Critical Collapse of Scalar Field in 3+1 Formalism

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Abstract

Over the years there has been continuing interest in numerical techniques and their use in solving complex equations. In General Relativity there are many equations that just from there complexity cannot or are extremely difficult to solve. In this paper we use 3+1 formalism to rewrite the metric in order to solve our field equations. We look at the static background black hole scattering problem. Here we find that if we send in a gaussian pulse, the amount of scattering is dependent on the width of the pulse. The second problem we look at is the dynamic background scalar field collapse from the Yang-Mills field. Here we find that there are two parameters to consider when feeding in the gaussian pulse, namely the amplitude and the width. We also try a kink function as the initial data and find that we can find static solutions to the field.

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Chapter 1

Introduction

Numerical Relativity is in essence a way to understand physically, problems brought up in classical General Relativity. The solutions of Einstein's equations have given us a greater understanding of the theory of General Relativity. Although it only gives us a "thin" understanding, in the sense that we know their geodesic, where they are singular and their causal and global structure. This is where I think Numerical Relativity comes in, to fill in the blanks. Numerical Relativity gives us solutions to equations that because of there complexity can be extremely difficult to solve.

The solution to partial differential equations are plentiful. They are however fraught with "what if"'s and instances. In order to obtain realistic physical solutions we need to have initial data and a domain of computation. In General relativity we also have many exact solutions. See "Exact solutions of Einstein field equations" [4]. A lot of these solutions however are pathological in nature. Many times they do not approximate realistic physical solutions.

Also, the objects that are of greatest interest to us at least on the astrophysical scale we still do not have solutions for. We do not have solutions to 2-body or n-body systems, interiors of rotating sources undergoing gravitational collapse, etc..

Many problems in Numerical Relativity involve the generation and propagation of gravitational radiation from a collapse of gravitational fields. We use scalar fields because of their simplicity and also because they can be used to model radiation in spherical symmetry. We can hypothesize, at best that we could be looking at some real field like an energy field or gravitational field, at worst it is a toy model.

In chapter 1 we discuss gravitational fields and the Einstein equations. We go onto discussing the scalar field and a numerical approach based on the work of Arnowitt, Deser and Misner. This so called 3+1 approach is the basis of numerical representation of the space-time at hand and is the most widely used approach.

In chapter 2 we introduce the specific field equations for the various models we are working with. It goes without saying that in order to be able to solve the set of field equations numerically you first have to understand the problem, including initial and boundary conditions and you have to know what your solution should look like at least in its most basic form.

In chapter 3 we discuss the various finite difference equations we use and talk about stability in our solution. This goes back to obtaining the correct field equations and initial and boundary conditions.

In chapter 4 we talk about the results we obtained from the different models we used. Specifically we talk about a massless scalar field on a static background with a black hole. The second model we discuss is the Yang-Mills field where we no longer have mass equal to zero.

Chapter 2

Theory

In this paper we are mainly concerned about the solution to field equations. We remain on the topic of the scalar field. The scalar field is a “toy model”, but hopefully it will give us some insight into more realistic phenomenon. Nevertheless, as said before numerical relativity’s most interesting topic is the propagation of gravitational waves and to that end we begin by giving an introduction to this topic before moving on to the the more general scalar field.

2.1 Gravitational Waves

Gravitational waves are comparable to light waves. Just like light waves consist of propagating electromagnetic fields generated by moving electrically charged particles, gravitational waves are propagating gravitational fields, ’ripples’ in the curvature of space-time generated by asymmetric motions of matter-energy. The former cause a strain of space-time which result in changes in the distance between points in space-time. However, these two entities are detected in different ways. The spatial variations in the phase of light waves (for example, differences received on different parts of the retina), allow us to perceive the direction and shape of the matter that they came from or in most cases reflected off from. Gravitational waves can be detected by devices which measure the induced changes in space-time. Waves with different frequencies are caused by different motions of mass and the difference in the phases of these waves allow us to perceive the direction to and the shape of the matter that generated them.

Because the fundamental force of gravity is generally weaker per unit mass than the fundamental electromagnetic force per unit charge, gravitational waves are much weaker than electromagnetic waves. There are two consequences because of this weakness:

1. only from very massive sources undergoing violent dynamics come detectable gravitational waves.

2. the waves are unaffected by scattering or absorption from intervening matter, so the waves are essentially untouched.

The detection of gravitational waves would give us experimental tests of fundamental physical laws which cannot be made by any other means. For example:

1. the prediction of GR that the waves only change distances perpendicular to the direction of propagation could be verified.

2. comparison of light and gravitational waves arrival times would test whether Einstein's prediction was right about gravitational waves traveling at the speed of light.

3. detection of gravitational waves would verify a fundamental prediction of General Relativity.

Gravitational waves will also offer a new window for observational astronomy, because they provide different information from that of the electromagnetic window. Also, because of the weakly interacting nature of gravitational waves they preserve detailed information about the source of the waves. This is different from electromagnetic waves, which can be scattered or screened by intervening matter. Violent activity in supernovae, galactic nuclei and quasars in theory are where the strongest gravitational waves are likely to originate.

2.2 The Field Equations

As Einstein developed the theory of gravity he also concluded that gravitational effect should propagate at the speed of light. There is also a "Newtonian" part of the gravitational field and in the linear limit (Weak Field) the Einstein equations reduce to a tensor wave equation relating the wave amplitude to source terms. Analogous to electromagnetic theory, the Einstein equations allow a propagating wave solution, which are transverse to the propagation direction, have two independent polarization and propagate at the speed of light. There are alternate relativistic theories of gravity which also predict existence of gravitational waves. However, these theories differ by polarization states, propagation speed and efficiency of wave generation. Systematic observation of these waves by LIGO and other detectors would test the predictions of General Relativity against these other theories.

2.3 Fundamental Scalar Field

A scalar field is a scalar quantity defined everywhere in space. An example is temperature; we can specify a temperature at every point and time and there by get a scalar field.

Prior to General Relativity, the force of gravity would have been considered a scalar field theory because it could be expressed as the gradient of a scalar quantity. But when GR came into the game, it was realized that gravity was a tensor theory. We will work with the "scalar field" first as a massless scalar field on a static background. So, we will observe the solution to a collapse of scalar field on a black hole. Second, we will look at massive scalar field collapse, specifically the Yang-Mills field [5]. The Yang-Mills field is given by a Yang-Mills potential, W. This is a dynamic collapse of space-time, to form a black hole.

2.4 ADM approach to geometrodynamics

The most frequently used formalism in Numerical Relativity is the 3+1 formalism of Arnowitt, Deser and Misner or ADM. From a geometrical viewpoint, it consists of a decomposition of space-time in which time is singled out as a privileged direction and space-time is foliated by 3-dimensional space-like hyper-surfaces corresponding to constant time slices. Since this decomposition is so important to Numerical Relativity, it will be discussed in concise detail.

We have a foliation of space-time into a family of hyper-surfaces $\sum$ given by,

$$\phi(x^\mu) = \text{constant} \tag{2.1}$$

the notation is important here, Roman indices can only be from 1 to 3 ("the spatial indices") and Greek indices can be from 0 to 4 (all space-time). Then we can define the covariant normal vector field to these surfaces as

$$\tau_\mu = \phi_{,\mu} \tag{2.2}$$

and the contravariant normal vector is given by

$$\tau^\mu = g^{\mu\nu}\tau_\nu = g^{\mu\nu}\phi_{,\nu} \tag{2.3}$$

Then we define the unit normal to the foliation, n^μ, which is proportional to 2.3 and satisfies

$$n^\mu n_\mu = -1 \tag{2.4}$$

We will have to decompose space-time tensors into hyper-surface-tangential ("spatial") and hyper-surface-orthogonal("temporal") pieces. In order to determine the temporal part of a tensor we simply contract it with n^μ on all indices of the tensor. In order to determine the spatial parts of tensors, we introduce the convenient *projection tensor*, which projects tensors onto the hyper-surface and is defined by

$$\perp^{\mu}{}_{\nu} \equiv \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu} \tag{2.5}$$

Notice the "+" between the identity tensor and $n^{\mu}n_{\nu}$. This comes from the Lorentzian signature (- + + +). We can now project the 4-metric $g_{\mu\nu}$ onto the foliation $\sum$ to obtain the induced 3-metric or spatial metric describing space on the foliations.

$$\gamma_{\mu\nu} = \perp_{\mu}{}^{\kappa}\perp_{\nu}{}^{\lambda}g_{\kappa\lambda} \tag{2.6}$$

using equations 2.4 and 2.5 we can prove that,

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_{\mu}n_{\nu} \tag{2.7}$$

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^{\mu}n^{\nu} \tag{2.8}$$

Now we adopt the following coordinates to incorporate into our foliations ϕ

$$(x^{\mu}) = (t, x^{i}) = (t, x^{1}, x^{2}, x^{3}) \tag{2.9}$$

Then our hyper-surfaces become those constant in time, $\sum(t)$,

$$t = \text{constant}$$

We consider any vector field that transvects (lies nowhere in) the foliation $\sum$. We decompose it into a component orthogonal and a component parallel to n^{μ},

$$\xi^{\mu} \equiv (\frac{\partial}{\partial t})^{\mu} = \alpha n^{\mu} + \beta^{\mu} \tag{2.10}$$

ξ^{μ} is a generalized time vector. α is the proportionality factor called the lapse which is the lapse of proper time per unit coordinate time for an observer moving normal to the slices and β^{i} is called the shift vector

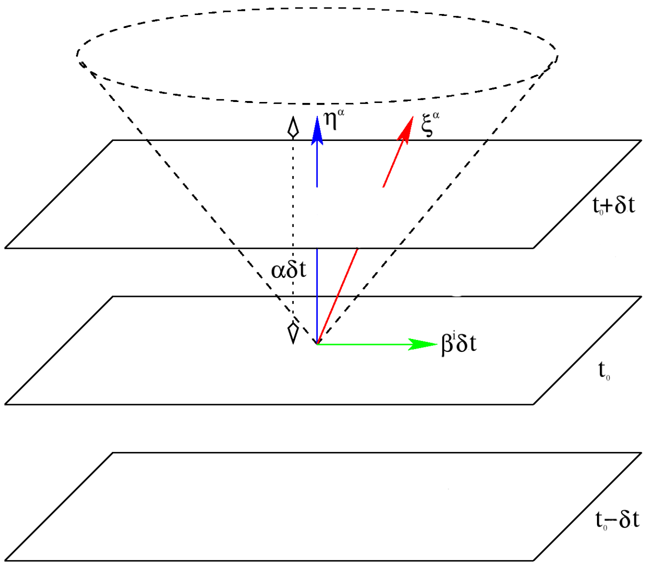

Figure 2.1: Shows space-time as hyper-surfaces in ADM formalism

describing the shifting of spatial coordinate relative to "normal propagation" (see figure 2.4). The shift must satisfy,

$$\beta^\mu n_\mu = 0 \tag{2.11}$$

then from 2.8 and 2.10 we have the following,

$$g^{\mu\nu} = -\frac{1}{\alpha}(\xi^\mu - \beta^\mu)\frac{1}{\alpha}(\xi^\nu - \beta^\nu)) + \gamma^{\mu\nu} \tag{2.12}$$

