Second-order in space systems

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- Second-order systems in GR
- Definitions of hyperbolicity
- Finite differencing

Gauge, constraints, and different formulations

Each gauge freedom generates one constraint.

Ingredients of a formulation of the Einstein equations:

- Well-posedness depends on the gauge choice.
- It can be achieved by adding constraints
- . . . and introducing redundant variables and their definition constraints.

Historical confusion between introducing **some** redundant variables to obtain hyperbolicity, and reducing to first order to prove it.

Some formulations

ADM: 2nd order, 12 variables γ_{ij} and K_{ij}

KST: 1st order, 18 auxiliary variables

$$d_{kij} \equiv \gamma_{ij,k}$$

NOR: 2nd order, 3 redundant variables

$$f_i \equiv \gamma^{jk}\gamma_{ij,k}$$

BSSN: 2nd order, $K_{ij} \to (K, \tilde{A}_{ij})$, $\gamma_{ij} \to (\phi, \tilde{\gamma}_{ij})$, plus $\tilde{\Gamma}^i \sim f_i$.

"BSSN-C": BSSN with algebraic constraints $\mathrm{tr}\tilde{A}_{ij} = 0$ and $\det\tilde{\gamma}_{ij} = 1$ imposed continuously: equivalent in the principal part to a variant of NOR.

Z4: 2nd order, redundant variables $Z^\mu \sim \Box x^\mu$.

Reduction to first order

CG & Martin-Garcia

Matrix notation: u is a vector of variables. Not all variables have second space derivatives:

$$u \equiv (v, w), \quad \dot{u} = \partial\partial v + \partial w + \text{lower order terms}$$

Reduction $d_i \equiv v_{,i}$ is possible only for

$$\begin{aligned} \dot{v} &= A_1^i v_{,i} + A_2 w + \text{l.o.} \\ \dot{w} &= B_1^{ij} v_{,ij} + B_2^i w_{,i} + \text{l.o.} \end{aligned}$$

(Counterexample $\dot{v} = v''$.)

Parameterise all ambiguities $v_{,i}$ or d_i, and $d_{i,j}$ or $d_{j,i}$. Hyperbolicity of the second-order system should be defined independently of these reduction parameters.

Evolution of auxiliary constraints $v_{,i} - d_i$ and $d_{i,j} - d_{j,i}$ closes $\Rightarrow$ we can restrict to the second order system.

Strong hyperbolicity

Definition: Second order system strongly hyperbolic $\Leftrightarrow$ there is a reduction that is strongly hyperbolic.

Theorem: $\Leftrightarrow$

$$\mathcal{A} \equiv \begin{pmatrix} B_2^n & B_1^{nn} \\ A_2 & A_1^n \end{pmatrix}$$

is uniformly diagonalisable for all n_i, where $A_1^n \equiv A_1^i n_i$ etc.

Lemma: $\Leftrightarrow$ second order system has a complete set of characteristic variables of the form $w + \partial v$.

Lemma: $\Leftrightarrow$ pseudo-differential reduction strongly hyperbolic

Idea of proof

A particular choice of reduction parameters gives the principal part

$$
\begin{aligned}
\dot{v} &\simeq 0 \\
\dot{w} &\simeq B_1^{ij} d_{k,i} + B_2^i w_{,i} \\
\dot{d}_j &\simeq \left(\delta_j{}^i A_1^k + i\mu\epsilon_j{}^{ik}\right) d_{k,i} + A_2 w_{,i}
\end{aligned}
$$

With $d_i \equiv (d_n, d_A)$, the principal part neglecting transverse derivatives is

$$
\begin{array}{rclcccc}
\dot{w} & \simeq & B_1^{nn} d_{n,in} & + & B_2^n w_{,n} & + & B_1^{nB} d_{B,n} \\
\dot{d}_n & \simeq & A_1^n d_{n,n} & + & A_2 w_{,n} & + & A_1^B d_{B,n} \\
\dot{w}_A & \simeq & & & & & i\mu\epsilon_A{}^{nB} d_{B,n}
\end{array}
$$

The lower diagonal block is diagonalisable with eigenvalues $\pm\mu$. Then the entire matrix is diagonalisable if (and in fact only if) the upper diagonal block $\mathcal{A}$ is diagonalisable. (We choose μ large enough so that $\pm\mu$ are not eigenvalues of $\mathcal{A}$.)

Symmetric hyperbolicity

Definition: Second order system symmetric hyperbolic $\Leftrightarrow$ there is a reduction that is symmetric hyperbolic.

Theorem: $\Leftrightarrow$ (1)

$$(\mathcal{H}\mathcal{A})^\dagger = \mathcal{H}\mathcal{A}$$

for all n_i, and (2)

$$H > 0$$

where

$$\mathcal{H} \equiv \begin{pmatrix} K & L^n \\ L^{\dagger n} & M^{nn} \end{pmatrix}, \quad H \equiv \begin{pmatrix} K & L^i \\ L^{\dagger j} & M^{ij} \end{pmatrix},$$

Theorem: $\Leftrightarrow$ second order system admits a conserved energy ϵ quadratic in $(w, v_{,i})$ and conserved in the sense that

$$\dot\epsilon = \phi^i{}_{,i}$$

Idea of proof

Step 1:

$$(\mathcal{H}\mathcal{A})^\dagger = \mathcal{H}\mathcal{A}$$

is necessary for

$$(HP^i)^\dagger = HP^i$$

for any first-order reduction (with principal part P^i).

Step 2: We can find reduction parameters, which depend on H, such that this condition is also sufficient.

Step 3: The energy for the second-order system is also the energy for the reduction (with $v_{,i} \leftrightarrow d_i$).

Constraint evolution

Theorem: Vector of constraints

$$c \equiv C_1^{ij} v_{,ij} + C_2^{i} w_{,i} + \text{l.o.} = 0$$

compatible with the evolution equations, and main system strongly hyperbolic $\Rightarrow$

Constraint system strongly hyperbolic, and characteristic variables in the direction n_i given by

$$\mathbf{c} \simeq \partial_n \mathbf{u} + \partial_A \ldots$$

where the $\mathbf{u}$ are some of the characteristic variables of the main system (and $\mathbf{c}$ and $\mathbf{u}$ have the same speed).

Finite differencing

- Interior: Semidiscrete version of some symmetric hyperbolic systems is unstable when using standard centered differences: shifted wave equation with $\beta > 1$ using $(\dot{\phi}, \phi)$ but not (Π, ϕ). Z4 but not NOR/BSSN.

- Boundaries: Summation by parts operators do not give a conserved semi-discrete energy even for the shifted wave equation in (Π, ϕ) form (too many separate summations by part required).

- Ad-hoc finite differencing methods give stable excision and timelike boundaries for the shifted wave equation with 2nd and 4th order accuracy (Calabrese & CG, in progress).

First order or second order?

- Astrophysics simulations so far only in BSSN.

- In testbeds, first-order formulations seem to require fine-tuning of parameters.

- First order allows “standard” finite differencing treatment of excision, outer, and multipatch boundaries, using summation by parts and projection methods.

- Second order simpler and probably more accurate (phase error, error growth from non-principal terms) but require
  - stable heuristic boundary treatments,
  - stable overlapping multipatch schemes.