Black Hole Critical Phenomena—Results

1 Initial Data Profiles

The basic idea is to choose a family of initial data parametrized by a single variable p and let the system evolve. Observing the dynamics, we note whether or not the outcome is formation of a black hole at some future time. We then successively adjust the value of p to the critical value p^* which marks the threshold of black hole formation: if $p > p^*$ the system forms a black hole, if $p < p^*$ it does not.

Now, since the scalar field is complex

$$\Phi = \phi_1 + i\phi_2 \ , \tag{1}$$

its conjugate momentum will also be complex

$$\Pi_\Phi = \Pi_{\phi_1} + i\Pi_{\phi_2} \ . \tag{2}$$

In terms of the scalar field and conjugate momenta, electromagnetic coupling parameter e, and radial metric function a, the electric charge density is explicitly

$$\rho_Q = -\frac{2e}{a}\left(\phi_1\Pi_{\phi_2} - \phi_2\Pi_{\phi_1}\right) \ . \tag{3}$$

Families of initial data can therefore possess a variety of charge density profiles, depending on the choice of ϕ_1, ϕ_2, Π_{ϕ_1}, Π_{ϕ_2} and e.

For a given value of e, we choose one of four different families for initial data and tune the free parameter to its critical value. The four families are tabulated below, and the tuning parameter is either a_1, a_2, δ_1, or δ_2.

family	ϕ_1, ϕ_2, Π_{ϕ_1}, Π_{ϕ_2}	$\rho_Q(e \neq 0)$
A	$\phi_1(r,t=0) = a_1 \exp(-((r-r_1)/\delta_1)^2)$, $\phi_2(r,t=0) = a_2 \exp(-((r-r_2)/\delta_2)^2)$, $\Pi_{\phi_1}(r,t=0) = a_3 \exp(-((r-r_3)/\delta_3)^2)$, $\Pi_{\phi_2}(r,t=0) = a_4 \exp(-((r-r_4)/\delta_4)^2)$	$\neq 0$
B	$\phi_1(r,t=0) = a_1 \exp(-((r-r_1)/\delta_1)^2)$, $\phi_2(r,t=0) = 0$, $\Pi_{\phi_1}(r,t=0) = 0$, $\Pi_{\phi_2}(r,t=0) = a_4 \exp(-((r-r_4)/\delta_4)^2)$	$= 0$
C	$\phi_1(r,0) = a_1\,(\tanh(r-r_1) - \tanh(r-\delta_1 r_1))$, $\phi_2(r,0) = a_2\,(\tanh(r-r_2) - \tanh(r-\delta_2 r_2))$, $\Pi_{\phi_1}(r,0) = a_3\,(\tanh(r-r_3) - \tanh(r-\delta_3 r_3))$, $\Pi_{\phi_2}(r,0) = a_4\,(\tanh(r-r_4) - \tanh(r-\delta_4 r_4))$,	$\neq 0$
D	$\phi_1(r,0) = a_1\,(\tanh(r-r_1) - \tanh(r-\delta_1 r_1))$, $\phi_2(r,0) = 0$, $\Pi_{\phi_1}(r,0) = 0$, $\Pi_{\phi_2}(r,0) = a_4\,(\tanh(r-r_4) - \tanh(r-\delta_4 r_4))$,	$= 0$

2 Results for $e = 0$, $m_\Phi = 0$

As expected, we recover Choptuik's original discretely self-similar and universal solution [1] in the limit $e \to 0$ and $m_\Phi \to 0$.

3 Results for $e \neq 0$, $m_\Phi = 0$

For relatively small values of $e \neq 0$ with $m_\Phi = 0$ we again observe a discretely self-similar and universal solution, but the solution now carries electric charge.

3.1 Discrete Self-Similarity and Universality

The solution represents itself as an infinite series of echoes as the scalar field repeatedly attempts to collapse to a black hole, but never quite makes it. With each failed attempt some scalar field is shed, and the succeeding echo occurs on an exponentially smaller spatial scale after an exponentially shorter interval of time.

