Black Hole Critical Phenomena—Equations

1 The Matter Lagrangian

The Lagrangian for a massive electromagnetically coupled complex scalar field is given by [1]

$$\mathcal{L}_{\mathcal{M}} = \sqrt{-g}[-(\nabla_\mu \Phi - ieA_\mu \Phi)(\nabla^\mu \Phi^* + ieA^\mu \Phi^*) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - {m_\Phi}^2 \Phi^* \Phi] \; . \tag{1}$$

The summation convention over repeated indices is implied (for Greek indices $\mu, \nu, ... = 0, 1, 2, 3$), m_Φ is the scalar field mass parameter, e sets the strength of electromagnetic coupling between the real and imaginary components of complex scalar field Φ, A_μ is the electromagnetic vector potential, and $F_{\mu\nu}$ is the antisymmetric electromagnetic field strength tensor defined according to

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \; . \tag{2}$$

In terms of its real and imaginary components (respectively ϕ_1 and ϕ_2) the scalar field is written

$$\Phi = \phi_1 + i\phi_2 \; . \tag{3}$$

An observer moving with four-velocity v^ν will measure an electric field

$$E_\mu = F_{\mu\nu} v^\nu \tag{4}$$

and a magnetic field

$$B_\mu = -\frac{1}{2}\epsilon_{\mu\nu}{}^{\gamma\delta} F_{\gamma\delta} v^\nu \; , \tag{5}$$

where $\epsilon_{\mu\nu\gamma\delta}$ is the totally antisymmetric tensor of positive orientation.[1]

2 The Matter Equations of Motion

The set of Euler-Lagrange equations for the system described by (1),

$$\frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial \Phi} - \nabla_\mu \left[\frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial(\nabla_\mu \Phi)} \right] = 0 \; , \quad \frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial \Phi^*} - \nabla_\mu \left[\frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial(\nabla_\mu \Phi^*)} \right] = 0 \; , \tag{6}$$

$$\frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial A_\nu} - \nabla_\mu \left[\frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial(\nabla_\mu A_\nu)} \right] = 0 \; , \tag{7}$$

yield equations of motion[2]

$$\nabla^\mu \nabla_\mu \Phi^* + 2ie(\nabla_\mu \Phi^*)A^\mu - e^2 \Phi^* A_\mu A^\mu + ie\Phi^* \nabla_\mu A^\mu - \frac{\partial({m_\Phi}^2 \Phi^* \Phi)}{\partial \Phi} = 0 \; , \tag{8}$$

$$\nabla^\mu \nabla_\mu \Phi - 2ie(\nabla_\mu \Phi)A^\mu - e^2 \Phi A_\mu A^\mu - ie\Phi \nabla_\mu A^\mu - \frac{\partial({m_\Phi}^2 \Phi^* \Phi)}{\partial \Phi^*} = 0 \; , \tag{9}$$

$$\nabla^\mu F_{\mu\nu} - ie(\Phi^* \nabla_\nu \Phi - \Phi \nabla_\nu \Phi^*) - 2e^2 \Phi \Phi^* A_\nu = 0 \; . \tag{10}$$

We have the freedom to choose an electromagnetic gauge condition. Our choice is the Lorentz gauge condition

$$\nabla_\mu A^\mu = \nabla^\mu A_\mu = 0 \; . \tag{11}$$

[1] $\epsilon_{\mu\nu\gamma\delta}$ has norm given by $\epsilon_{\mu\nu\gamma\delta}\epsilon^{\mu\nu\gamma\delta} = -24$, and $\epsilon_{0123} = 1$ in a right-handed orthonormal basis.

[2] Under decomposition (3) it is clear that (8) and (9) yield equivalent equations of motion.

This provides us with an equation of evolution for the temporal component of A_μ.

Furthermore, since it can be demonstrated that

$$\nabla^\mu \nabla^\nu F_{\mu\nu} = 0 \ , \tag{12}$$

we conclude

$$-j_{Q\nu} \equiv \nabla^\mu F_{\mu\nu} = ie(\Phi^* \nabla_\nu \Phi - \Phi \nabla_\nu \Phi^*) + 2e^2 \Phi \Phi^* A_\nu \tag{13}$$

is a conserved current.[3]

3 Matter Evolution in a Time-Dependent Spherically Symmetric Spacetime

3.1 The Spherically Symmetric Metric

We use the spherically symmetric metric in polar areal coordinates. It is described by the line element

$$ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2 d\Omega^2 \ , \tag{14}$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$.

