Critical Collapse of Scalar Field in the 3+1 Formalism

Pal Sandhu

Supervisor: Dr. Choptuik

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Outline

- Introduction
  - Motivation
  - Fundamental Scalar Field
  - Gravitational Radiation
  - ADM Formalism
- Scalar Field Analysis
  - Scalar Field Scattering on Static Background
  - The Yang-Mills Field

Motivation

- Numerical Relativity
  - Solutions to complex equations
  - Realistic Physical Solutions
  - No solutions to
    - 2-body
    - n-body
    - **gravitational collapse**

Scalar Field

- defined everywhere
- temperature
- “Toy Model”
- Simplicity
- Symmetric
- Our Models
  - Scalar Field Scattering on Static Background
  - Yang-Mills Field

Gravitational Radiation

- "ripples" in space-time

- propagating gravitational fields

- generated by spherically asymmetric motions of matter-energy

- sources

  – violent dynamics of massive sources

- untouched waves

Gravitational Radiation

- Tests
  - perpendicular to direction of propagation
  - gravitational waves at speed of light?
  - General Relativity
- Detection
  - strain distortion of space

3+1 Formalism

- metric in 3+1 Formalism

$$ds^2 = (-\alpha^2 + a^2\beta^i\beta_i)dt^2 + 2\beta_j dt dx^j + \gamma_{ij}dx^i dx^j$$

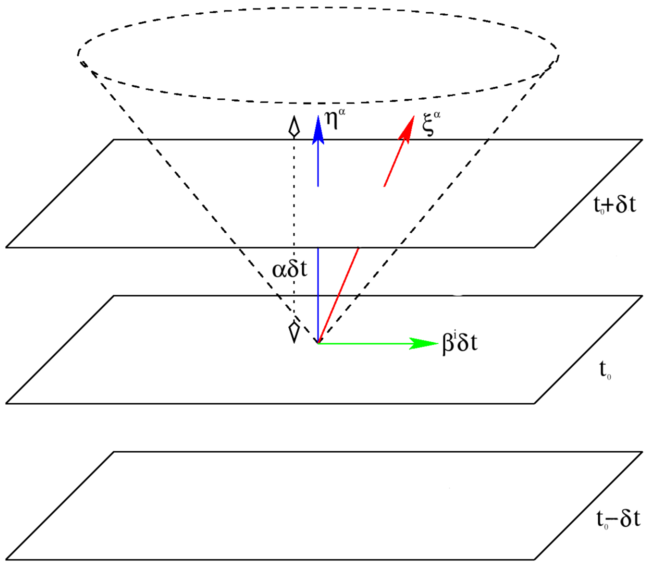
- foliations of space-time

$$\sum(t) = constant$$

- α=$\alpha(t, x^i)$: the lapse
  - lapse of proper time per unit coordinate time for an observer moving normal to the slices

- β^j=$\beta^j(t, x^i)$: the shift vector
  - 3-vector describing the shifting of spatial coordinate relative to "normal propagation"

Hypersurfaces in 3+1 Formalism



Scalar Field Scattering on Static Background

- static background
- black hole fixed in time and space
- Eddington-Finkelstein Coordinates
- massless scalar field

$$\Box\phi(x^\mu) = 0$$

- our metric

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2 d\Omega^2$$

Eddington-Finkelstein coordinates

- Schwarzschild space-time has coordinate singularity at $r = 2M$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- we adapt a time-like coordinate $\tilde{t}$

$$\tilde{t} = v - r = t + 2M \ln\left(\frac{r}{2M} - 1\right)$$

- the ingoing Eddington-Finkelstein form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \frac{4M}{r} d\tilde{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

$\tilde{t}$, the null coordinate

- derived from constraint on β

- by adapting $\tilde{t}$ we maintain areal radial coordinates

- $\tilde{t}$ chosen so ingoing tangent vector is null
$$\left(\frac{\partial}{\partial \tilde{t}}\right)^a - \left(\frac{\partial}{\partial r}\right)^a$$

