# High order methods for multi-block evolutions in Numerical Relativity

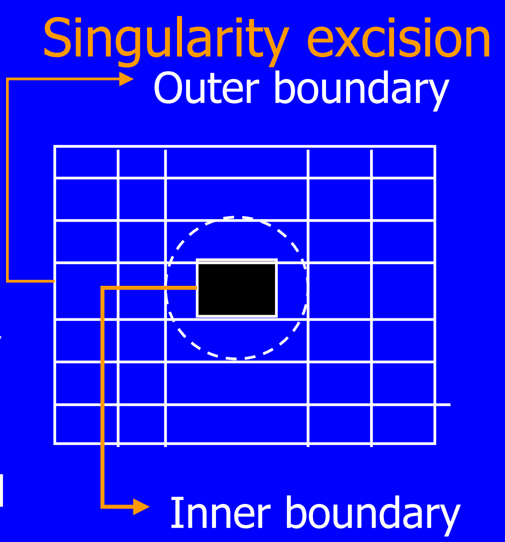
Manuel Tiglio
Dept of Physics & Astronomy, and
Center for Computation and Technology
Louisiana State University

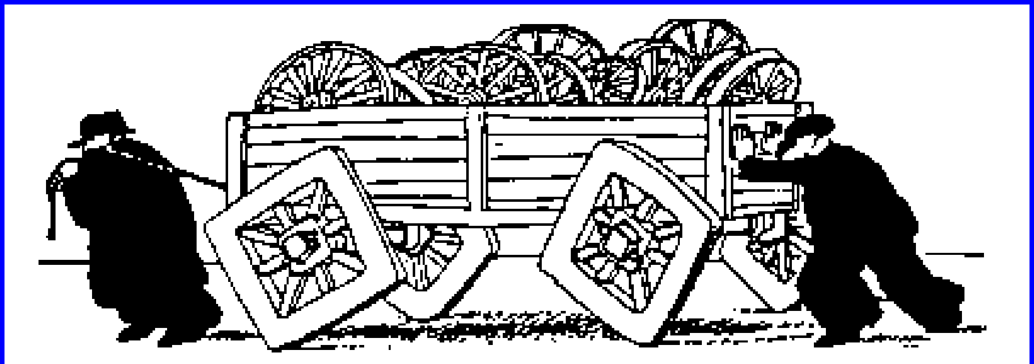
# Work in collaboration with

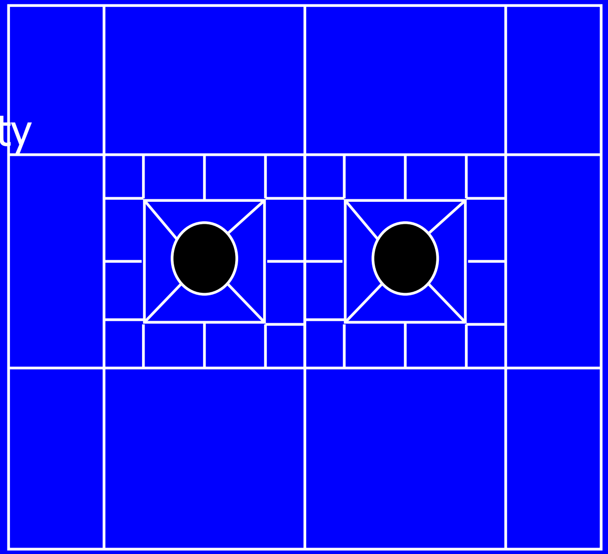
- Peter Diener (LSU)
- Nils Dorband (LSU)
- Luis Lehner (LSU)
- Oscar Reula (University of Cordoba)
- Erik Schnetter (Albert Einstein Institute)

# Motivation

- Need to have non-regular geometries, but simple enough that a semi-structured approach can be followed.
- Some examples: black hole excision (Choptuik's talk), outer spherical boundaries (e.g., compactified approach, Friedrich's talk), co-rotating coordinates.
- Break the domain into subdomains that are topologically cubes and glue them together.
- Numerical energy estimates through difference operators of arbitrary high order satisfying summation by parts (SBP) and penalty terms for the interfaces.
- As an extra, one gets some kind of (non nested) fixed adaptivity

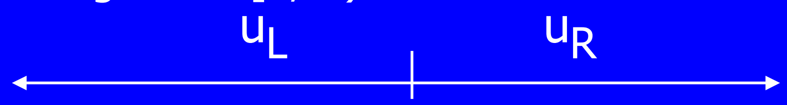




# Matching technique and numerical stability: energy estimates for symmetric systems through penalty terms [Carpenter, Nordstrom and Gottlieb '98]

- Say you want to discretize the advection equation $u_t = cu_x$, in two domains. The Left one covers (…,0], and the Right one [0,…)

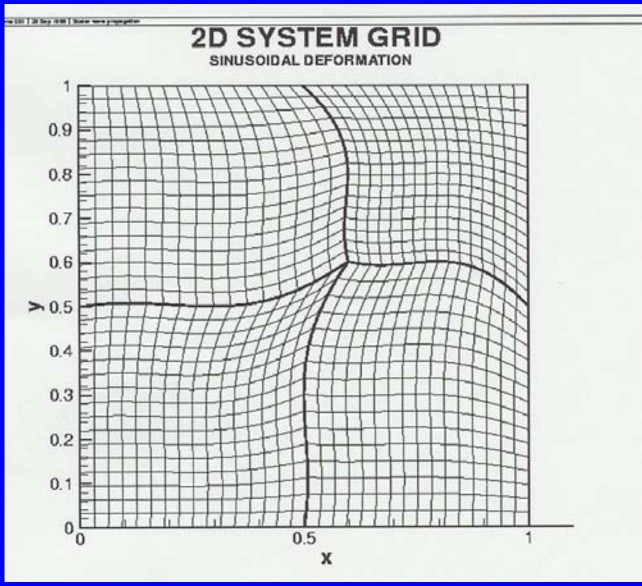



- We use two fields to describe u, $u_L$ and $u_R$. At x=0 the two fields are defined, and the solution is multivalued.