To explain what might have been confusing before about the indices, since the induced 3-metric and shift vector are spatial we have,

$$\beta^\mu = \delta_i{}^\mu \beta^i \tag{2.13}$$

Then from 2.12 and 2.13 we can write the contravariant metric as,

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{2.14}$$

We can find the induced covariant metric, γ_{ab} as,

$$\gamma_{ij}\gamma^{jk} = \delta_i{}^k \tag{2.15}$$

this is so we can raise and lower indices in our 3-space.

$$\beta_i = \gamma_{ij}\beta^j \tag{2.16}$$

so our covariant metric is

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^k\beta_k & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{2.17}$$

Now 2.14 and 2.17 satisfy,

$$g^{\mu\nu}g_{\nu\sigma} = \delta_\mu{}^\sigma \tag{2.18}$$

so we can write out the 3+1 or ADM form of the metric line element,

$$ds^2 = (-\alpha^2 + \beta^k\beta_k)dt^2 + 2\beta^i dt dx^i + \gamma_{ij}dx^i dx^j \tag{2.19}$$

2.4.1 Spherical Symmetric with Areal Coordinates Specialization

We can simplify equation 2.19 immensely if we have spherical symmetry. When we have a spherically symmetric collapse we can basically through away the angular terms and work our problem in 1+1 dimensions. This greatly simplifies the equations of motion and it goes without saying greatly shortens the computation time. In that case our foliations $\sum$ are not hyper-surfaces but hyper-lines if you will. Our shift vector β^i becomes,

$$\beta^i = (\beta, 0, 0) \tag{2.20}$$

so,

$$\beta_i = \zeta_{ij}\beta^j = (a^2\beta, 0, 0) \tag{2.21}$$

and the line element in our general form having chosen areal spatial coordinates so that area of $r = constant = 4\pi r^2$, becomes

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2 d\Omega^2 \tag{2.22}$$

Chapter 3

Scalar Field Analysis

Of course the most important thing in solving a PDE is having the correct equation initial conditions and boundary conditions. So, in-depth discussion of the scalar field is very important. These scalar field equations in their most simplest form become the wave equation with one non-rigid barrier as the boundary. For example, our black hole could be interpreted as a heavy string and our scalar field as a light string. When we set a pulse forth in our light string as the it hits the heavy string it gets reflected and absorbed. Unfortunately this is as far as this analogy can go. In this chapter we discuss our equations of motion and perform a characteristic analysis. Also, we look at the initial and boundary conditions for the background computation, that is when we are scattering on a pre-formed static black hole and for the self-gravitating case where we deal with scalar field collapse to form a black hole.

3.1 Equations of Motion

From analytical mechanics we know that we can obtain our equations of motion from taking the variation of the action $\partial \mathcal{S} = 0$. Where the action, $\mathcal{S}$, is the integral over all space-time of the lagrangian, $\mathcal{L}$. A detailed formulation of the equations of motion using lagrangian formalism is given in appendix C.

Here we will start with the Einstein-Field equations,

$$G_{\mu\nu} = 8\pi\mathcal{T}_{\mu\nu} = 8\pi(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}) \tag{3.1}$$

where the comma denotes differentiation with respect to.

A more complete derivation of the equation of motion is given in appendix C. We have the Klein-Gordon equation

$$\Box\phi = m^2\phi \tag{3.2}$$

Where the $\Box$ represents the D'Alembertian operator,

$$\Box\phi = \phi^{,\alpha}\phi_{,\alpha} \tag{3.3}$$

When m=0, i.e. we have the massless scalar field equation 3.2

$$\Box\phi = 0 \tag{3.4}$$

In keeping with the discussion in section 2.4 we can introduce for convenience the following auxiliary functions.

$$\Phi = \phi' \tag{3.5}$$
$$\Pi = \frac{a}{\alpha}(\dot{\phi} - \beta\phi') \tag{3.6}$$

Then, we can write the massless scalar field equations as two first order partial differential equations in terms of Φ and Π. The evolution equations for massless scalar field are

$$\dot{\Phi} = (\beta\Phi + \frac{\alpha}{a}\Pi)' \tag{3.7}$$
$$\dot{\Pi} = \frac{1}{r^2}(r^2(\beta\Pi + \frac{\alpha}{a}\Phi))' \tag{3.8}$$

For a field coupled to gravity, that is when m$\neq$0, our equations can become a lot more difficult to come by. In addition we have constraint equations to consider.

For the Yang-Mills field the matter Lagrangian scalar is given by

$$\mathrm{L_M} = -\left(\frac{\mathrm{g}^{\mu\nu}\partial_\mu \mathrm{W} \partial_\nu \mathrm{W}}{r^2} + \frac{1}{2}\frac{(1-\mathrm{W}^2)^2}{r^4}\right) \tag{3.9}$$

So our total Lagrangian is

$$\mathcal{L} = \mathcal{L}_\mathrm{G} + \alpha_\mathrm{M}\mathcal{L}_\mathrm{M} = \sqrt{-\mathrm{g}}(\mathrm{R} + \alpha_\mathrm{M}\mathrm{L_M}) \tag{3.10}$$

Where α_M is some coupling constant. So our Stress Tensor can be written

$$\mathcal{T}_{\mu\nu} = \frac{\alpha_\mathrm{M}}{8\pi}\left(-\frac{\partial \mathrm{L}}{\partial \mathrm{g}^{\mu\nu}} + \frac{1}{2}\mathrm{g}_{\mu\nu}\mathrm{L_M}\right) \tag{3.11}$$

Now we define the auxiliary functions Φ and Π as,

$$\Phi = \mathrm{W}' \tag{3.12}$$

$$\Pi = \frac{a}{\alpha}\dot{\mathrm{W}} \tag{3.13}$$

To obtain the equation of motion for Π we take the variation with respect to W of the action. The equation of motion for Φ is easily obtained by taking the derivative with respect to time of equation 3.12. Generally the evolution equation for a in polar/areal coordinates is given by

$$\dot{a} = 4\pi r a \mathcal{T}_{tr} \tag{3.14}$$

So now we can write down the evolution equations for Π, Φ and a.

$$\dot{\Pi} = \left(\frac{\alpha}{a}\Phi\right)' + \frac{a\alpha}{r^2}\mathrm{W}(1-\mathrm{W}^2) \tag{3.15}$$

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)' \tag{3.16}$$

$$\dot{a} = \frac{2\alpha}{r}\Pi\Phi \tag{3.17}$$

Now we also have to consider the constraint equations, namely the polar-slicing constraint that constrains the lapse α at all instants in time and the Hamiltonian constraint that constrains a radially. In general the polar slicing constraint in polar/areal coordinates is written like this

$$\frac{\alpha'}{\alpha} - \frac{a^2 - 1}{2r} - 4\pi r a^2 \mathrm{S}^r_r \tag{3.18}$$

where S^i_j are the stress components defined by,

$$\mathrm{S}^i_j = \mathrm{g}^{ik}\mathrm{S}_{kj} = \mathrm{g}^{ik}\mathcal{T}_{kj} \tag{3.19}$$

The Hamiltonian constraint takes the form

$$\frac{a'}{a} + \frac{a^2 - 1}{2r} - 4\pi r a^2 \rho \tag{3.20}$$

where ρ is the energy density defined by,

$$\rho = n^\mu n^\nu \mathcal{T}_{\mu\nu} \tag{3.21}$$

where n^μ is the vector normal to our time-like hyper-surfaces defined in section 2.4. Then the Hamiltonian and polar-slicing constraint equations become,

$$\frac{a'}{a} + \frac{a^2 - 1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1 - \mathrm{W}^2)^2\right) = 0 \tag{3.22}$$

$$\frac{\alpha'}{\alpha} - \frac{a^2 - 1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1 - \mathrm{W}^2)^2\right) = 0 \tag{3.23}$$

3.2 Characteristic Analysis

Space-time is always divided into time-like and space-like separations. In principle anything with mass must move along a time-like curve. The boundary line between space-like and time-like separations is the null light cone (figure 3.2).

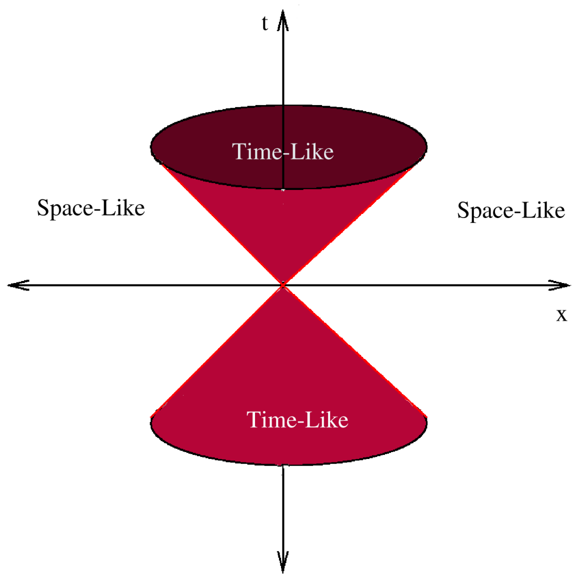

Figure 3.1: Null light cone seperating space-like and time-like

In spherical symmetry these null paths or characteristics can be determined by solving for directions $\frac{dr}{dt}$ such that $ds^2 = 0$. The characteristics tell us how small "disturbances" in the scalar field propagate. In general our metric,

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta^2 dt dr + a^2 dr^2 + r^2 d\Omega^2 \tag{3.24}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

The characteristics are given by,

$$ds^2 = 0 \Rightarrow \left(\frac{dr}{dt}\right) = -\beta \pm \frac{\alpha}{a} \tag{3.25}$$

Now the analysis becomes more specific. In the case of a massless scalar field given by equation 3.2 and on a Schwarzschild background we can find exactly what α, β and a are. We consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2\mathrm{M}}{r}\right) dt^2 + \left(1 - \frac{2\mathrm{M}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.26}$$

We can see right away though that we have a coordinate singularity at r=2M in addition to the actual singularity at r=0. We can remove this coordinate singularity by using in-going Eddington-Finkelstein coordinates,

$$\tilde{t} = v - r = t + 2\mathrm{M}\ln\left(\frac{r}{2\mathrm{M}} - 1\right) \tag{3.27}$$

where v is the in-going null coordinate. Then we can write the Schwarzschild metric in in-going Eddington-Finkelstein (IEF) form

$$ds^2 = -\left(1 - \frac{2\mathrm{M}}{r}\right)dt^2 + \frac{4\mathrm{M}}{r}dtdr + \left(1 + \frac{2\mathrm{M}}{r}\right)dr^2 + r^2 d\Omega^2 \tag{3.28}$$

In terms of the 3+1 decomposition 2.22, we have

$$\alpha = \left(\frac{r}{r+2\mathrm{M}}\right)^{\frac{1}{2}} \tag{3.29}$$

$$\beta = \left(\frac{2\mathrm{M}}{r+2\mathrm{M}}\right)^{\frac{1}{2}} \tag{3.30}$$

$$a = \left(\frac{r}{r+2\mathrm{M}}\right)^{\frac{-1}{2}} \tag{3.31}$$

Evaluating the characteristic using equations 3.29, 3.30 and 3.31 we get that the ingoing characteristics are the same as they are in Minkowski flat space-time, but the outgoing take some logarithmic form.

$$\frac{dr}{dt} = -1 \quad \text{ingoing} \tag{3.32}$$

$$\frac{dr}{dt} = \frac{r-2\mathrm{M}}{r+2\mathrm{M}} \quad \text{outgoing} \tag{3.33}$$

The ingoing and outgoing characteristics, 3.32 and 3.33 are plotted in figure 3.2. A slightly more informative graph is shown in figure 3.2, which shows the collapse of a spherically symmetric and uniform pressure-less sphere of mass to a black hole.