- most important that all metric components are well behaved at $r \geq 2M$

3+1 components

- $\alpha{=}\alpha(t,x^i){=}\alpha(r)$: the lapse

$$\alpha = \left(\frac{r}{r+2M}\right)^{\frac{1}{2}}$$

- $\beta^j{=}\beta(x^i){=}\beta(r)$: the shift vector

$$\beta = \frac{2M}{r+2M}$$

- $a(r)$, is the inverse of α

$$a = \alpha^{-1} = \left(\frac{r}{r+2M}\right)^{-\frac{1}{2}}$$

Characteristic Analysis

- null line or characteristic
- propagation path of space-time disturbances
- separation of time and space like paths

$$ds^2 = 0 \Rightarrow \left(\frac{dr}{dt}\right)_{\pm} = \beta \pm \frac{\alpha}{a}$$

Characteristic Analysis cont.

- In Eddington-Finkelstein Coordinates

$$\frac{dr}{dt} = -1 \quad \text{ingoing}$$
$$\frac{dr}{dt} = \frac{r - 2\mathrm{M}}{r + 2\mathrm{M}} \quad \text{outgoing}$$

- $r = 2M$ is null

The Equations of Motion

- auxiliary functions Φ and Π

$$\Phi(r,t) = \phi'(r,t)$$

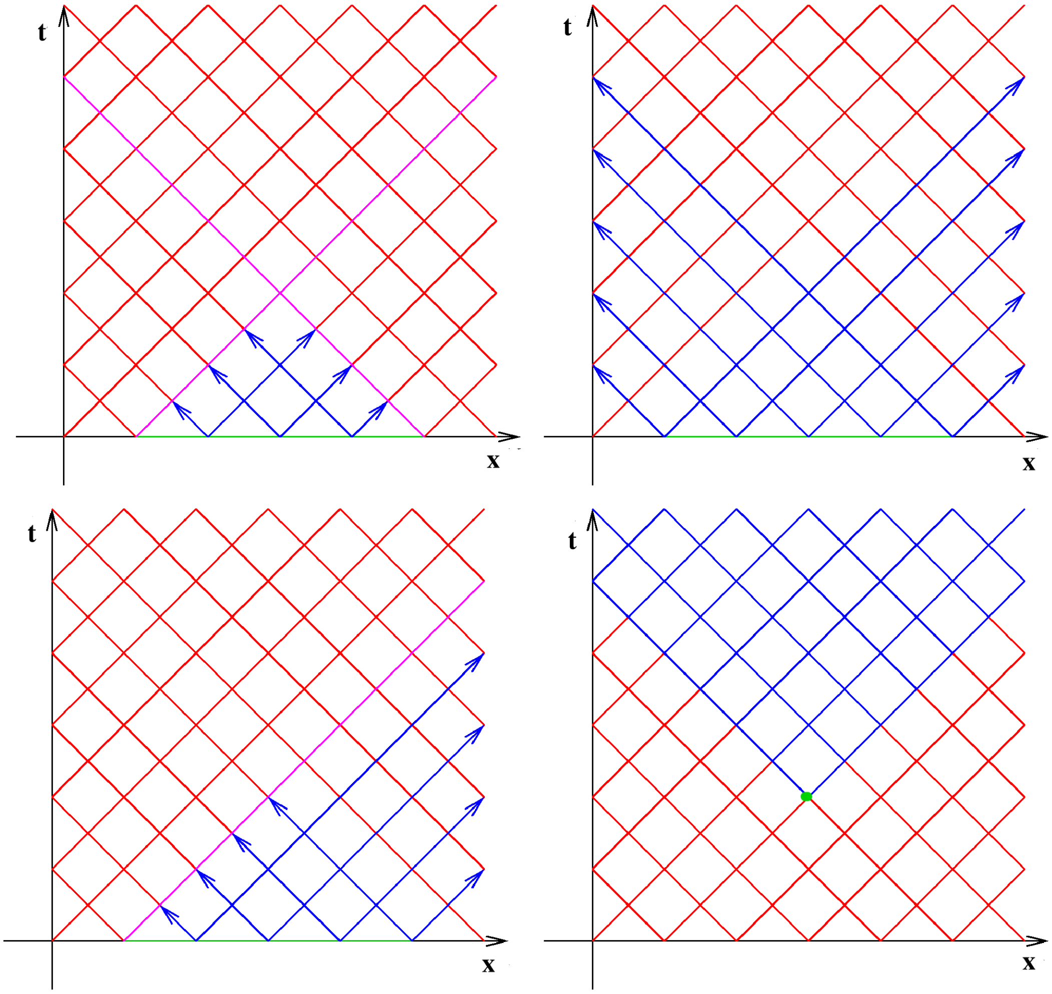
$$\Pi(r,t) = \frac{a}{\alpha}(\dot{\phi} - \beta\phi')$$