- Now discretize using penalty terms:

$$\frac{d}{dt}u_j^L = cDu_j^L - \frac{S^L}{h\sigma_{00}}(u_j^L - u_j^R)$$

$$\frac{d}{dt}u_j^R = cDu_j^R - \frac{S^R}{h\sigma_{00}}(u_j^R - u_j^L)$$




- And use <u>any</u> operator D satisfying the summation by parts property:

$$(u, Dv)_\Sigma + (v, Du)_\Sigma = \tfrac{1}{2} uv\Big|_0^1$$

$$(u,v)_{\Delta x} = \Delta x \sum_{i,j} < u_i, Hv_j > \sigma_{ij}$$

- Define the energy $$E = (u,u)^L_\Sigma + (u,u)^R_\Sigma$$

$$(u,v)^L_\Sigma = h\sum_{-\infty}^{0} \sigma_{ij} u_j v_j \qquad (u,v)^R_\Sigma = h\sum_{0}^{\infty} \sigma_{ij} u_j v_j$$

Take its time derivative and use the SBP property to get

$$\frac{d}{dt}E = (c-2S^L)(u_0^L)^2 - (c+2S^R)(u_0^R)^2 + 2(S^L+S^R)u_0^L u_0^L$$

- If c>0, choosing $S^L = c+\delta$, $S^R = \delta$ gives $$\frac{d}{dt}E = -(u_0^L - u_0^R)^2(c+2\delta)$$

- And the energy estimate $\frac{d}{dt}E \leq 0$ follows if $\delta >= -\lambda/2$

- Using $\delta = -\lambda/2$ results in a "non-disipative" scheme, E=constant.

  Using $\delta > -\lambda/2$ "dissipates", but only the difference between $u_L$ and $u_R$ at x=0.

  Using $\delta > 0$ any mismatch asymptotically decays to zero.

- Can do the same for any linear, variable coefficients symmetric hyperbolic system.

# Very high order difference operators satisfying SBP, and associated dissipations [Kreiss and Scherer '74, Strand '94, Mattsson, Svard and Nordstrom 2004]

- Diagonal and full restricted norms.

$$(u,v)_{\Delta x} = \Delta x \sum_{i,j} < u_i, H v_j > \sigma_{ij}$$

  The norm is diagonal if $\sigma_{ij} = \sigma_i \delta_{ij}$ , full restricted if $\sigma_{0i} = 0 \quad for \quad i \neq 0$

- In the diagonal (full restricted) case, the order of the derivative is 2n in the interior and n (2n-1) at and close to boundaries.

- There are some issues in the non-diagonal case.

- Derivatives with minimum bandwith are not necessarily the optimal ones, as they might have a large spectral radius associated -> severe restrictions on the Courant limit.

- Inventory of high order derivatives we have analyzed/whose spectral radius we have "minimized" (notation: order in the interior – order at and close to boundaries):

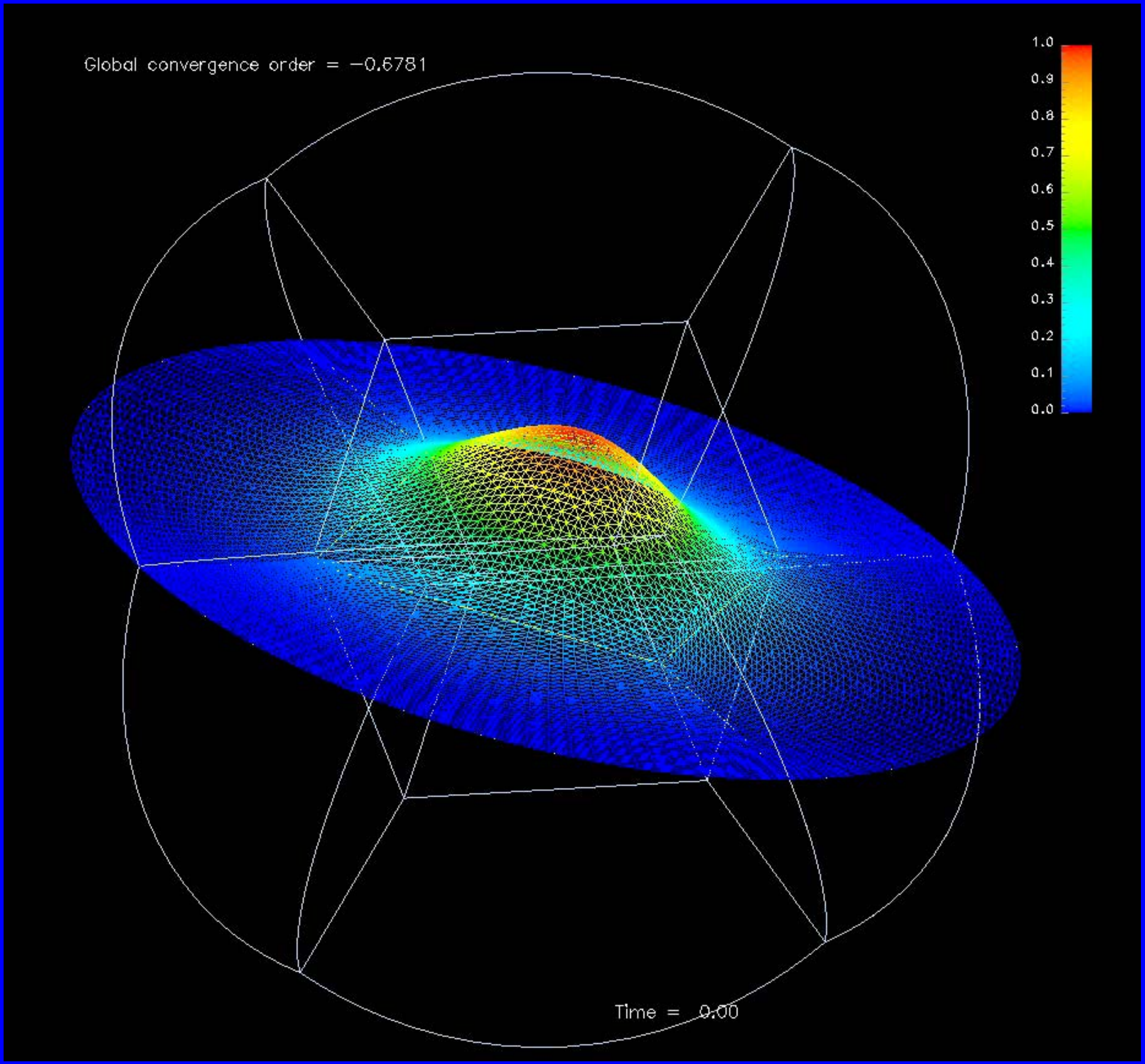
  * 2-1, 4-2, 6-3, 8-4 (diagonal case)
  * 4-3, 6-5, 8-7 (full restricted case)

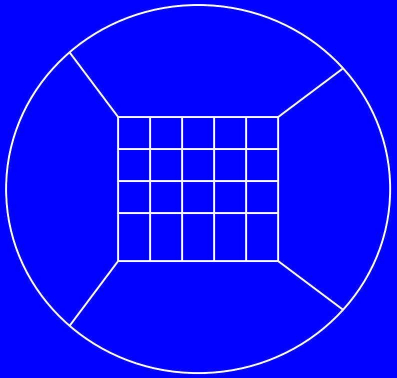
- Dissipations: need to be non-positive definite with respect to the SBP scalar product. Mattsson's solution: a prescription for all norms.