The discrete self-similarity of the critical solution is apparent in successive maxima of the quantity

$$2m/r \equiv (1 - a^{-2}) + {Q_r}^2/(4\pi r^2) \ , \tag{4}$$

where a is the radial metric function and Q_r is the net charge enclosed by a sphere of radius r. For example, we take the profile of $2m/r$ at the maximum of its third echo for family A and spatially rescale it by a factor $\exp(\Delta)$, $\Delta \approx 3.473$. We then find the rescaled profile matches that of the preceding (second) echo. This result is displayed in the figure below.

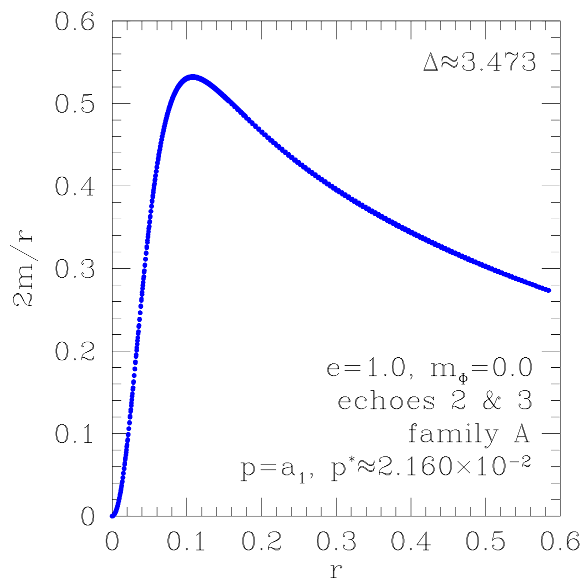

Figure 1: Discrete self-similarity for family A as demonstrated by spatially rescaling the third echo of $2m/r$ by $\exp(\Delta)$ and comparing it to the unscaled second echo.

The universality of the solution is manifest in the facts that all critical solutions exhibit like profiles and that the scaling exponents Δ are essentially the same for all families.

3.2 Mass and Charge Scaling

As the critical solution is approached from above, the black holes we form are arbitrarily small. Their masses observe a power-law behavior

$$\ln(m_{BH}) = \gamma_m \ln\left((p - p^*)/p^*\right) + c1, \tag{5}$$

where $c1$ is a family dependent constant. For family A we find a value $\gamma_m \approx 0.3849$.

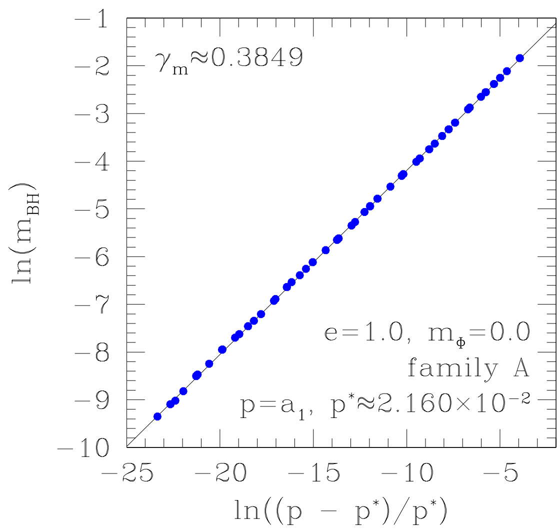

Figure 2: Power-law scaling of the black hole mass m_{BH} as the critical solution is approached from above for family A. We find a power $\gamma_m \approx 0.3849$.

Not only do the black hole masses observe a power-law relationship with respect to the critical parameter, but so do the black hole charges. The power-law has the functional form

$$\ln(Q_{BH}) = \gamma_Q \ln\left((p - p^*)/p^*\right) + c2, \tag{6}$$

where $c2$ is another family dependent constant. However, $\gamma_Q > \gamma_m$ so the black hole sheds charge more rapidly than mass with each successive echo as the critical solution is approached from above. For family A we find $\gamma_Q \approx 0.8541$.