- wave equation, become a set of first order in time equations

$$\dot{\Phi} = (\beta\Phi + \frac{\alpha}{a}\Pi)'$$

$$\dot{\Pi} = \frac{1}{r^2}(r^2(\beta\Pi + \frac{\alpha}{a}\Phi))'$$

Initial and Boundary Conditions



Initial Conditions

- Initial Pulse

$$\phi(r,0) = \phi_0 \exp\left(-\left(\frac{r-r_0}{\delta}\right)^2\right)$$

- ϕ_0 arbitrary

- scattering dependent on δ

Conservation of Mass

- conserved quantities

$$p^\mu_{;\mu} = 0 \to \frac{1}{\sqrt{-g}}(\sqrt{-g}p^\mu)_{,\mu} = 0 \tag{1}$$

- integrate over all space and from t_1 and t_2

$$Q(t) = \int_0^\infty (4\pi r^2 \alpha a p^t) dr$$

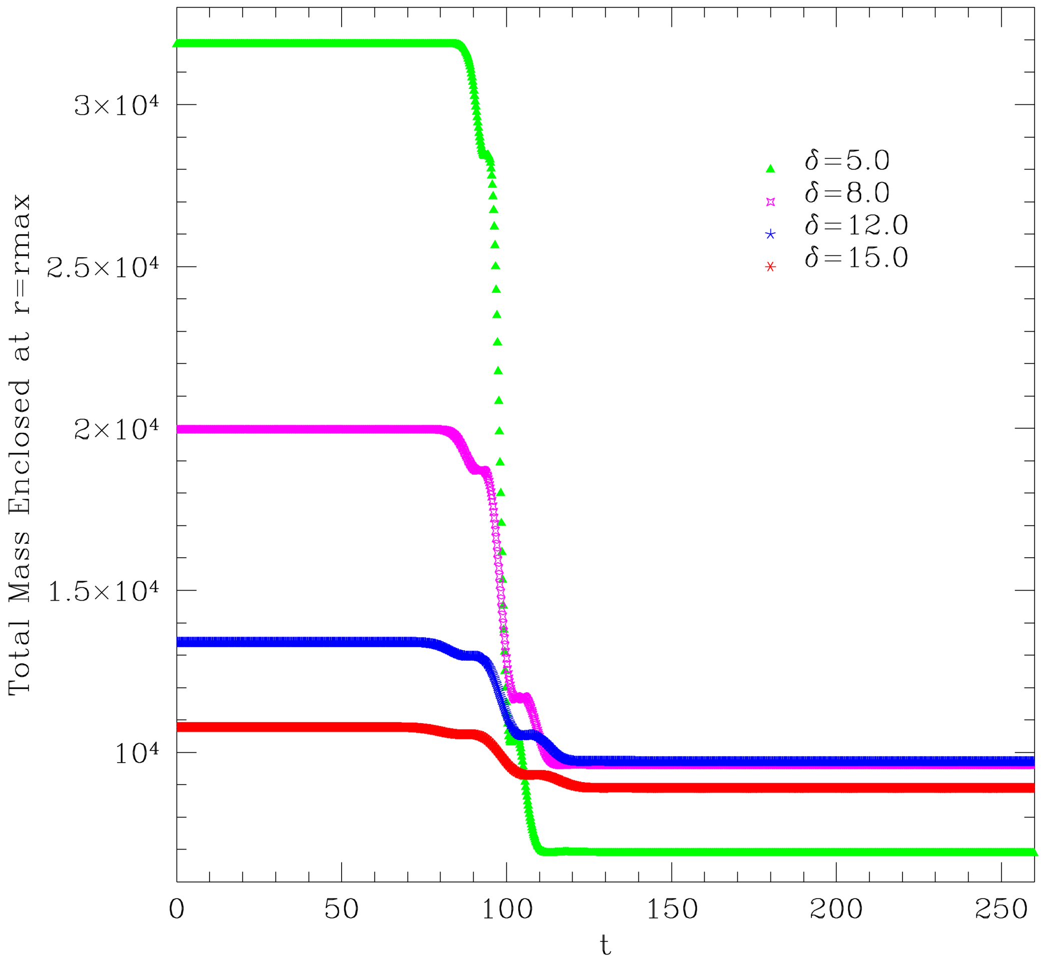
- p^μ from stress-energy tensor

$$p^\mu = T^{\mu\nu}\xi_\nu$$

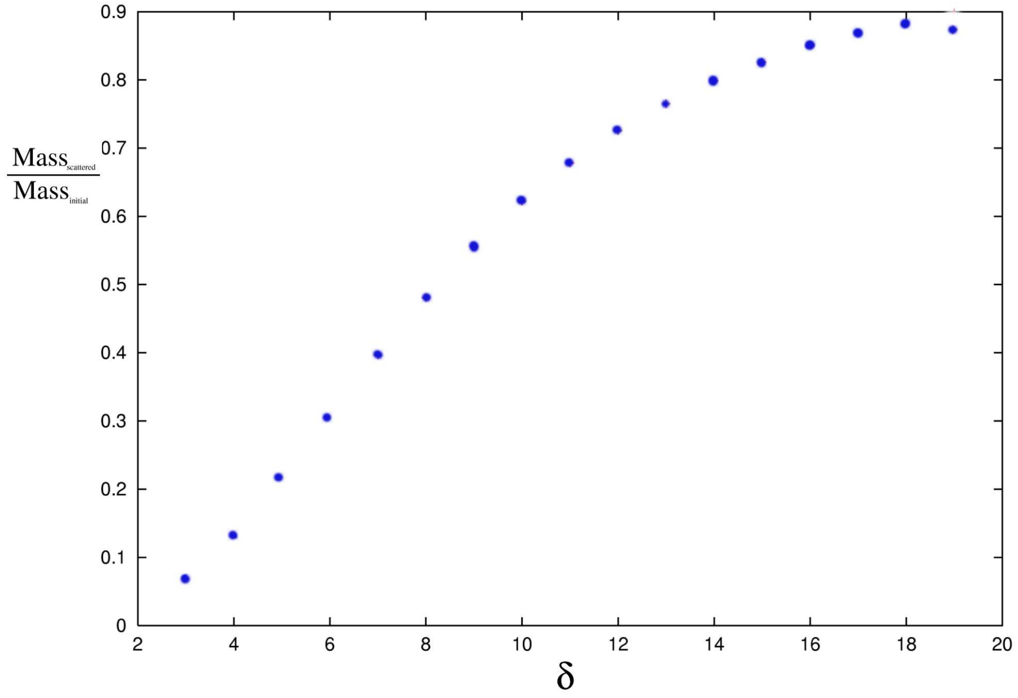
- conserved mass

$$\frac{dm}{dr} = \left(\frac{\Pi^2 + \Phi^2}{a^2} + \frac{2\beta\Phi\Pi}{\alpha a}\right) 2\pi r^2 \alpha a$$

Total Mass Enclosed



Scattering Fraction



Yang-Mills Field

- dynamic background

- collapse of self-gravitating field

- our metric

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$$

Equations of Motion

- Evolution Equations

$$\begin{aligned}
\dot{\Pi} &= \left(\frac{\alpha}{a}\Phi\right)' + \frac{a\alpha}{r^2}W(1-W^2) \\
\dot{\Phi} &= \left(\frac{\alpha}{a}\Pi\right)' \\
\dot{a} &= \frac{2\alpha}{r}\Pi\Phi
\end{aligned}$$

- Constraint Equation

$$\begin{aligned}
\frac{a'}{a} + \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) &= 0 \\
\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) &= 0
\end{aligned}$$