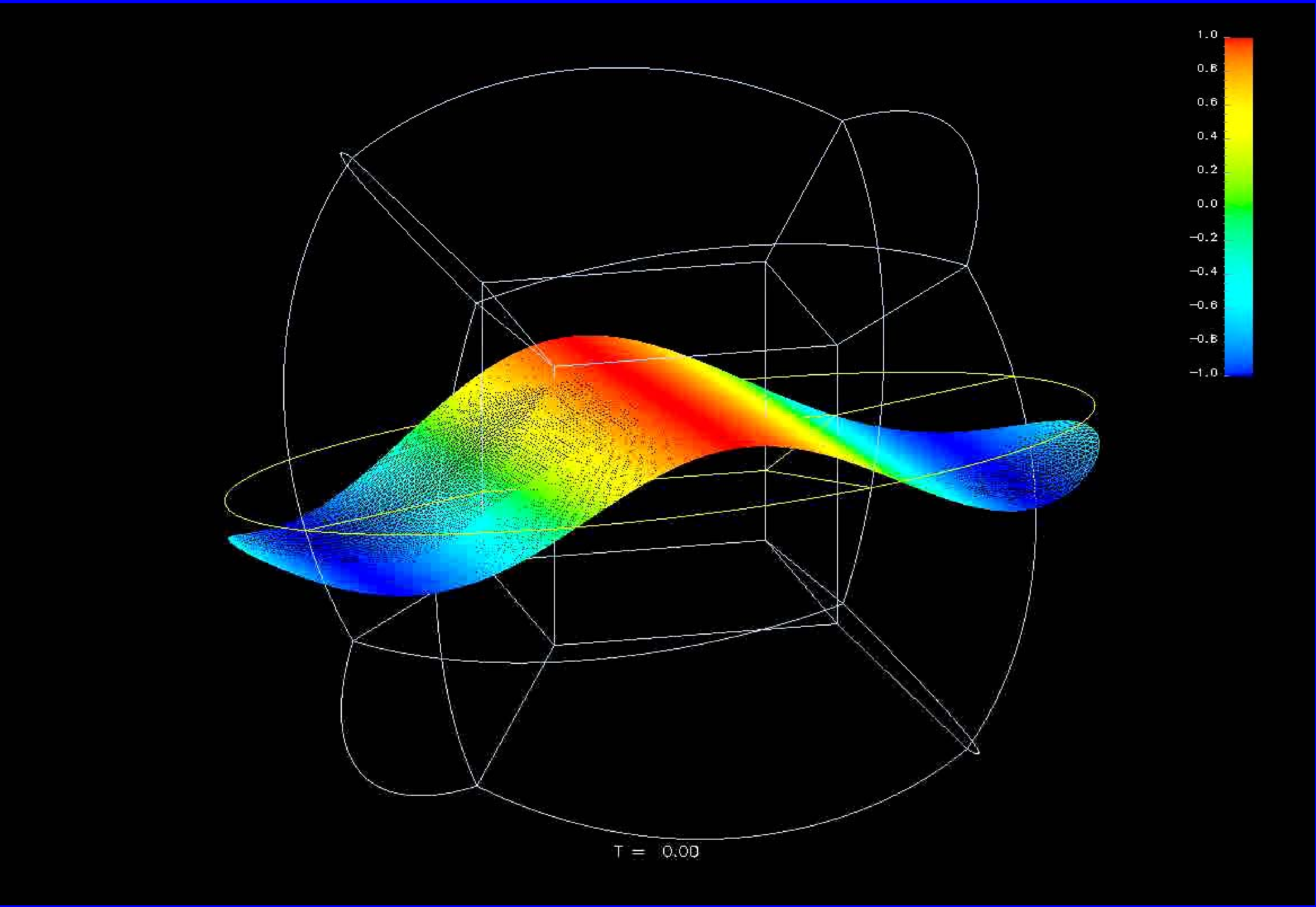
# Computational infrastructure

- Parallel, modular infrastructure for the CACTUS framework (www.cactuscode.org)
- Uses Erik Schnetter's parallel driver for CACTUS CARPET (www.carpetcode.org).
- SBP thorns with all the derivatives and dissipations just described.
- Modular infrastructure: derivatives, geometries and equations being solved are completely independent of each other. Can choose at runtime different geometries, - derivatives, etc.
- If you have a CACTUS code for a first order hyperbolic system, you can use the multipatch infrastructure essentially out of the box.
- Because of its modular nature, this infrastructure has opened the door to many applications (described below).
- The infrastructure allows for overlapping patches, but we haven't exploited it so far.

