EXCISION STRATEGY



EXCISION GRID STRUCTURE



n^μ = unit timelike normal to Cauchy hypersurfaces

t^μ = evolution vector $\quad t^\mu \partial_\mu = \frac{\partial}{\partial t}$

$$t^\mu = \alpha n^\mu + \beta^\mu$$

α ↑ lapse

β^μ ↑ shift

PDE

EINSTEIN'S EQUATIONS: $G_{\mu\nu}(g_{\alpha\beta}) = 0$

Gauge freedom: $x^\mu \to y^\mu(t, x^i)$

Uniqueness - Harmonic coordinates:

$$\Box_g x^\mu = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta x^\mu) = 0 \qquad \partial_\beta x^\mu = \delta^\mu_\beta$$

subject to initial and boundary conditions.

Note: $g = \det(g_{\alpha\beta})$ and $g^{\alpha\beta}g_{\beta\mu} = \delta^\alpha_\mu$

Reduced Einstein equations:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta(\sqrt{-g}g^{\mu\nu}) = S^{\mu\nu}$$

spatial metric

$$g^{\alpha\beta} = h^{\alpha\beta} - n^\alpha n^\beta$$

$$g^{ij} = h^{ij} - n^i n^j$$

Harmonic constraints:

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}) = 0$$

Principle part is wave operator - Well-Posed

Flux conservative form:

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta\sqrt{-g}g^{\mu\nu}) = \hat{S}^{\mu\nu} \qquad \partial_\alpha F^\alpha = \partial_t F^t + \vec{\nabla}\cdot\vec{F}$$

D. Garfinkle, *Phys.Rev.*, **D65**, 044029 (2002)

B. Szilágyi and J. Winicour, *Phys. Rev.*, **D68**, 041501 (2003).

F. Pretorius, *Class.Quant.Grav.* **22**, 425 (2005)

GAUGE WAVE TEST

Standardized tests: www.appleswithapples.org

In harmonic coordinates, there exists pure gauge solutions (flat metrics)

$$ds^2 = \left(1 + A\sin 2\pi(t-x)\right)(-dt^2 + dx^2) + dy^2 + dz^2,$$

describing traveling waves with periodic boundary conditions.

Their simulation is complicated by the existence of exponentially growing gauge waves