3.3 Initial and Boundary conditions

Without proper initial conditions and boundary conditions we can not be sure we obtained the correct solution. So we must be careful to choose

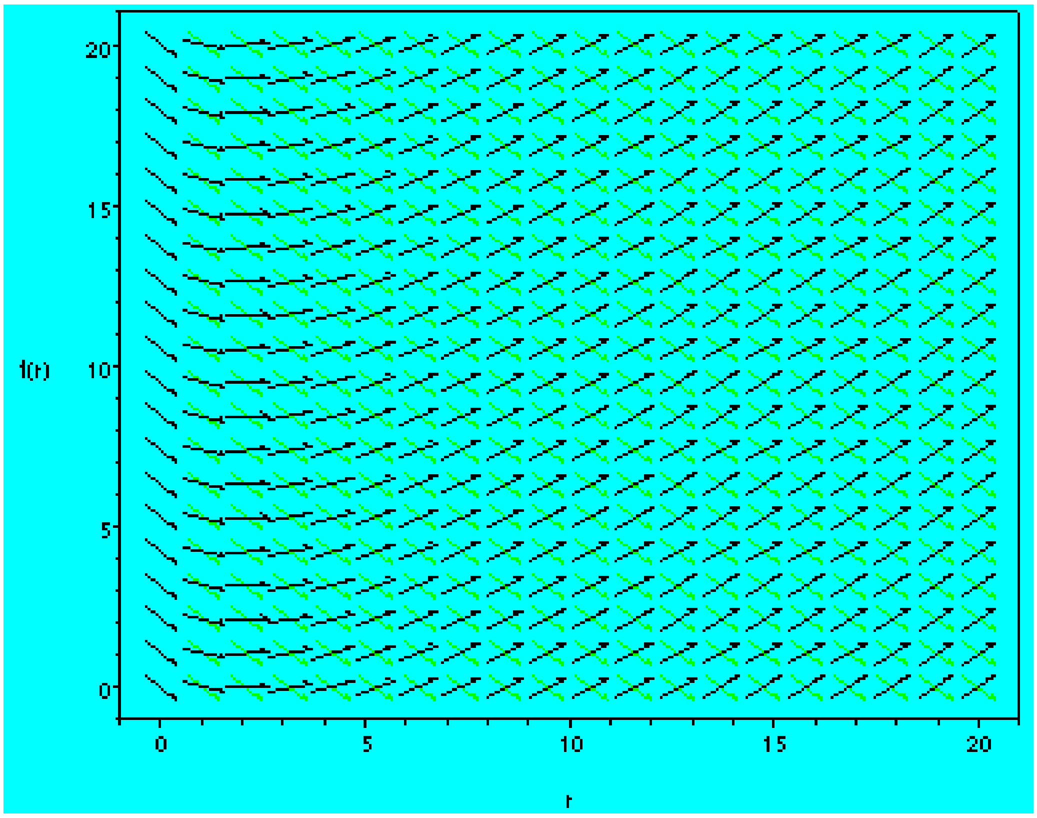

Figure 3.2: The slope of the in-going propagation and the outgoing propagation in Eddington-Finkelstein coordinates

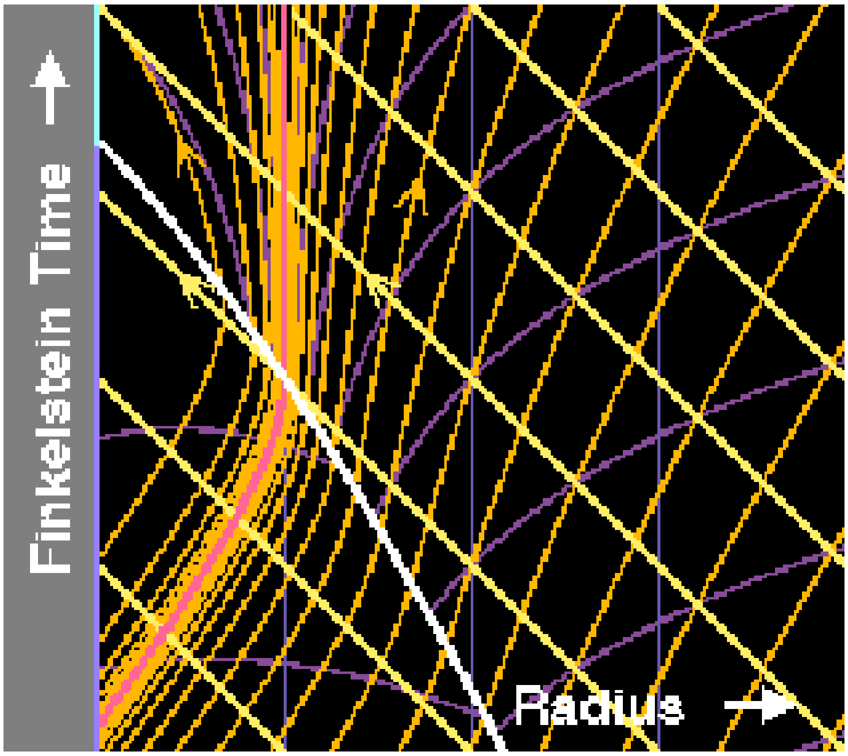

Figure 3.3: Radially infalling light rays in yellow. Wordline of surface of the collapsing sphere in white. The absolute horizon, defined as the surface from which photons can never escape to the outside in pink-red. The singularity forms in cyan. Lines of constant Schwarzschild time outside the collapsing sphere and of constant Friedmann-Robertson-Walker time inside the collapsing sphere in dark-purple. Constant circumferential radius in dark-blue. [7]

suitable conditions. Figure 3.3 shows how different choices of initial and boundary conditions has a different "zone of influence".

For solutions to our problems we have the following criteria. We have boundary conditions at finite r (approximate). We have boundary conditions at $r = 2M$. This condition is not needed because from equation 3.25 $r = 2M$ is a characteristic. We also have to enforce regularity conditions. This means that although we have curved space, we also have locally flat space. Specifically at $r = 0$, we don't want singularity except for coordinate singularity.

Initial conditions must be chosen so that they satisfy the Hamiltonian constraint (3.20) and the polar slicing constraint (3.18). We also can enforce the constraint of being t $\rightarrow$ -t symmetric, so the evolution backward in time is identical to the evolution forward in time. The initial configuration of the field takes the form of a gaussian "pulse",

$$\phi(r,0) = \phi_0 e^{\left(-\left(\frac{r-r_0}{\Delta}\right)^2\right)} \tag{3.34}$$

3.3.1 Background Computation

For a static background case, we note that from equation 3.25, that r=2M is a characteristic, so we can declare boundary conditions at initial time and at r=2M and then propagate them to later times.

As r $\rightarrow \infty$ $\alpha, a \rightarrow 1$ and $\beta \rightarrow 0$, so our metric 2.22, turns into the Minkowski flat space metric,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{3.35}$$

and the Klein-Gordon equation for massless scalar field 3.4 becomes

$$\partial_{rr}(r\phi) = \partial_{tt}(r\phi) \tag{3.36}$$

In general our solution is of the form

$$(r\phi) \sim f(t-r) + g(t+r) \tag{3.37}$$

$f(t-r)$ is the outgoing wave and $g(t+r)$ is the ingoing wave.In order to get the outgoing or outgoing radiation boundary conditions we demand that,

$$\lim_{r\to\infty}(r\phi)(t,r) \to f(t-r) \tag{3.38}$$

So, we have only outgoing radiation. In terms of Φ and Π our boundary conditions at the outer edge of our grid become,

$$\partial_r\Phi + \partial_t\Phi + \frac{\Phi}{r} = 0 \tag{3.39}$$

$$\partial_r\Pi + \partial_t\Pi + \frac{\Pi}{r} = 0 \tag{3.40}$$

We must also be careful now to pick an r_{max} so we do actually get Minkowski space time.

3.3.2 Self-Gravitating Case

The boundary conditions for the self-gravitating case is more involved than for the static background case. It will be discussed in some detail here.

First we note that the Yang-Mills potential, W has precisely two vacuum states,

$$\mathrm{W}(r,t) = \pm 1 \tag{3.41}$$

So we demand that during the evolution the field stays in one of these configurations at the spatial end points. So we can write

$$W(0,t) = +1 = W_0 \tag{3.42}$$

that means,

$$\lim_{r\to\infty} W(r,t) = +1 \tag{3.43}$$

or

$$\lim_{r\to\infty} W(r,t) = -1 \tag{3.44}$$

From regularity conditions we have,

$$\lim_{r\to 0} a(r,t) = a_0(t) + r^2 a_2(t) + O(r^4) \quad (3.45)$$
$$\lim_{r\to 0} \alpha(r,t) = \alpha_0(t) + r^2 \alpha_2(t) + O(r^4) \quad (3.46)$$

this gives,

$$a(0,t) = 1 \quad (3.47)$$
$$a'(0,t) = 0 \quad (3.48)$$
$$\alpha(0,t) = 0 \quad (3.49)$$

From equation 3.42 we get that at the inner boundary, r=0

$$\Phi(0,t) = W_0' \quad (3.50)$$
$$\Pi(0,t) = \dot{W}_0 \quad (3.51)$$

and as r$\to\infty$ we again acquire Mankowski flat space-time 3.35 and our field equation becomes,

$$\mathrm{W}'' - \ddot{\mathrm{W}} \quad (3.52)$$

and our boundary conditions for Φ and Π become,

$$\partial_r\Phi + \partial_t\Phi = 0 \quad (3.53)$$
$$\partial_r\Pi + \partial_t\Pi = 0 \quad (3.54)$$

Also there are conditions for α to consider. At the outer boundary we have,

$$\lim_{r\to\infty} \alpha(r,t) = \frac{1}{a(r,t)} \quad (3.55)$$

The technique is to start with a(0,t) = 1 and then integrate the Hamiltonian constraint to some large r=R, the maximum radius of our spatial domain. Then we can get α via equation 3.55.