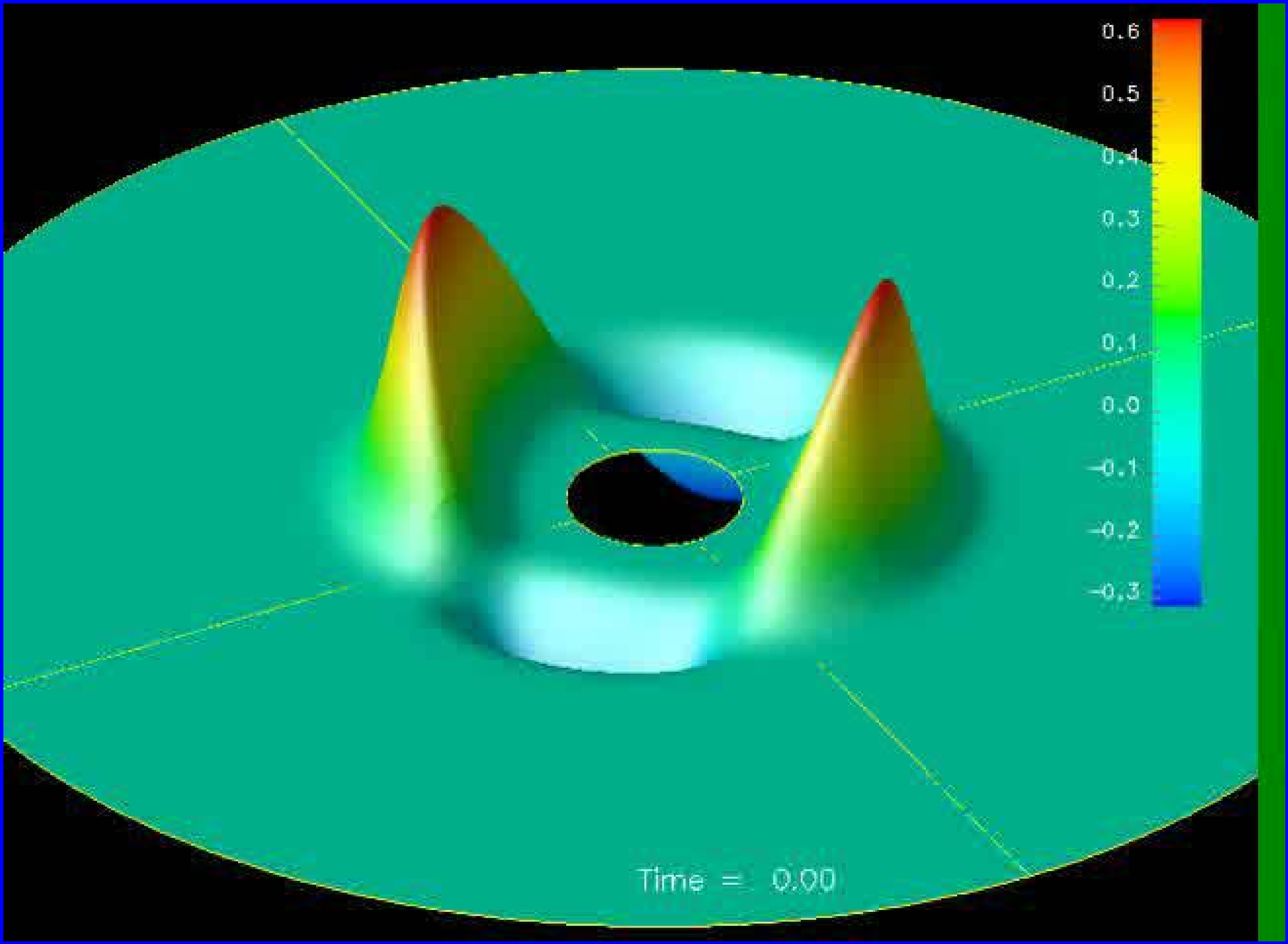
# Examples

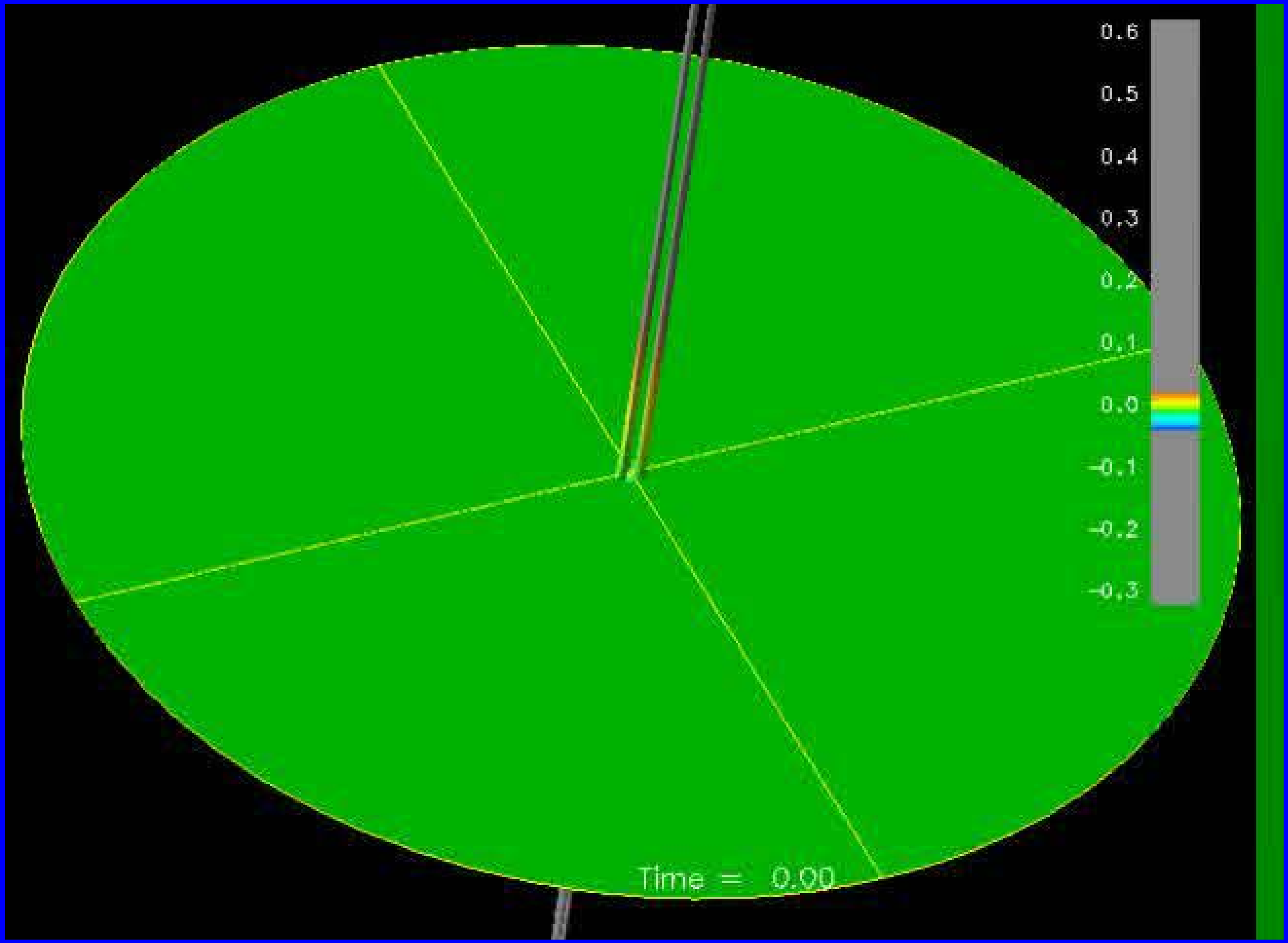

Global convergence order = −0.6781
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Time = 0.00





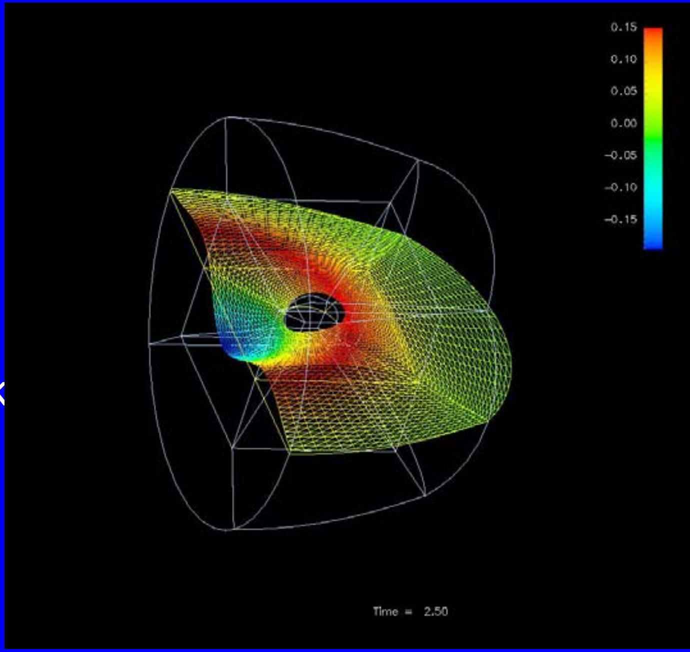
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
T = 0.00


