High-resolution finite volume methods for hyperbolic PDEs on manifolds

Randy LeVeque and James Rossmanith

Reference: J.A. Rossmanith, D.S. Bale and R.J. LeVeque,
J. Comput. Phys. 199 (2004), pp. 631-662.

`http://www.amath.washington.edu/~rjl/pubs/manifolds/`

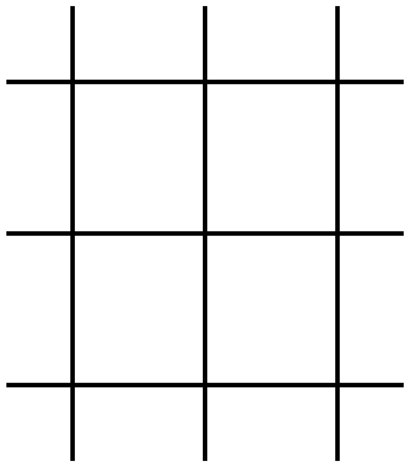
More complete slides available from a lecture at KITP:

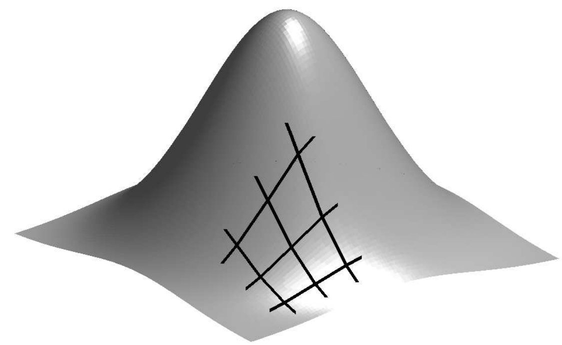
`http://online.kitp.ucsb.edu/online/gravity03/leveque/`

Overview

- High-resolution methods for first-order hyperbolic systems
- Shock waves in nonlinear problems
- Heterogeneous media with discontinuous properties
- Godunov-type methods based on Riemann solvers
- Second-order correction terms with limiters to minimize dissipation and dispersion

Acoustics on a manifold





$$\frac{\partial}{\partial t}\,p + \frac{1}{\sqrt{h}}\,\frac{\partial}{\partial x^k}\left(\sqrt{h}\,u^k\right) = 0$$

$$\frac{\partial}{\partial t}\,u^m + \frac{1}{\sqrt{h}}\,\frac{\partial}{\partial x^k}\left(\sqrt{h}\,T^{km}\right) = -\Gamma^m_{nk}T^{kn}\,.$$

$p =$ pressure, $u^m =$ velocity vector in computational space.
This models compressional waves on a surface, e.g. acoustics in a metal sheet.

At each cell edge:

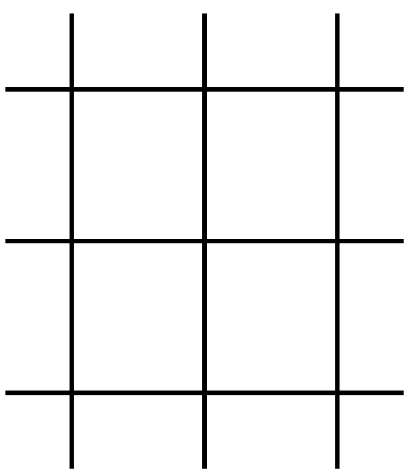
- Parallel transport cell-centered velocities to edge,
- Change coordinates to a local orthonormal frame at cell edge to obtain normal and tangential velocities,
- Solve 1d Riemann problem normal to cell edge (assuming locally flat)
- Scale resulting waves by length of side, transform back to cell-centered coordinates,
- Update cell averages.

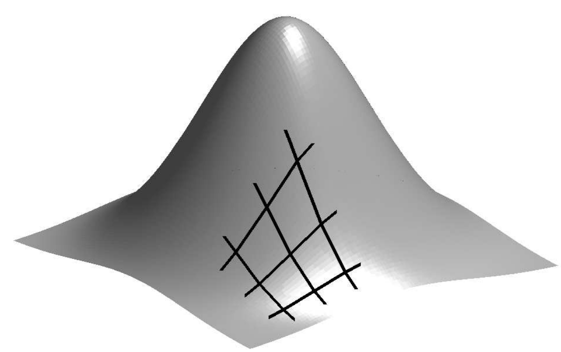
Parallel Transport

$$\frac{\partial}{\partial x^k} u^m + \Gamma^m_{nk} u^n = 0 \quad \text{or} \quad \frac{\partial}{\partial x^k} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \begin{bmatrix} \Gamma^1_{1k} & \Gamma^1_{2k} \\ \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = 0.$$

Approximate using Taylor series:

$$u^m\left(x^k \pm \frac{1}{2}\Delta x^k\right) \approx u^m(x^k) \mp \frac{1}{2}\Delta x^k \Gamma^m_{nk} u^n(x^k).$$





Parallel Transport for acoustics

For acoustics with $q = (p, u^1, u^2)^T$, solve Riemann problem with

$$q^{\ell}_{i-1/2,j} = \left(I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix} \right) q_{i-1,j}$$

and

$$q^{r}_{i-1/2,j} = \left(I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix} \right) q_{i,j}$$

f-wave formulation

Split jump in fluxes ΔF into waves.

$$\Delta F^1 = \left(I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix}_{ij} \right) f^1_{ij} - \left(I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix}_{i-1,j} \right) f^1_{i-1,j}$$

$$= f^1_{ij} - f^1_{i-1,j} - \Delta x^1 \psi^1_{i-1/2,j}$$

where

$$\psi^1_{i-1/2,j} = -\frac{1}{2} \left(\begin{bmatrix} 0 \\ \Gamma^1_{11} T^{11} + \Gamma^1_{12} T^{21} \\ \Gamma^2_{11} T^{11} + \Gamma^2_{12} T^{21} \end{bmatrix}_{ij} + \begin{bmatrix} 0 \\ \Gamma^1_{11} T^{11} + \Gamma^1_{12} T^{21} \\ \Gamma^2_{11} T^{11} + \Gamma^2_{12} T^{21} \end{bmatrix}_{i-1,j} \right)$$

This is $n = 1$ portion of the source term

$$\psi = -\Gamma^m_{nk} T^{kn}.$$

Acoustics on a manifold

$$\frac{\partial}{\partial t} p + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, u^k \right) = 0$$

$$\frac{\partial}{\partial t} u^m + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, T^{km} \right) = -\Gamma^m_{nk} T^{kn} \,.$$

With finite volume formulation,

- Source term is automatically incorporated by parallel transport of fluxes,
- Covariant divergence is handled by use of edge lengths and cell volume,
- Parallel transport and orthonormalization allows use of standard flat-space Riemann solver at interface.

CLAWMAN software

`www.amath.washington.edu/~claw/clawman.html`

Currently only 2d.
AMR available by request.

Requires metric tensor H

- 2×2 matrix as function of x^1 and x^2,
- Used to compute scaling factors for edge lengths, cell areas,
- Used for orthonormalization at cell edges.

Christoffel symbols are needed for parallel transport

- Computed by finite differencing H.

AMR on a manifold