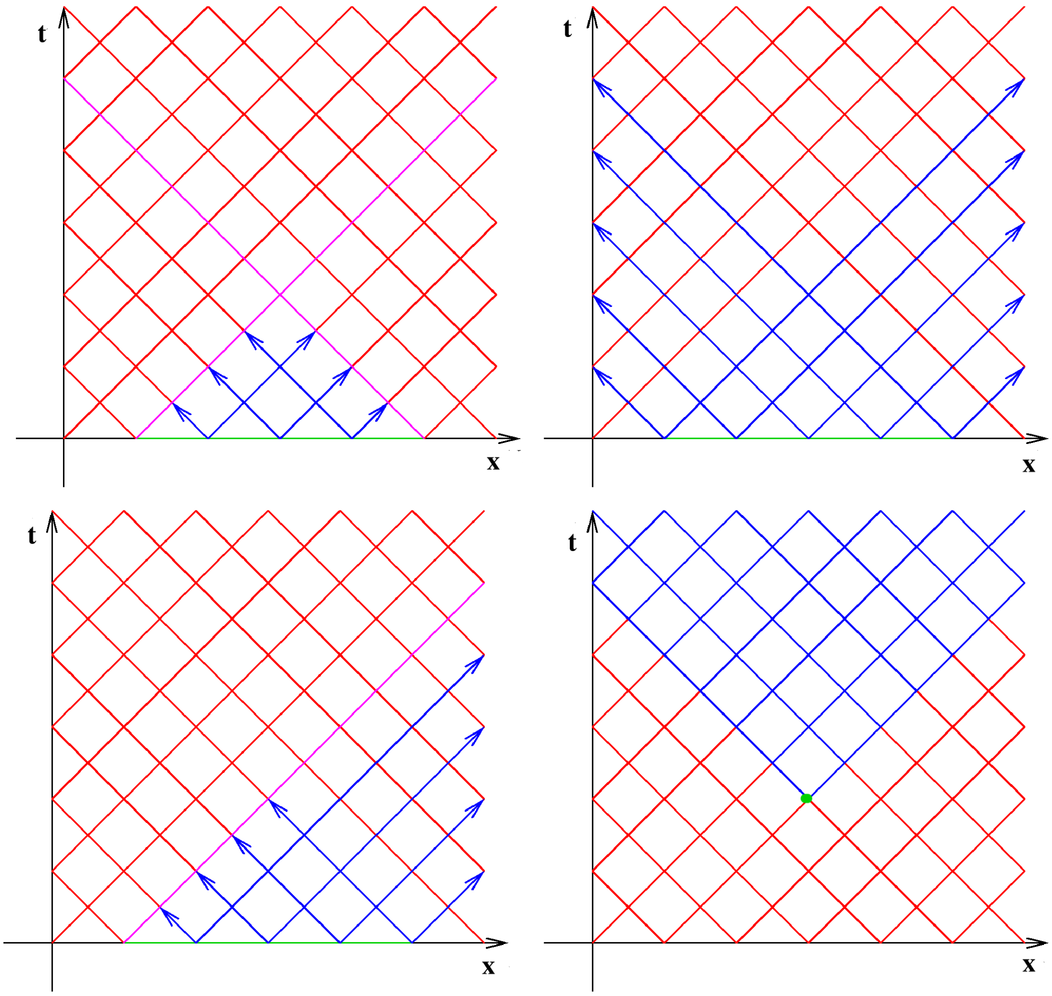

Figure 3.4: Shows zone of influence on appropriate boundary and initial conditions. a) Boundary Conditions for both ends and initial conditions for all space. b) Initial Conditions only. c) Boundary condition for one end and initial conditions for all space. d) How a single point directly or indirectly influences other parts domain. (green is used for boundary conditions, magenta for initial conditions and blue is the path of propagation)

Chapter 4

Numerical Analysis

Our general approach is to specify our space-time or equations that make up our space-time at some space-like slice ($t = t_1 =$ constant) with their time derivative and use the field equations to compute the space-time at some future time ($t = t_2 > t_1$). Also if we have well constructed code it can provide us with a sort of laboratory to do experimental (although theoretical) relativity. The requirement of this is that our computational method and computer is fast enough and that our code is general enough to satisfy different initial conditions. The computer speed requirement many times can not be helped, but generalizing code to the point where different initial conditions can be inputed and even changes can be made to suit other problems easily while not wasting memory and computation time is essential. As mentioned earlier working with spherically symmetric conditions drastically helps the latter.

4.1 Finite Difference Approach

Now we consider finite difference approximation of partial differential equations or PDE's. Divide our continuum grid into spatial subintervals, h or equivalently dx and k or equivalently dt. Later when we look at physi-

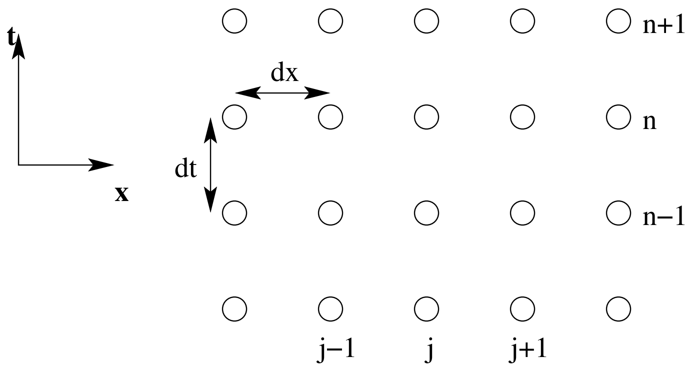

Figure 4.1: Finite Difference Mesh in 1+1 dimensions

cal problems dx becomes dr. In discretizing our continuum we get a finite difference mesh or grid (fig. 4.1).

We let,

$$Du = f \tag{4.1}$$

denote a differential system. Where D is a differential operator (ex. $\partial_{tt} - \partial_{xx}$), u is some unknown solution and f is some specified function . Then let

$$D^h u^h = f^h \tag{4.2}$$

denote a finite differenced system. Where D^h is a finite difference operator, u^h is a approximate solution and f^h is a function defined on our finite difference mesh. We then demand that as h approaches 0, equation 4.2 approaches equation 4.1.

The finite difference operators are listed in table (4.1).

Forward and Backward Difference Operators	$\Delta^x_\pm f^n_j \pm \frac{f^n_{j\pm1}-f^n_j}{\Delta x} = f'^n_{j\pm\frac{1}{2}} + \mathrm{O}(\Delta x^2)$ $\Delta^t_\pm f^n_j \pm \frac{(f^{n\pm1}_j - f^n_j)}{\Delta t} = f'^{n\pm\frac{1}{2}}_j + \mathrm{O}(\Delta t^2)$
Central Difference Difference	$\Delta^x_0 f^n_j = \frac{f^n_{j+1}-f^n_{j-1}}{2\Delta x} = f'^n_j + \mathrm{O}(\Delta x^2)$ $\Delta^t_0 f^n_j = \frac{f^{n+1}_j - f^{n-1}_j}{2\Delta t} = f'^n_j + \mathrm{O}(\Delta t^2)$ $\Delta^x_+\Delta^x_- f^n_j = \Delta^x_- f^n_j \Delta^x_+ = \frac{f^n_{j+1}-2f^n_j+f^n_{j-1}}{\Delta x^2} = f''^n_j + \mathrm{O}(\Delta x^2)$
Averaging Operators	$\mu^x_\pm = \frac{f^n_j + f^n_{j\pm1}}{2} = f^n_{j\pm\frac{1}{2}} + \mathrm{O}(\Delta x^2)$ $\mu^t_\pm = \frac{f^n_j + f^{n\pm1}_j}{2} = f^{n\pm\frac{1}{2}}_j + \mathrm{O}(\Delta t^2)$

Table 4.1: Differential Operators

4.2 Error Analysis

The solution error is defined by e^h=u-u^h. So our finite difference schemes gives us the following approximations and demands.

$$u \longrightarrow u^h + h^2 e^h_2 \tag{4.3}$$

$$D \longrightarrow D^h + h^2 e^h_2 \tag{4.4}$$

$$Du \longrightarrow D^h u^h + O(h^2) \tag{4.5}$$

Specifically if we know the solution to a problem, $\hat{u}$ and in addition have the discrete solution on finite difference mesh of level h and $2h$. Then we can define a convergence factor, c_h, as

$$c_h = \frac{\|u^{2h} - \hat{u}\|}{\|u^h - \hat{u}\|} \tag{4.6}$$

then if the expected solution has error of the form $O(h)$, as $h \to 0$, $c_h \to 2$. Simularily if expected solution has error of form $O(h^2)$ then as $h \to 0$, $c_h \to 4$.

But what if we do not know the solution on the finite difference mesh. Then we can define the same convergence factor as

$$c_h = \frac{\|u^{4h} - u^{2h}\|}{\|u^{2h} - u^h\|} \tag{4.7}$$

So as before, if the expected error of our solution is of the form $\mathrm{O}(h)$, as $h \to 0$, $\mathrm{c}_h \to 2$ and if the expected error of our solution is of the form $\mathrm{O}(h^2)$, as $h \to 0$, $\mathrm{c}_h \to 4$.

In order for our solution to be correct the preceding convergence tests are essential. However, the best these convergence tests can do is tell us our difference schemes are working correctly. They can not tell us that we have obtained the correct solution. The solution will always be a product of our equations of motion, our boundary and initial conditions and our constraint equations. So if these are not correct our solution will not be correct, even though the convergence tests are.

4.3 Stability

There is also a stability to consider which is the growth of errors. Truncation analysis gives us a magnitude of discretization errors which depend on step sizes h and k, but we can add to this, because the behavior of discretization errors exhibit great regularity, which we can quantify by notions of numerical dissipation and dispersion. So even though we can estimate the magnitude of the discretization and rounding error (due to machine precision), it is still advantageous to look at these in more detail. One reason is that someone more familiar with these types of errors is less likely to mistake spurious features of a numerical solution for something physical. Another important reason is to help us design schemes with special properties like low dispersion or dissipation.

Although not particularly for the results we obtained, the fact that finite difference operators have a non-trivial dispersion relation is fascinating and we summarize some of the results we found in appendix A.