0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
Time = 0.00


0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
Time = 0.00

# Going beyond proof of concept

- Single distorted black hole simulations with fixed shift (Nis Dorband et al)
- Incorporating and coding better "driver" shift conditions into the Z4 system (Carlos Palenzuela et al) and into our current symmetric hyperbolic system for binary black hole evolutions.
- Accretion processes (Burkhard Zink et al)
- Revisiting Cauchy-perturbative matching (Enrique Pazos et al)
- High order multigrid elliptic solver, possibly for multi-block scenarios (Mark Miller et al).
- In the meantime using parallel, adaptative finite element solver to provide initial data (Matt Andersson et al)



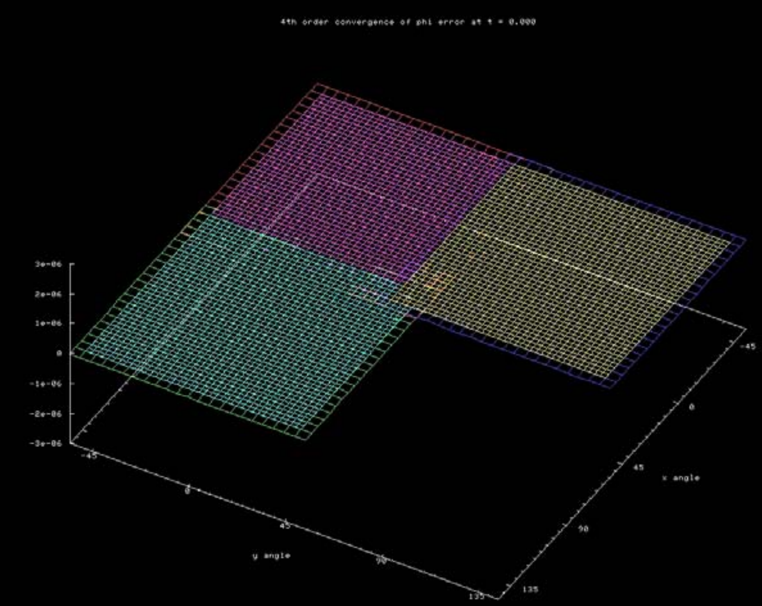

- Visualization for multiple patches (Werner Benger et al).
- Do mesh refinement on each block/patch (Schnetter et al).

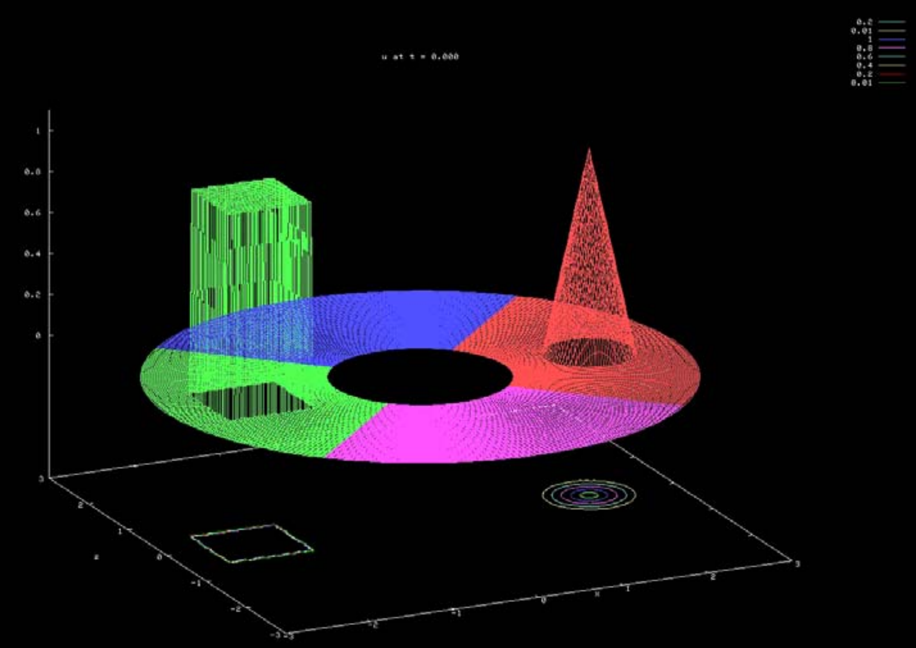
# Other efforts I: Spectral Einstein Code (SpEC)

- Lawrence Kidder (Cornell), Harald Pfeiffer (Caltech), and Mark Scheel (Caltech)
- Multidomain pseudospectral method. Standalone parallel infrastructure.
- Domains can be overlapping or touching.Each individual domain mapped to cube or spherical shell.
- Basis functions are tensor products of Chebyshev, Fourier (for periodic dimensions) or spherical harmonics (for spheres).
- Uses first order strongly hyperbolic systems.
- Outgoing characteristic fields provide boundary conditions on incoming fields of neighboring domains. Use spherical excision boundaries and outer boundaries.
- Excision boundary is outflow boundary (no bc needed). Outer boundary use constraint-preserving boundary conditions.

# Other efforts II: high order methods, high resolution shock capturing methods and overlapping patches

- Jonathan Thornburg and Ian Hawke (Albert Einstein Institute)
- Cactus code. Also uses Carpet as underlying parallel driver.