- W, Yang-Mills Potential

Initial Conditions

- initial Pulse
  - gaussian

$$W(r,0) = W_0 \exp^{\left(-\left(\frac{r-r_0}{\delta}\right)^2\right)}$$

  - "kink"

$$W_0(r;r_0,\delta) = \frac{1+\frac{r_0^2-r^2}{\delta^2}}{\sqrt{(1+\frac{r_0^2-r^2}{\delta^2})^2+4r^2}}$$

- W_0 and δ determine mass-energy of system

- black hole creation dependent on W_0 and δ

Critical Parameters

- gaussian critical parameters

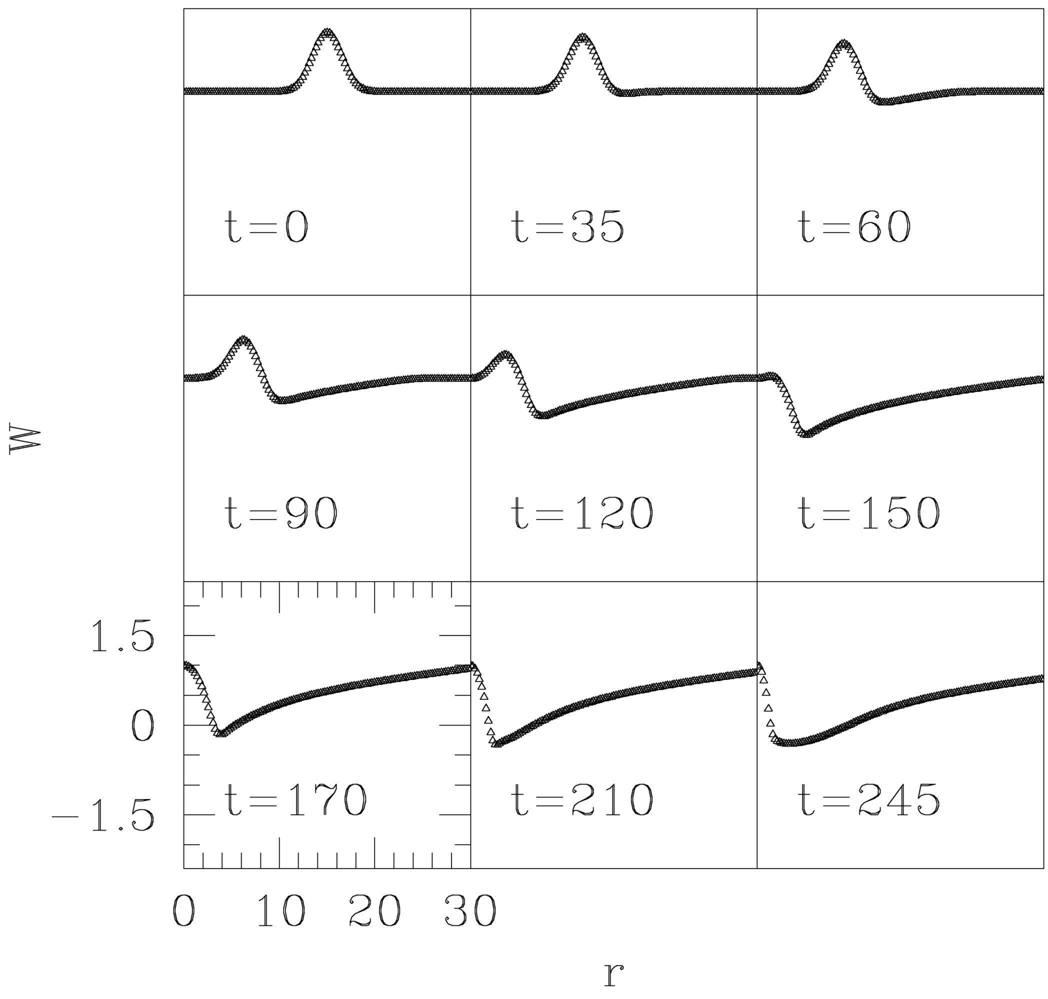
$$1.4616189 \le \delta \le 1.461619$$

$$0.96787529 \le W_0 \le 0.967875385$$

- "kink" critical parameter

$$1.64268059 \le \delta \le 1.6426806$$

Evolution of Gaussian Pulse



Evolution of “Kink” Function