The PDE may conserve energy in the norm but its finite difference approximation may lose energy as t increases especially if the grid size is of comparable length as the wavelength. This is called numerical dissipation and it tends to help with unwanted oscillations and instability. It is so advantageous at times that sometimes artificial dissipation is added to no-dissipative equations. For example, we can discretize the advection equation 4.8

$$u_t = a u_x \ (a > 0) \tag{4.8}$$

as,

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \tag{4.9}$$

So, if we have a solution at u_j^n then we can solve for advanced times by equation 4.10.

$$u_j^{n+1} = u_j^{n-1} + a\lambda(u_{j+1}^n - u_{j-1}^n) \tag{4.10}$$

Where λ is the courant factor, which is the ratio of the temporal step size to the spatial step size. We can then add the dissipation operator to equation 4.10 to get equation 4.11.

$$u_j^{n+1} = u_j^{n-1} + a\lambda(u_{j+1}^n - u_{j-1}^n) - \frac{\epsilon}{16}(u_{j+2}^{n-1} - 4u_{j+1}^{n-1} + 6_j^{n-1} - 4u_{j-1}^{n-1} + u_{j-2}^{n-1}) \tag{4.11}$$

This is a special type of dissipation which has been used very effectively in Numerical Relativity.

This is all for now about stability except for just stating that by utilizing dispersion analysis and dissipation techniques gives us even better approximation to our solution.

4.4 Specific finite differences

In addition to the finite difference operators 4.1 we introduce the additional spatial difference operator,

$$D_{-}^{x} u_{j}^{n} = \frac{3u_{j}^{n} - 4u_{j-1}^{n} + u_{j-2}^{n}}{2\Delta x} \tag{4.12}$$

Equation 4.12 is used only at the boundary points of our spatial grid.

Chapter 5

Results

5.1 Scalar "scattering" on BH background in IEF Coordinates

To refresh your memory as to what this model is about. We are talking about a scattering problem. We have a static background with a black hole of arbitrary mass. The amplitude is also arbitrary in this case since our wave equation takes the form of the massless Klein-Gordon equation (3.4). So if ϕ is a solution to our equation then so is ϕ+constant. We start the gaussian pulse at the middle of our spatial mesh. The parameter that is of interest to us in this case is δ. Our initial conditions for Φ and Π are,

$$\Phi_0(r) = \partial_r \phi_0(r) \tag{5.1}$$

$$\Pi_0(r) = \frac{a}{\alpha}\left(\frac{\phi_0(r)}{r}\right) + \partial_r \phi_0(r) \tag{5.2}$$

Convergence tests for ϕ and total conserved mass is shown in figures 5.1 and 5.1.

Using these initial conditions and boundary conditions given by equations 3.39, 3.40 we obtain a solution. As mentioned before δ is a parameter of great interest to this particular problem and we shall see, can also function as a solution check.

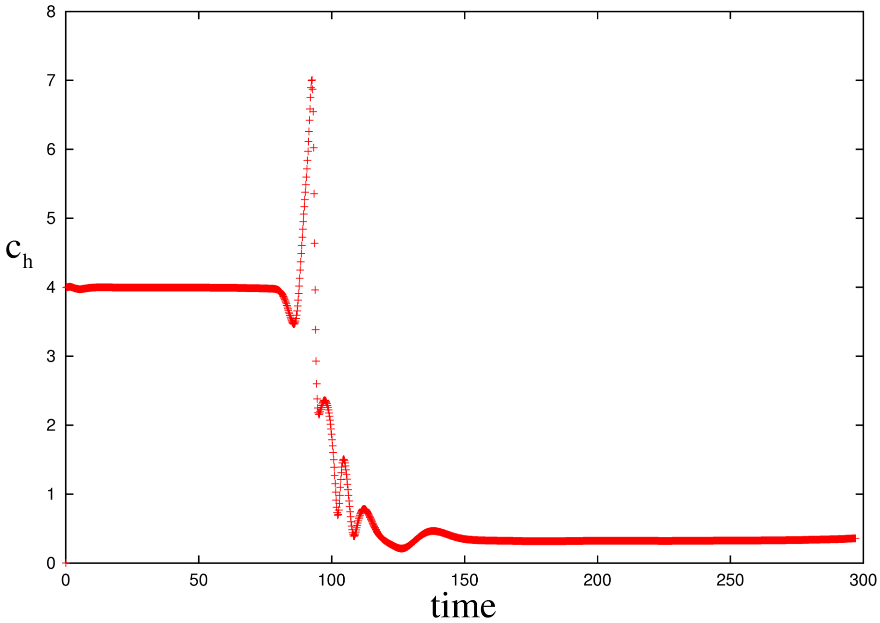

Figure 5.1: Convergence test for scalar field ϕ

We can find out how much of the scalar field actually falls into the black hole and how much scatters from it. To that end we can derive a conserved mass function, m(r,t), which is conserved in the limit r $\to \infty$. The quantity is derived in detail in appendix B. The conserved mass is given by,

$$\frac{dm}{dr} = [\frac{\Pi^2 + \Phi^2}{a^2} + \frac{2\beta\Phi\Pi}{\alpha a}]2\pi r^2 \alpha a \tag{5.3}$$

This conserved mass serves as a solution check because we know what it should look like. If we look at the massed contained inside a certain radius r=R. Done by integrating equation 5.3 and looking at the evolution of this over time and at a radius r=R, we should get a constant value at first and then as the field hits the black hole there should be a drastic “swallowing of mass” and then a second plateau as some of the mass scatters off to infinity. Incidently there should be a third plateau signifying the scalar field

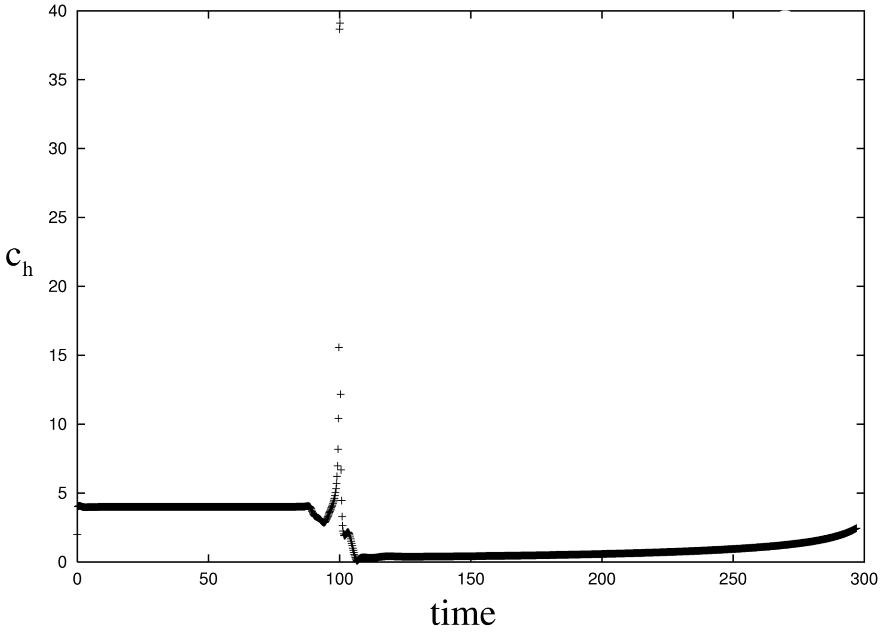

Figure 5.2: Convergence test for total conserved mass

actually leaving our "zone of control", but we can not do anything about this. Theoretically, if we had an infinite zone of control this third plateau would not exist. We show these plateau's in figure 5.1 for several δ. We see from figure 5.1 that the height of these plateau's are not the same for each delta. To examine this closer we plot the fraction of mass scattered versus δ in figure 5.1.

This shows us that a wider pulse has trouble fitting into the radius of the black hole and is thus only partly absorbed.

5.2 Yang-Mills Field

The Yang-Mills Field is more difficult to describe numerically than the scattering case discussed in section 5.1. The field is self gravitating, as a result of this there is back-scattering, which basically means that the field tends to scatter itself. We do have two parameters to consider in this case. The amplitude and the width of the pulse or in the case of input data resembling a kink type function (5.11) we have to consider the steepness of the kink. We also look at the existence of static solutions when the self-gravitation of the field and the repulsive self-interaction of the field are equal. This was numerically observed first by Bartnik and McKinnon [5] and are the so-called the Bartnik and McKinnon solutions of type n=1.

Our initial and boundary conditions for the Yang-Mills Field are detailed in section 3.3.2, for Φ and Π we have,

$$\Phi_0(r) = \partial_r \mathrm{W}_0(r) \tag{5.4}$$

$$\Pi_0(r) = \partial_r \mathrm{W}_0(r) \tag{5.5}$$

and the boundary conditions are,

$$\Phi(0,t) = 0 \tag{5.6}$$

$$\Pi(0,t) = 0 \tag{5.7}$$

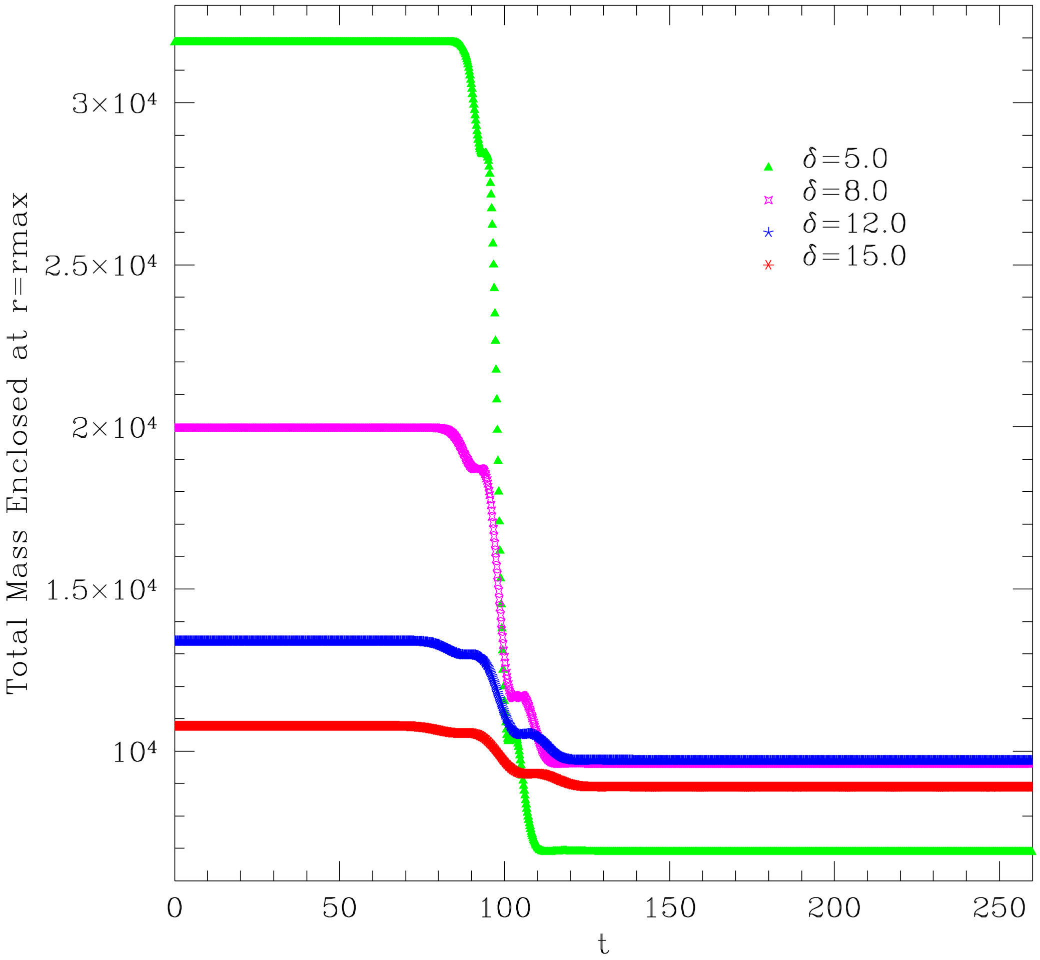

Figure 5.3: Plot of amount of mass contained at r_max of our spatial grid. The plot is for δ equal to 5.0, 8.0, 12.0 and 15.0