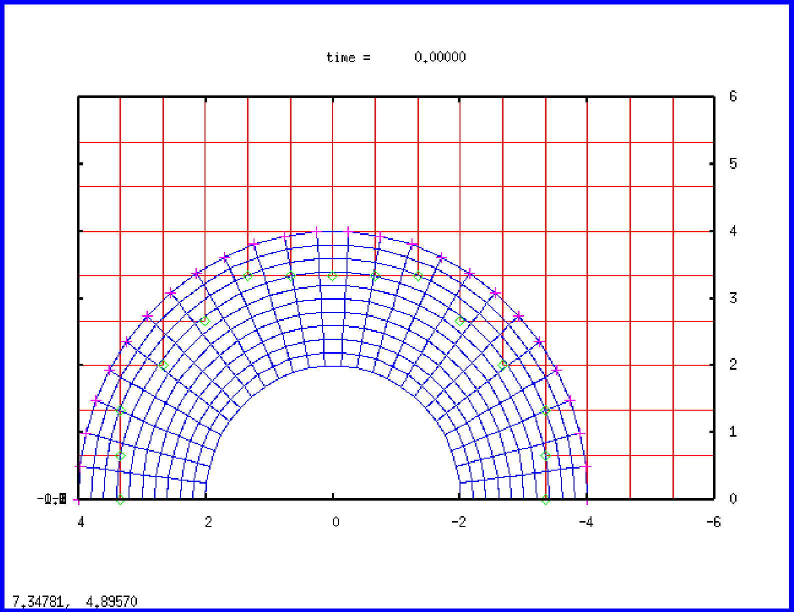


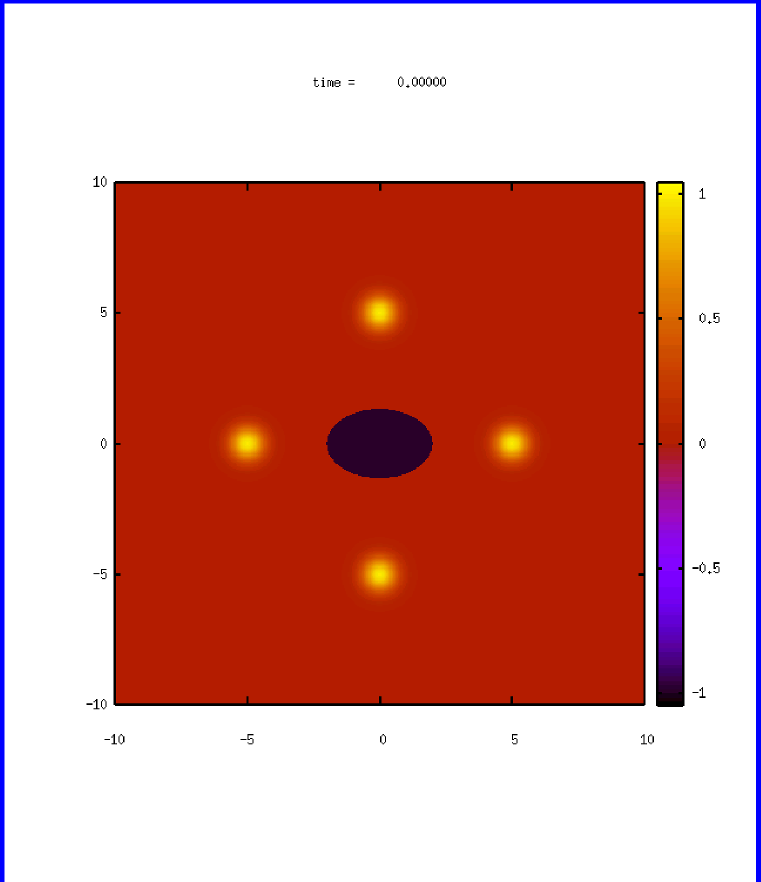




- Overlapping patches communicated through interpolation.
- Can therefore in principle handle first or second order formulations.
- Fourth order vacuum code.
- Uses BSSN formulation of the Einstein's equations.
- HRSC code for fluid part.

# Other efforts III: high order methods and overlapping, moving patches

- Gioel Calabrese (Southampton University) and Dave Neilsen (BYU).
- Wave equation in an axisymmetric boosted rotating black hole background.
- Fourth order code.
- Data between patches communicated via n-th order Lagrangian interpolation for all fields. Outer boundary conditions imposed through Olsson's orthogonal projections. A pinch of artificial dissipation gives (experimental) stability.



# High order numerical schemes:

- Let's consider diagonal metrics: $\sigma_{ij} = \sigma_i \delta_{ij}$

- In the absence of boundaries standard centered operators of order 2n satisfy SBP.

- In the presence of boundaries these operators have to be modified at and near boundaries in order to satisfy SBP.

- The modification at the boundary can be shown to be, necessarily, of order n.

- Second and fourth order cases (n=1, n=2): there is a unique modification near boundaries.
- Sixth order case (n=3): mono-parametric family of modifications.
- Eighth order case (n=4): three-parametric familly.

- The standard choice is to pick up a preferred operator by choosing the one that has the minimum bandwith.

## The spectral radius of the evolution equation and the region of absolute stability of the time integrator

- For an ordinary differential equation $$\frac{d}{dt}u = cu$$

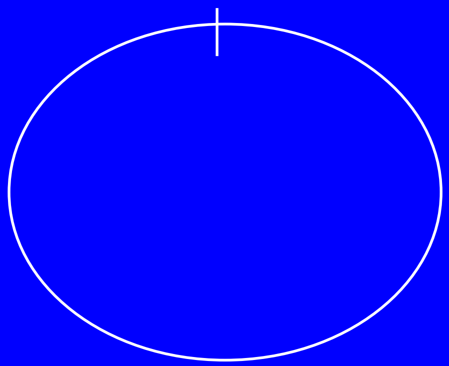
  the region of stability in complex space is the set of c's for which no exponential growth occurs.

- For a differential equation, say $$\frac{\partial}{\partial t}u = A\frac{\partial}{\partial x}u$$

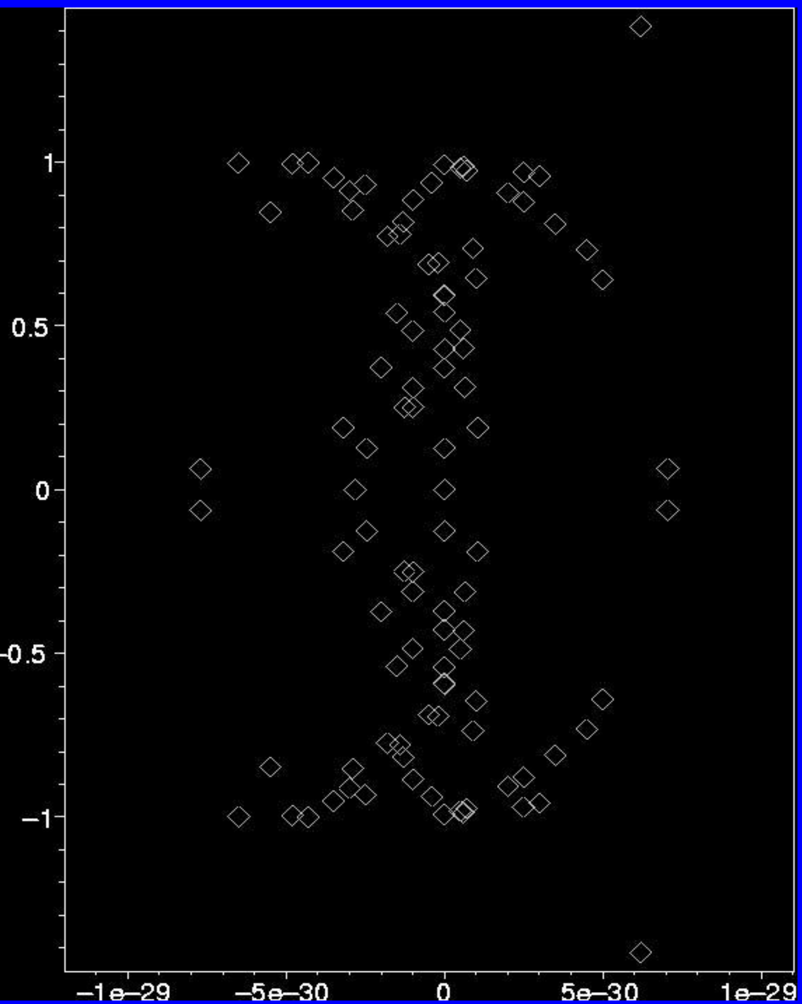
  the maximum eigenvalue of A has to be inside this region of absolute stability, otherwise the scheme is <u>numerically</u> unstable.