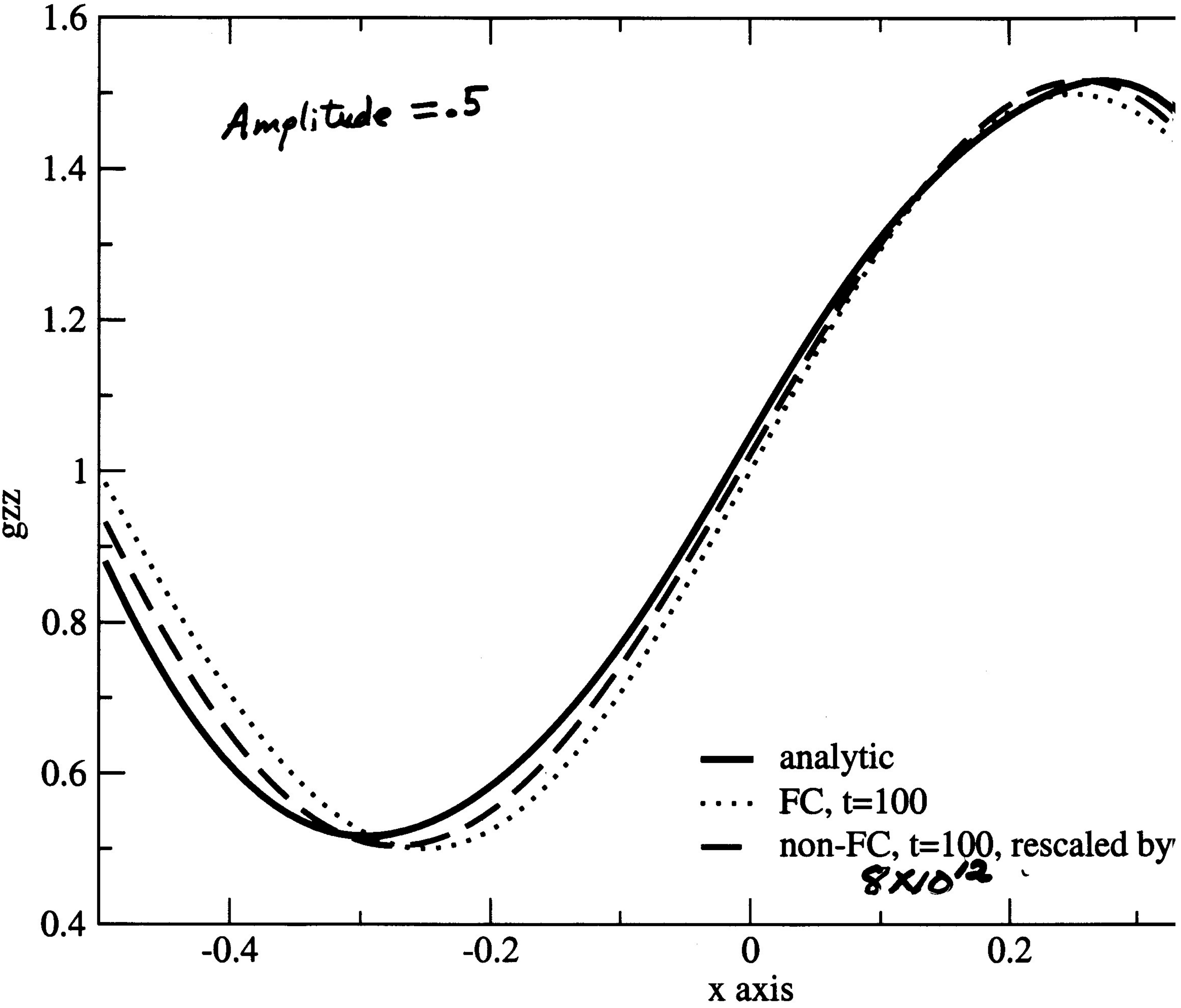
$$ds^2_\lambda = e^{\lambda t}\left(1 + A\sin 2\pi(t-x)\right)(-dt^2 + dx^2) + dy^2 + dz^2,$$

which solve the same equations. For $\lambda \approx 0$, they have approximately the same Cauchy data as the test gauge wave

Accurate long term simulation requires suppressing the exponential mode by using a semi-discrete conservation law, e.g. flux conservative form or SBP

"Some mathematical problems in numerical relativity", M. Babiuc, B. Szilágyi and J. Winicour, gr-qc/0404092

CONVERGENCE, STABILITY & CONSTRAINT PROPAGATION **IS NOT ENOUGH** FOR GOOD LONG TERM ACCURACY IN THE NONLINEAR REGIME



THE SCHWARZSCHILD EXCISION PROBLEM

Wave equation on Schwarzschild background: in Eddington-Finkelstein coordinates

$$- \ (1+\frac{2M}{r})\Phi_{tt} + \frac{4M}{r}\Phi_{tr} + (1-\frac{2M}{r})\Phi_{rr}$$
$$+ \ \frac{1}{r^2}(\Phi_{\theta\theta} + \frac{1}{\sin^2\theta}\Phi_{\phi\phi}) = \textit{lower order terms.}$$