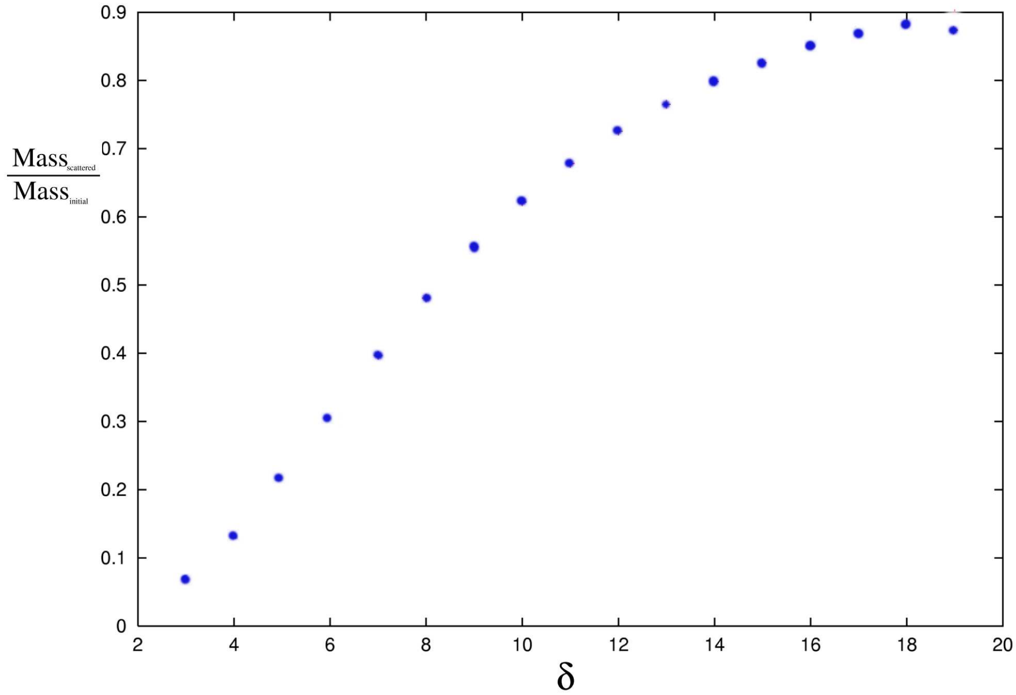

Figure 5.4: Plot of the pulse width versus fraction of scattered field for massless scalar field

We can also compute a black hole function Z(r,t), defined by,

$$\mathrm{Z}(r,t) = \frac{2m(r,t)}{r} \tag{5.8}$$

where the mass aspect function, m(r,t) is defined as,

$$m(r,t) = \frac{1}{2} r(1 - a^{-2}) \tag{5.9}$$

The black hole function signals a black hole formation when it becomes 1 at some radius r_{BH}. r_{BH} is the radius of the black hole horizon.

The initial W_0 we will be using is as mentioned before, a gaussian pulse and a kink.

$$W_0(r; r_0, \delta) = 1 + W_0 \exp^{\left(-\left(\frac{r-r_0}{\delta}\right)^2\right)} \tag{5.10}$$

$$W_0(r; r_0, \delta) = \frac{1 + \frac{r_0^2 - r^2}{\delta^2}}{\sqrt{(1 + \frac{r_0^2 - r^2}{\delta^2})^2 + 4r^2}} \tag{5.11}$$

There are also critical parameters of interest, namely δ in the case of the kink and δ and W_0 in the case of the gaussian (see table 5.1). Movies of the evolutions of the gaussian and the "kink" can be found at [6]. An evolution survey for the gaussian is shown in figure 5.2 and an evolution survery for the "kink" can be found in figure 5.2. Convergence tests for the field itself and Φ are shown in figures 5.2 and 5.2.

The basic principle as to what causes the black hole to form is as follows. For example in the case of the gaussian when we increase the amplitude and/or width of the pulse we effectively are constructing a pulse with more mass-energy and so it tends to gravitate more profusely and as we pass the critical parameter a black hole forms.