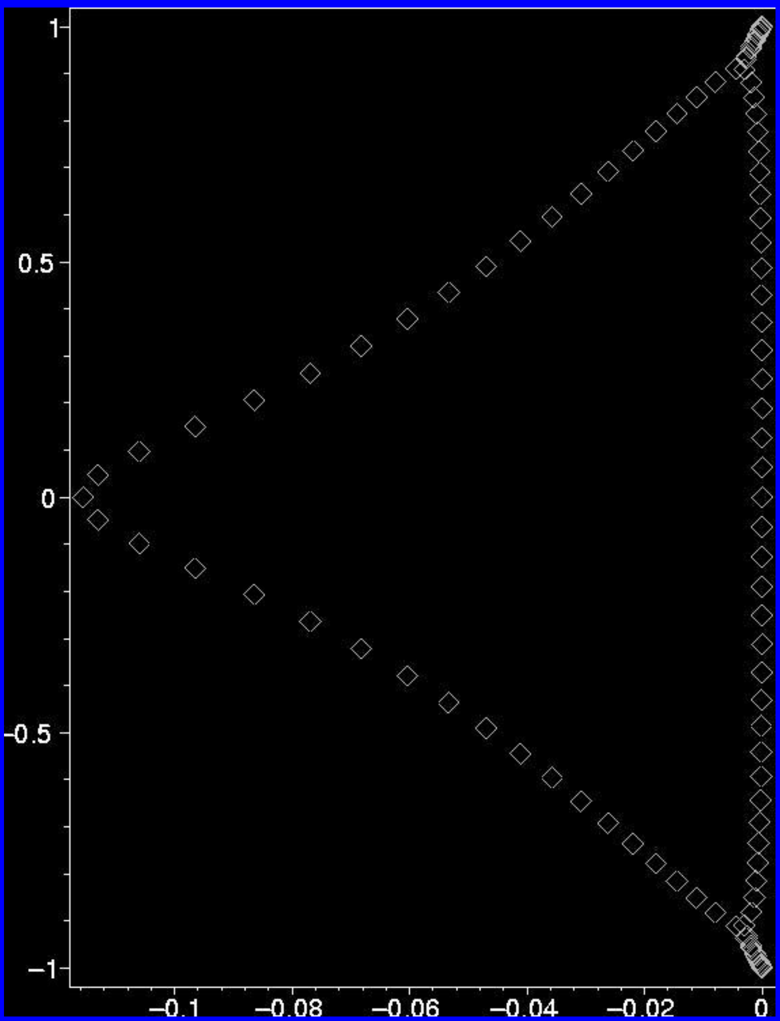
- Let's take a look at the spectrum for a toy model: $$\frac{\partial}{\partial t}u = \frac{\partial}{\partial x}u$$

  in a periodic domain, divided by an interaface,
  with penalties used for the matching.