Well-posed for $r > 0$

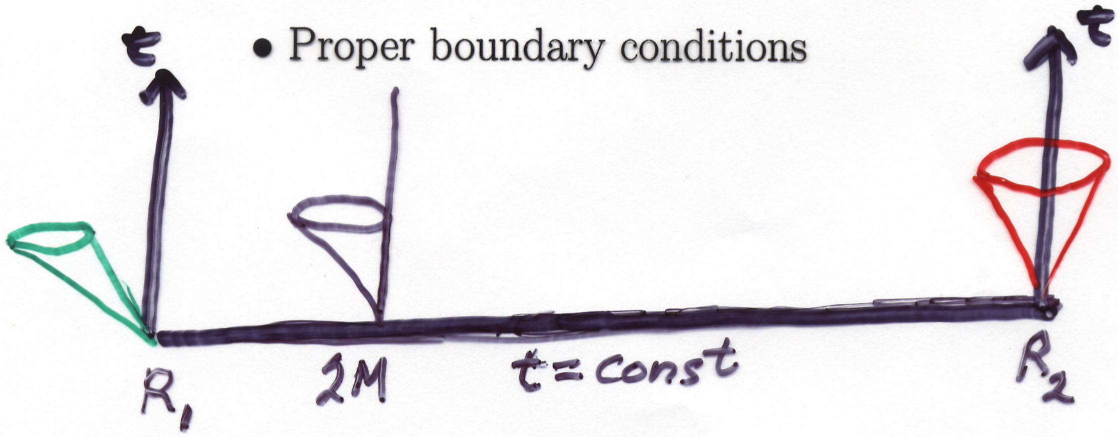
Complications for numerical treatment:

- Mixed derivative Φ_{tr} due to *shift*
- Sign change in $(1-2M/r)\Phi_{rr}$. Evolution with $r = const$ grid points is superluminal inside horizon.
- Conserved energy $E = \int \mathcal{E} dv$ on $t = const$ Cauchy hypersurfaces where

$$\mathcal{E} = \frac{1}{2\sqrt{1+\frac{2M}{r}}}\left((1+\frac{2M}{r})\Phi_t^2 + (1-\frac{2M}{r})\Phi_r^2 + \frac{1}{r^2}\Phi_\theta^2 + \frac{1}{r^2\sin^2\theta}\Phi_\phi^2\right)$$

$\mathcal{E}$ is positive-definite only for $r > 2M$

- Proper boundary conditions



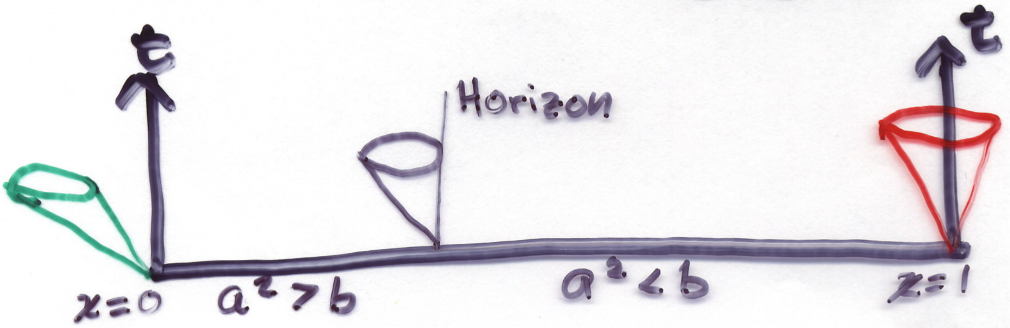
THE MODEL EXCISION PROBLEM

1D system with all the geometric features of the Schwarzschild problem.

In $0 \leq x \leq 1$, we consider

$$-\partial_t(\frac{1}{\Phi}\partial_t\Phi) + \partial_t(\frac{a}{\Phi}\partial_x\Phi) + \partial_x(\frac{a}{\Phi}\partial_t\Phi) + \partial_x(\frac{b-a^2}{\Phi}\partial_x\Phi) = 0$$

where a and b are smooth and give picture



The non-linearity models gauge wave problem: Growing waves $\Phi = e^{\lambda t} f(x-t)$.

Proof of the stability of a finite difference algorithm for the IBVP in the linearized case.

The system is treated as second differential order. Main complication: Shift term inside horizon.

Calabrese, *Phys.Rev. D* **71**, 027501 (2005)

Modeling the Black Hole Excision Problem, B. Szilagyi, H-O. Kreiss, J. Winicour gr-qc/0412101

NONLINER WAVE