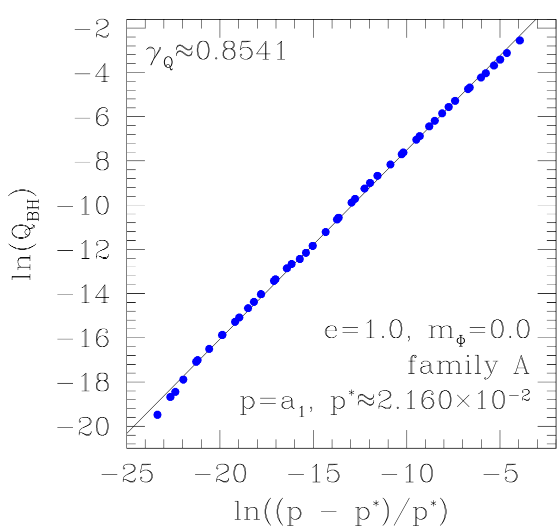

Figure 3: Power-law scaling of the black hole charge Q_{BH} as the critical solution is approached from above for family A. We find a power $\gamma_Q \approx 0.8541$.

4 Results for $e \neq 0$, $m_\Phi \neq 0$

When the scalar field mass parameter m_Φ is sufficiently larger than zero, the critical solution ceases to be discretely self-similar. Rather, it becomes periodic—an oscillating perturbed boson star with nonzero electric charge. This is apparent from the evolution of $2m/r$. Figure 4 shows the maximum of $2m/r$ in time for family A. Figure 5 shows the location of the maximum of $2m/r$ for the same evolution. An MPEG animation for a portion of the evolution is available at `http://laplace.physics.ubc.ca/People/petryk/periodic.mpeg` .

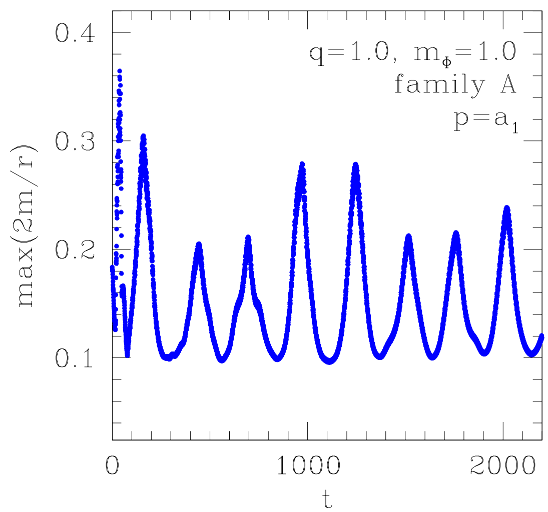

Figure 4: Maximum of $2m/r$ in coordinate time t for family A shows the critical solution is periodic.

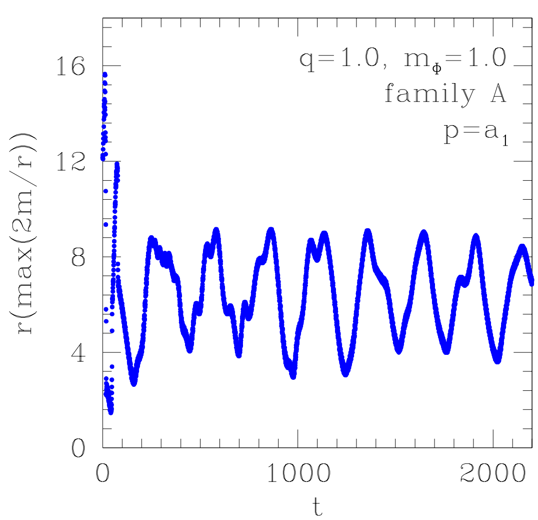

Figure 5: Location of the maximum of $2m/r$ as a function of coordinate time t for family A.

References

[1] M. W. Choptuik, *Universality and scaling in gravitational collapse of a massless scalar field*, Phys. Rev. Lett. **70**, 9 (1993).