# Second order case. Maximum = 1.414

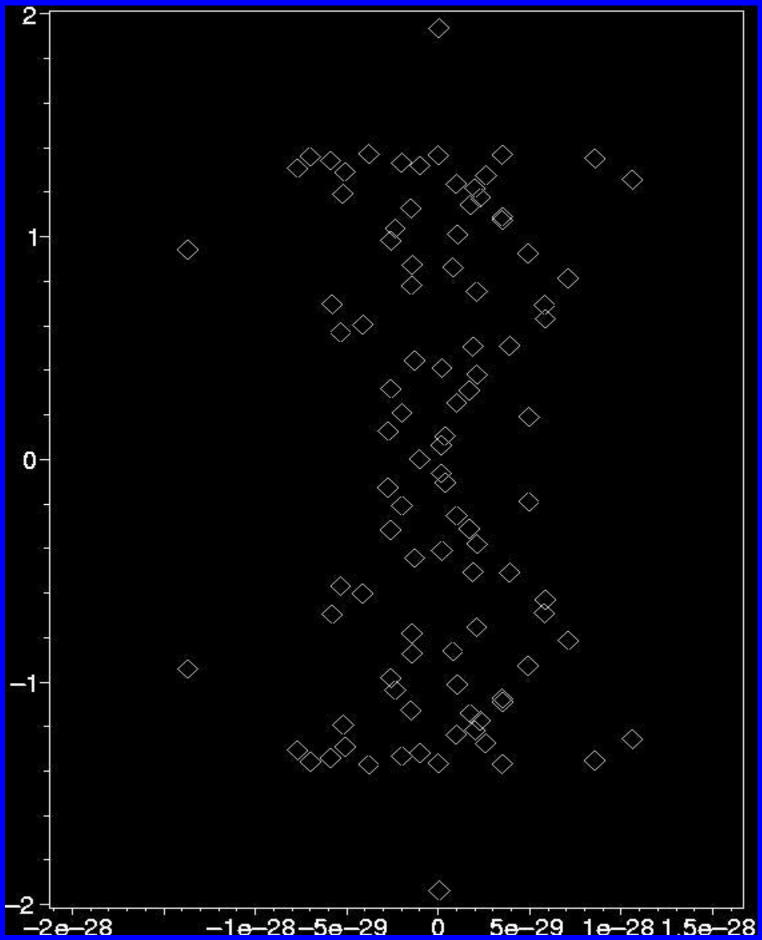


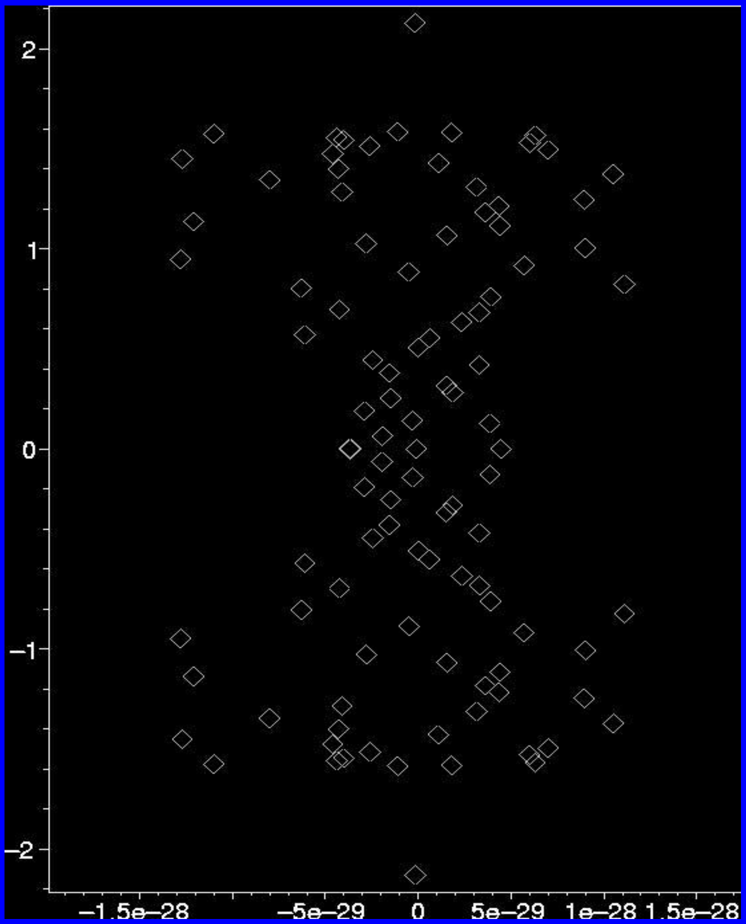

The spectrum is purely imaginary, as it should be. The maximum eigenvalue, 1.414, is associated with the operator near the boundary.

Adds a negative real part to the spectrum, but the maximum in the imaginary axis remains essentially unchanged.

# Fourth and sixth order cases

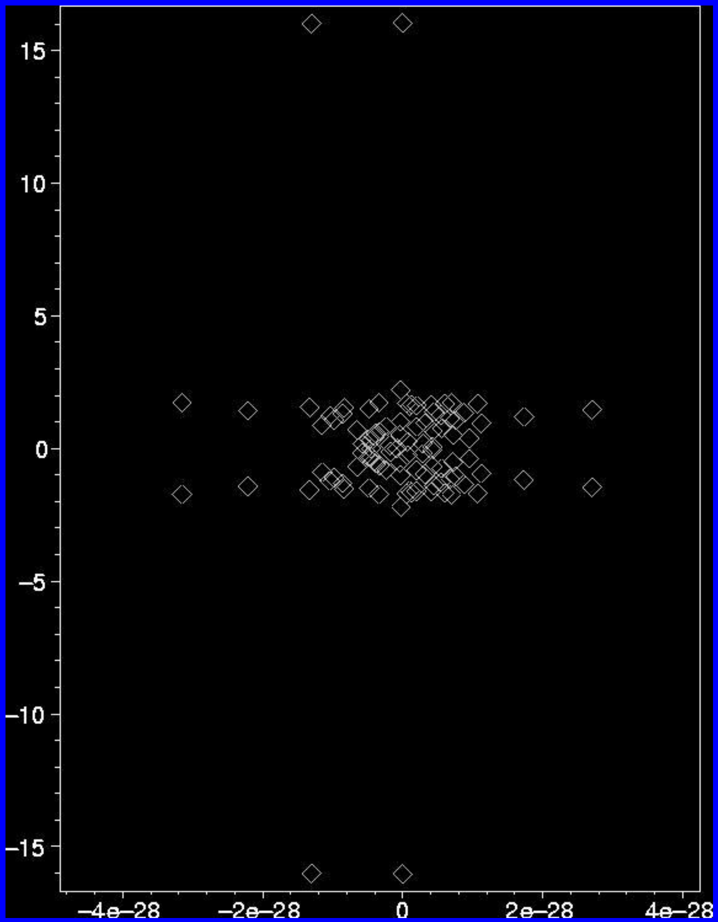


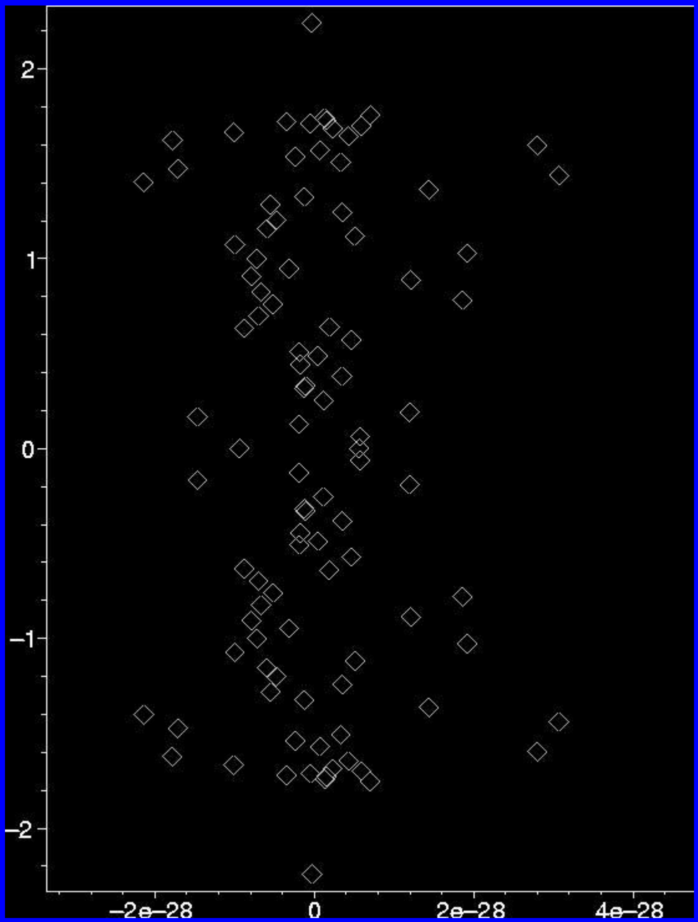

Maximum 1.936




Minimum bandwith operator
Maximum 2.129

# Eight-th order case




Minimum bandwith operator:

Maximum 16.04!

Optimized operator:

Maximum 2.242