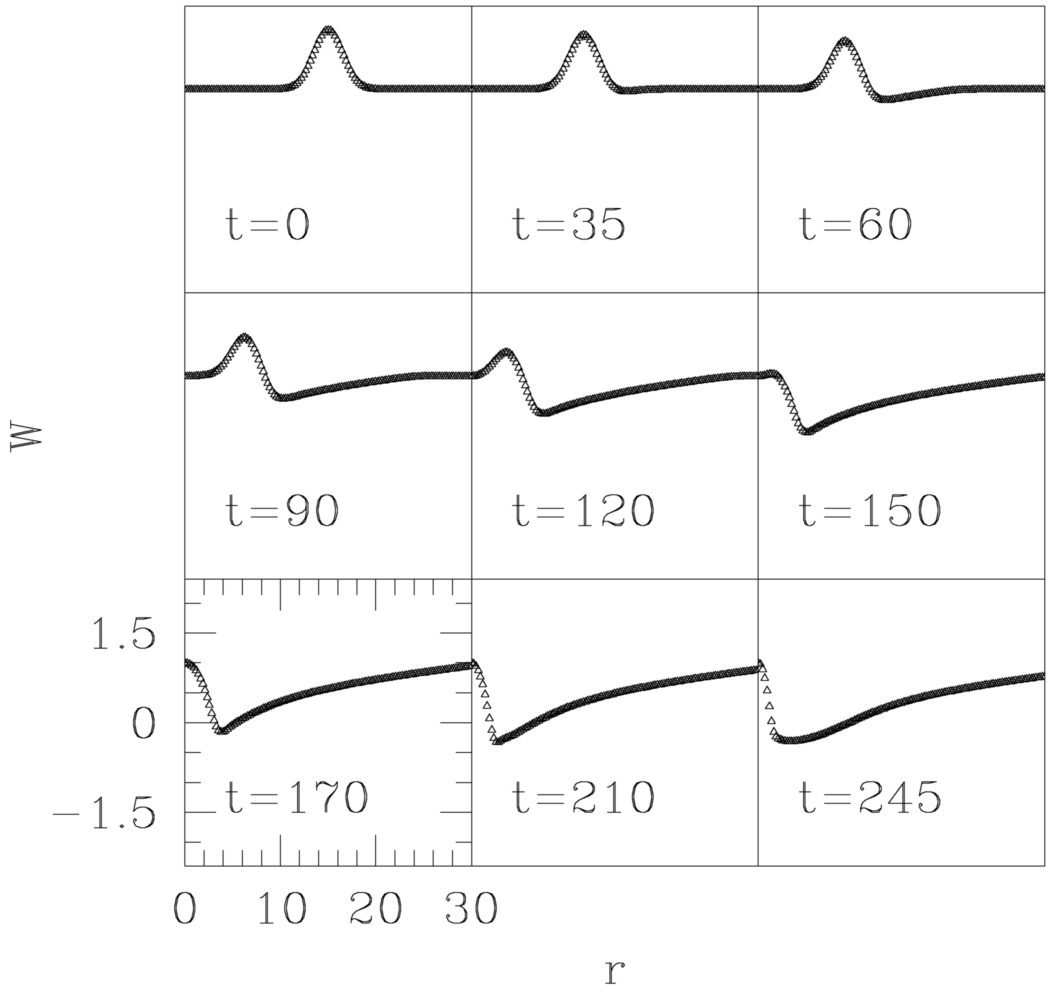

Figure 5.5: gaussian evolution in the Yang-Mills Field

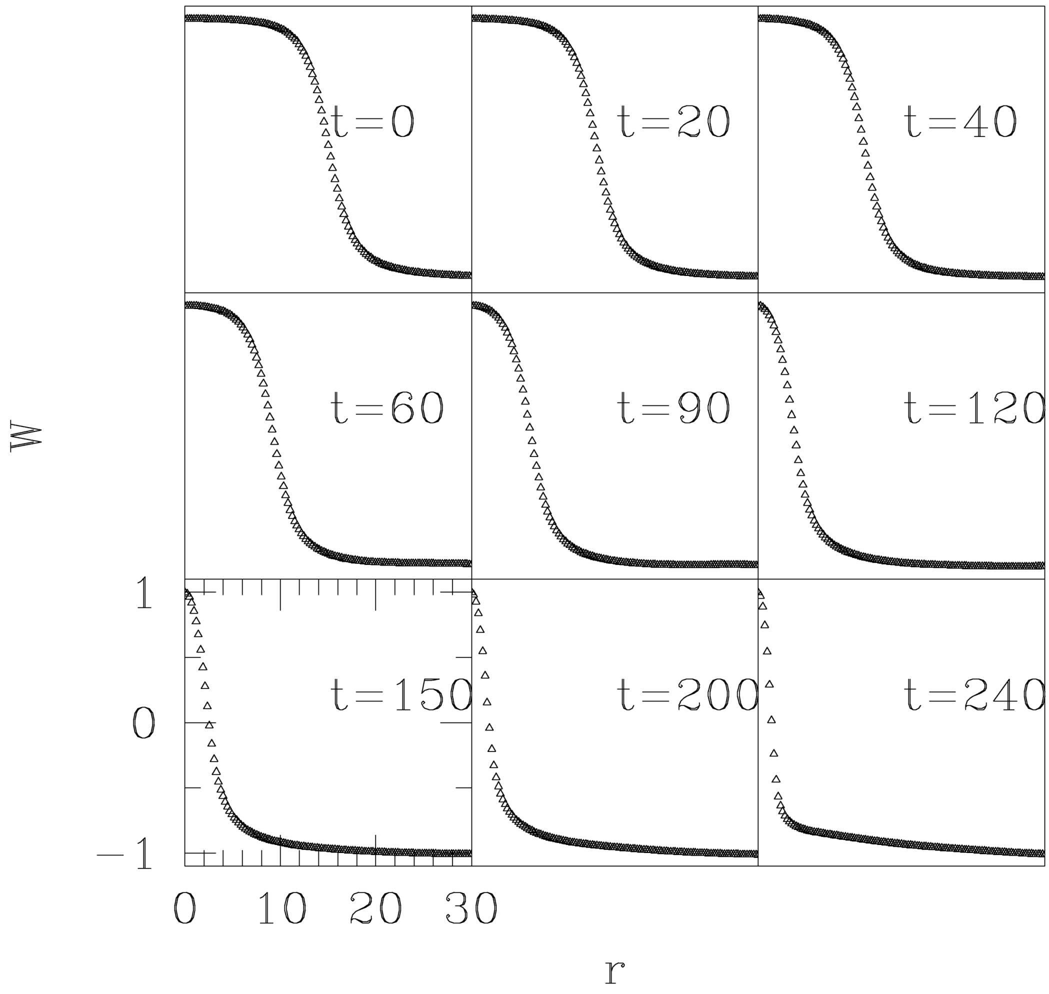

Figure 5.6: gaussian evolution in the Yang-Mills Field

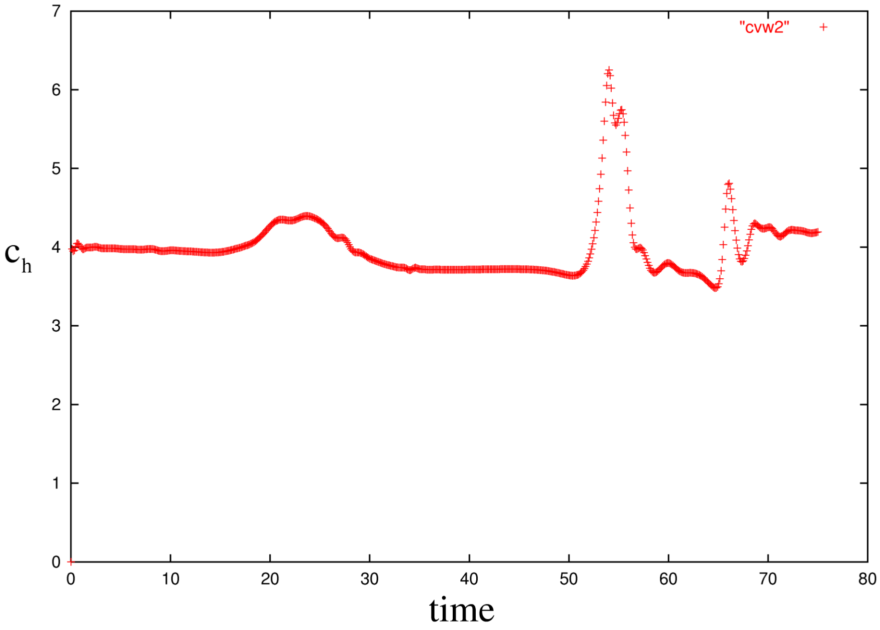

Figure 5.7: Convergence test for the Yang-Mills Potential, W

Initial data	critical δ	critical amplitude (only for gaussian)
Gaussian	$1.4616189 \leq \delta \leq 1.461619$	$0.96787529 \leq W_0 \leq 0.967875385$
Kink	$1.64268059 \leq \delta \leq 1.6426806$	no amplitude

Table 5.1: The critical parameters for the gaussian 5.10 and kink 5.11 initial data. (Note that for the gaussian, when attempting to find one critical parameter the other is fixed.)

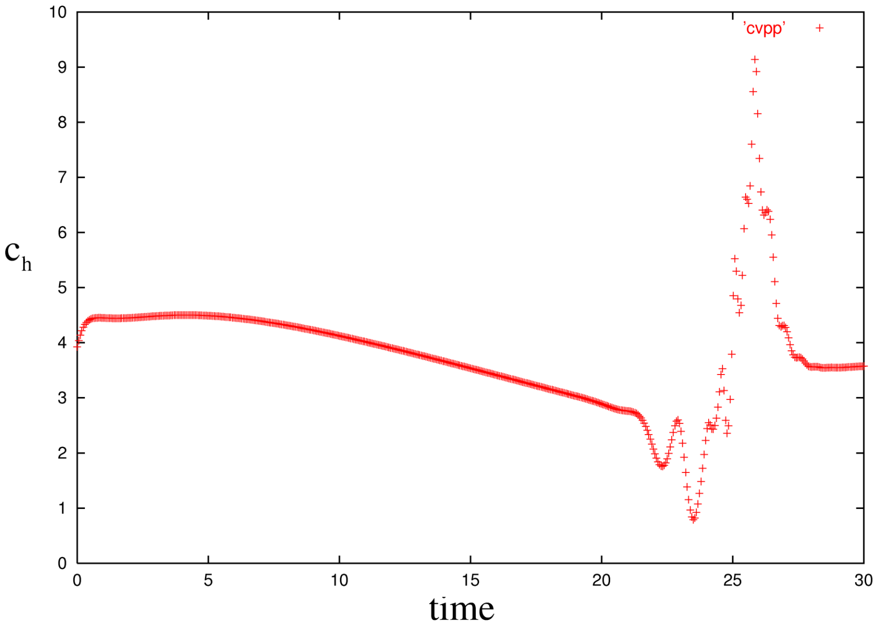

Figure 5.8: Convergence test for Φ

Chapter 6

Conclusion

In conclusion we state that numerical relativity is by no means a way of explaining the un-explainable or a last resort, rather it is a way of understanding more clearly a problem that can only be explained vaguely. Numerical relativity gives us the only way to solve the most complex field equations.

We find that in the case of a static black-hole background there is scattering associated with the width of the initial pulse and in the non-extreme case there is a linear relationship with the amount of absorption and width of the pulse. We also find that the amplitude is completely arbitrary and has no effect on the solution.

We find for the self-gravitating, Yang-Mills field that there is additional constraints we have to consider. Our initial data cannot be arbitrarily chosen but has to satisfy constraints 3.20, 3.18. For the case of a gaussian pulse, by setting the width and the amplitude of any initial pulse we effectively set the amount of mass-energy associated with that pulse and by doing so we can see at what critical values of these parameters we obtain black hole formation. In the case of the kink function (5.11) we find that there is a Bartnik-McKinnon solution associated with this. That is, the solution tends to stay static for a length of time determined by parameter δ and we can measure this systematically. The critical parameters are shown in table 5.1.

Further work in the field of numerical relativity will be concerned with solutions of the n-dimensional scalar field collapse and also one of oscillating field in spherically symmetric Klein-Gordon Model.

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Appendix A

Dispersion

To give a better understanding of dispersion let us consider the following examples. First lets consider the advection equation given by equation A.1,

$$u_t = au_x \ (a > 0) \tag{A.1}$$

which admits plain wave solutions of the form,

$$u(x,t) = e^{\xi x + \omega t} \tag{A.2}$$

where ξ is the wave number and ω is the frequency. For each value of ξ not all values of ω can be taken in equation A.2. Instead the PDE imposes a relation between ξ and ω

$$\omega = \omega(\xi) \tag{A.3}$$

which is known as the **dispersion relation**. In general each wave number ξ corresponds to m frequencies ω. Where m is the order of differential equation with respect to t. Dispersion relations for a few model equations are given in table (A.1).

Discrete approximations also admit plane wave solutions A.2, if the grid is uniform. That means that they too have dispersion relations. From A.1

Equations	Continuum Dispersion Relation	Discretized Dispersion Relation
$u_t = u_x$	$\omega = \xi$	$\omega = \frac{1}{h}\sin \xi h$
$u_{tt} = u_{xx}$	$\omega = \pm\xi$	$\omega = \pm\frac{2}{h}\sin\frac{\xi h}{2}$
$u_t = u_{xx}$	$\omega = \imath\xi^2$	$\omega = \imath\frac{4}{h^2}\sin^2\frac{\xi h}{2}$
$u_t = \imath u_{xx}$	$\omega = -\xi^2$	$\omega = -\frac{4}{h^2}\sin^2\frac{\xi h}{2}$

Table A.1: Dispersion Relation for continuum and discrete PDE's

we can see that the discrete dispersion relations are only good for small ξ, which corresponds to many grid points per wavelength.

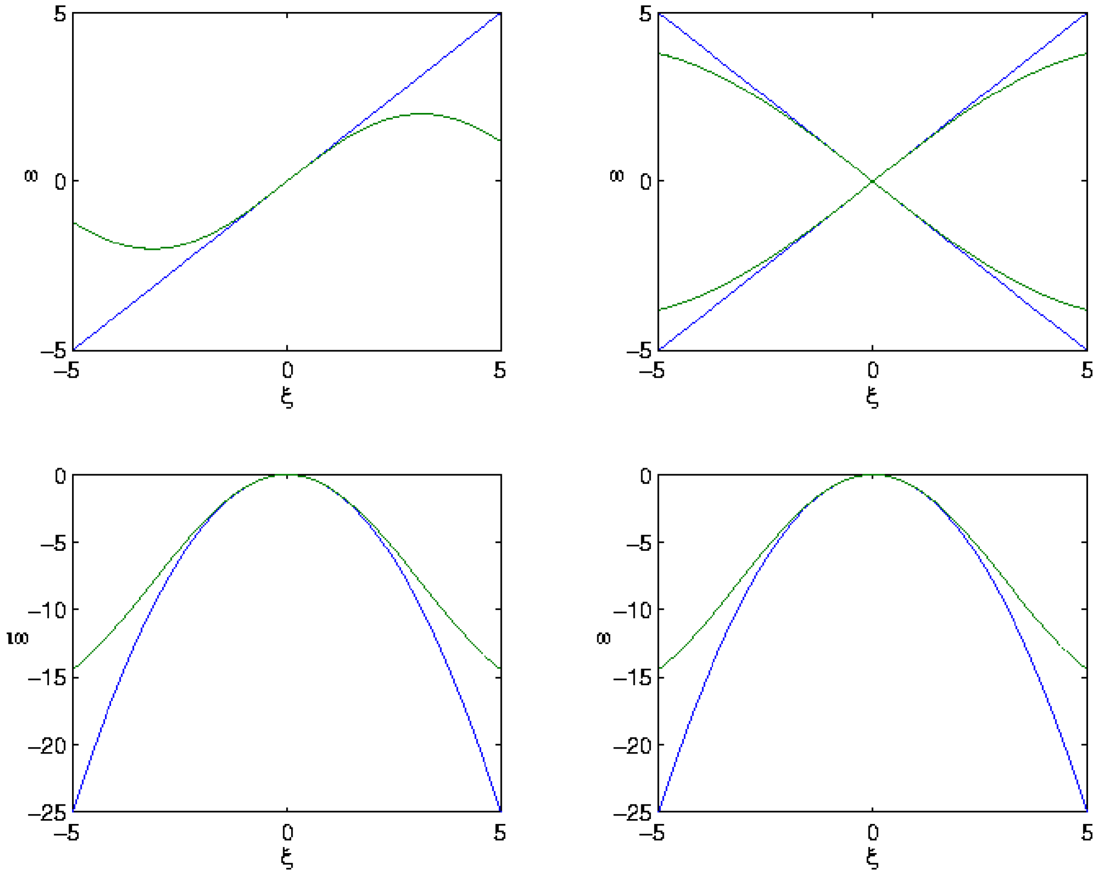

Figure A.1: Graphs of the dispersion relations A.1. Blue: Continuum Dispersion. Green: Discretized Dispersion.

Appendix B

Conservation of Mass

To get the conserved mass we need the killing vector given by,

$$\xi^\mu = (1,0,0,0)$$

so then $p^\mu = T^{\mu\nu}\xi_\mu$ is a conserved 4-vector if

$$p^\mu_{;\mu} = 0 \qquad \longrightarrow \qquad \frac{1}{\sqrt{-g}}(\sqrt{-g}p^\mu)_{,\mu} = 0$$

so we integrate over all 3-space, and from t=t$_1$ to t=t$_2$

$$0 = \int_V (p^\mu_{;\mu})dV = \int_{t_1}^{t_2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{1}{\sqrt{-g}}(\sqrt{-g}p^\mu)_{,\mu}\sqrt{-g}drd\theta d\phi dt$$

Our metric $g_{\mu\nu}$ is given by the following,

$$\mathbf{g}_{\mu\nu} = \begin{pmatrix} -\alpha^2 + a^2\beta^2 & a^2\beta & 0 & 0 \\ a^2\beta & a^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and the inverse metric $g^{\mu\nu}$ we get,

$$\mathbf{g}_{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta}{\alpha^2} & 0 & 0 \\ \frac{\beta}{\alpha^2} & \frac{\alpha^2 - a^2\beta^2}{a^2\alpha^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

and $\sqrt{-g} = r^2 \sin\theta\alpha a$. So if we integrate over θ and ϕ to get,

$$0 = 4\pi \int_{t_1}^{t_2} \int_0^{\infty} (\alpha a r^2 p^{\mu})_{,\mu} dt dr$$

the scalar field is not changing in θ and ϕ so they contribute nothing to the integral. We sum over t and r,

$$\begin{aligned} 0 &= 4\pi \int_{t_1}^{t_2} \int_0^{\infty} (\alpha a r^2 p^t)_{,t} dt dr + 4\pi \int_{t_1}^{t_2} \int_0^{\infty} (\alpha a r^2 p^r)_{,r} dr dt && \text{(B.1)} \\ &= 4\pi \int_{t_1}^{t_2} \int_0^{\infty} (\alpha a r^2 p^t)_{,t} dt dr + 4\pi \int_{t_1}^{t_2} (\alpha a r^2 p^r|_0^{\infty}) dt && \text{(B.2)} \end{aligned}$$

the second term has to be zero so we get,

$$\int_{t_1}^{t_2} [\int_0^{\infty} (4\pi r^2 \alpha a p^t) dr]_{,t} dt = 0$$

t_1 and t_2 are arbitrary so we get $\dot{Q}$(t)=0 where

$$Q(t) = \int_0^{\infty} (4\pi r^2 \alpha a p^t) dr$$

so we need p^t and we obtain it in the following way,

$$\begin{aligned} p^{\mu} &= T^{\mu\nu}\xi_{\nu} && \text{(B.3)} \\ &= T^{\mu}_{\nu}\xi^{\nu} && \text{(B.4)} \\ &= T^{\mu}_{0} && \text{(B.5)} \end{aligned}$$

so,

$$\begin{aligned} p^0 &= T^{\mu}_{0} && \text{(B.6)} \\ &= g^{0\nu}T_{\nu 0} && \text{(B.7)} \\ &= g^{00}T_{00} + g^{01}T_{10} && \text{(B.8)} \end{aligned}$$

and

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}$$

it is convenient to work out the following equations

$$\phi_{,0} = \beta\Phi + \frac{\alpha}{a}\Pi \tag{B.9}$$

$$\phi_{,1} = \Phi \tag{B.10}$$

$$\phi^{,\alpha}\phi_{,\alpha} = \frac{\Theta^2 - \Pi^2}{a^2} \tag{B.11}$$

$$\phi_{,0}\phi_{,0} = \beta^2\Phi^2 + \frac{\alpha^2}{a^2}\Pi^2 + \frac{2\beta\alpha\Phi\Pi}{a} \tag{B.12}$$

$$\phi_{,0}\phi_{,1} = \beta\Phi^2 + \frac{\alpha\Phi\Pi}{a} \tag{B.13}$$

$$\phi_{,1}\phi_{,0} = \phi_{,0}\phi_{,1} \tag{B.14}$$

$$\phi_{,1}\phi_{,1} = \Phi^2 \tag{B.15}$$

and obtain for T_{00} and T_{10},

$$T_{00} = \phi_{,0}\phi_{,0} - \frac{1}{2}g_{00}\phi_{,\alpha}\phi^{,\alpha} \tag{B.16}$$

$$= \frac{1}{2}[\beta^2(\Phi^2 + \Pi^2) + \frac{\alpha^2}{a^2}(\Phi^2 + \Pi^2) + \frac{4\beta\alpha\Pi\Phi}{a}] \tag{B.17}$$

and

$$T_{01} = \phi_{,0}\phi_{,0} - \frac{1}{2}g_{00}\phi_{,\alpha}\phi^{,\alpha} \tag{B.18}$$

$$= \frac{1}{2}[\beta(\Phi^2 + \Pi^2) + \frac{2\alpha\Phi\Pi}{a}] \tag{B.19}$$

Now, we can obtain an expression for p^0

$$p^0 = g^{00}T_{00} + g^{01}T_{10} \tag{B.20}$$

$$= -\frac{1}{2}[\frac{\Pi^2 + \Phi^2}{a^2} + \frac{2\beta\Phi\Pi}{\alpha a}] \tag{B.21}$$

so finally,

$$\int \sqrt{-g}p^0 d\vec{r} = \int \frac{1}{2}[\frac{\Pi^2 + \Phi^2}{a^2} + \frac{2\beta\Phi\Pi}{\alpha a}]r^2 \sin\theta a\alpha d\theta d\phi dr \tag{B.22}$$

$$= \int [\frac{\Pi^2 + \Phi^2}{a^2} + \frac{2\beta\Phi\Pi}{\alpha a}]2\pi r^2 \alpha a dr \tag{B.23}$$

$$\frac{dm}{dr} = [\frac{\Pi^2 + \Phi^2}{a^2} + \frac{2\beta\Phi\Pi}{\alpha a}]2\pi r^2 \alpha a$$

Appendix C

Lagrangian Formalism

From analytical mechanics we know that we can obtain our equations of motion from taking the variation of the action C.1, $\partial\mathcal{S} = 0$. From this we get the Euler-Lagrange equation C.2.

$$\mathcal{S} = \int \mathrm{L}dt \tag{C.1}$$

$$\frac{d}{dt}(\frac{\partial \mathrm{L}}{\partial \dot{q}_i}) - \frac{\partial \mathrm{L}}{\partial q_i} = 0 \tag{C.2}$$

Where q_i represents the position of a particle, say.

In General Relativity we always seek covariant equations where position and time are given equal status. Equation C.2 is clearly not covariant because of the special emphasis put on time via the q_i and $\frac{d}{dt}(\frac{\partial \mathrm{L}}{\partial \dot{q}_i})$.

So we replace q_i with $\phi = \phi(x^\mu)$. We also modify the equation for action C.1 so it too is covariant.

$$\mathrm{S} = \int \mathcal{L}d^4x \tag{C.3}$$

Then $\mathrm{L} = \int \mathcal{L}d^3x$. $-\frac{\partial\mathcal{L}}{\partial \Pi_i}$ gets replaced by the covariant term $-\frac{\partial\mathcal{L}}{\partial\phi(\S^\mu)}$. And of course any time derivative is replaced by $\partial_\mu = \frac{\partial}{\partial \mathrm{x}^\mu}$. So then the covariant generalization is,

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{C.4}$$

If the Lagrangian density does not depend explicitly on the coordinates, then its space-time variation is given by the change of the field only.

$$\frac{\partial\mathcal{L}}{\partial \mathrm{x}^\mu} = \frac{\partial\mathcal{L}}{\partial\phi}\phi_{,\mu} + \frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\phi_{,\nu\mu} = \frac{\partial}{\partial \mathrm{x}^\nu}(\frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\phi_{,\mu}) \tag{C.5}$$

where

$$,\mu = \partial_\mu \tag{C.6}$$

Now we can define the the momentum density as,

$$\Pi^\mu \equiv \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} \tag{C.7}$$

so the Euler-Lagrange equations C.4 become,

$$\Pi^\mu,\mu = \frac{\partial\mathcal{L}}{\partial\phi} \tag{C.8}$$

Then the canonical momentum is defined as

$$\Pi \equiv \Pi^t = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{C.9}$$

where the dot represents "the temporal derivative of" and later on a prime denotes "the spatial derivative of". The energy momentum tensor is

$$\mathcal{T}_{\mu\nu} = \Pi_\mu\phi_{,\nu} - \mathrm{g}_{\mu\nu}\mathcal{L} \tag{C.10}$$

Now we look at the massive Klein-Gordon field defined with the Lagrangian density. The covariant momentum density can more easily be evaluated by writing the Lagrangian as

$$\mathcal{L}_{KG} = \frac{1}{2}\mathrm{g}^{\mu\nu}(\phi,\mu\phi,\nu - m^2\phi^2) \tag{C.11}$$

So we can write the C.7 as,

$$\Pi^{\mu} = \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \tag{C.12}$$

$$= \frac{1}{2} g^{\mu\nu} (\delta^{\alpha}_{\mu} \phi_{,\nu} + \phi_{,\mu} \delta^{\alpha}_{\nu}) \tag{C.13}$$

$$= \frac{1}{2} (\phi^{,\mu} + \phi^{,\mu}) \tag{C.14}$$

$$= \phi^{,\mu} \tag{C.15}$$

So for the Klein-Gordon field we have

$$\Pi^{\mu} = \phi^{,\mu} \tag{C.16}$$

The canonical momentum is the temporal component of Π^{α}.

$$\Pi = \Pi^{0} = \phi^{,0} = \phi_{,0} = \dot{\phi} \tag{C.17}$$

Then by evaluating $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$ the Euler-Lagrange equations give the equation of motion.