3.2 The Matter Equations of Motion

We can rewrite equation of motion (9) as

$$\nabla^\mu \left(\nabla_\mu \Phi - ieA_\mu \Phi\right) - ieA^\mu \left(\nabla_\mu \Phi - ieA_\mu \Phi\right) - \frac{\partial ({m_\Phi}^2 \Phi^* \Phi)}{\partial \Phi^*} = 0 \ . \tag{15}$$

Then, computing covariant derivatives of (15) for metric (14), our equation of motion is a relatively simple expression. We choose to re-express them in first order form. Using

$$n_\mu = (-\alpha, 0, 0, 0) \ , \quad n^\mu = (1/\alpha, 0, 0, 0) \ , \tag{16}$$

and defining

$$\begin{aligned} \Pi_\Phi(r,t) &\equiv n^\mu a \left(\nabla_\mu \Phi - ieA_\mu \Phi\right) \\ &= \frac{a}{\alpha}\left(\dot{\Phi} - ieA_t \Phi\right) \ , \qquad (17) \\ \Phi_r(r,t) &\equiv \partial_r \Phi \ , \qquad (18) \end{aligned}$$

where the overdot $(\dot{})$ designates partial differentiation with respect to coordinate t, equation of motion (15) becomes the following set of three equations

$$\dot{\Pi}_\Phi = \frac{1}{r^2}\partial_r \left[r^2 \frac{\alpha}{a}\left(\Phi_r - ieA_r \Phi\right)\right] - ie\left[\frac{\alpha}{a}\left(\Phi_r - ieA_r\Phi\right)A_r - \Pi_\Phi A_t\right] - \alpha a \left[\frac{\partial ({m_\Phi}^2 \Phi, \Phi^*)}{\partial \Phi^*}\right] \tag{19}$$

$$\dot{\Phi}_r = \partial_r \left[\frac{\alpha}{a}\left(\Pi_\Phi\right) + ieA_t \Phi\right] \ , \tag{20}$$

$$\dot{\Phi} = \left[\frac{\alpha}{a}\left(\Pi_\Phi\right) + ieA_t \Phi\right] \ . \tag{21}$$

[3]The choice of sign and normalizing factor on $j_{Q\nu}$ are somewhat arbitrary. Our choice of sign will later introduce a factor of -1 in the value of the electric charge density ρ_Q.

We can similarly obtain the equations of motion for field Φ^*.

We now focus on the gauge field equations of motion. As with the scalar field, we choose to write and evolve the spatial part of A_μ in first order form.[4] Defining

$$\Pi_{A_r}(r,t) \equiv n^\mu \frac{1}{a}\left(\nabla_\mu A_r - \nabla_r A_\mu\right) \quad (22)$$
$$= \frac{1}{\alpha a}\left(\dot{A}_r - A_{tr}\right) , \quad (23)$$
$$A_{tr}(r,t) \equiv \partial_r A_t , \quad (24)$$

the Euler-Lagrange equation for the spatial component of A_μ becomes the following set of three equations

$$\dot{\Pi}_{A_r} = \frac{\alpha}{a}\left(j_{Qr}\right) , \quad (25)$$
$$\dot{A}_r = \left[\alpha a\left(\Pi_{A_r}\right) + A_{tr}\right] , \quad (26)$$
$$A_{tr} = \partial_r\left(A_t\right) , \quad (27)$$

where (13) gives

$$-j_{Qr} = ie(\Phi^*\Phi_r - \Phi\Phi_r^*) + 2e^2\Phi\Phi^* A_r . \quad (28)$$

Defining

$$B_{A_t}(r,t) \equiv n^\mu a\left(A_\mu\right)$$
$$= \frac{a}{\alpha}\left(A_t\right) , \quad (29)$$

and

$$\Pi_{A_t} \equiv \dot{B}_{A_t} , \quad (30)$$

the Lorentz gauge condition can then be written as the set of three equations

$$\Pi_{A_t} = \frac{1}{r^2}\partial_r\left[r^2\frac{\alpha}{a}\left(A_r\right)\right] ,$$
$$\dot{B}_{A_t} = \Pi_{A_t} ,$$
$$A_t = \frac{\alpha}{a}\left(B_{A_t}\right) . \quad (31)$$

3.3 The Electromagnetic Constraint Equation

Providing the electromagnetic constraint is satisfied at initial time, the evolution of our system is guaranteed to uniquely satisfy the Maxwell equations (up to a choice of gauge) for all future time. To solve the constraint equation we will implement a procedure outlined by York and Piran [2], described as follows. For this purpose we temporarily switch from the language of 4-vector potentials A_ν to the language of (spatial) 3-vector fields $\vec{E}$.

Since the constraint equation is essentially Gauss' law for the electric field, we start by decomposing the electric field into it's longitudinal (divergenceful) and transverse (divergenceless) parts (respectively $\vec{E}^L$ and $\vec{E}^T$),

$$\vec{E} = \vec{E}^L + \vec{E}^T . \quad (32)$$

[4]It is easy to see that components A_θ and A_ϕ completely decouple from the system of equations. We therefore freely set them both to zero.

Now, we introduce a scalar field U and define it to be the potential for the electric field,

$$\vec{E}^L = \vec{\nabla} U \ . \tag{33}$$

Then, clearly,

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \left(\vec{E}^L + \vec{E}^T \right) = \Delta U \ , \tag{34}$$

where $\Delta \equiv \nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$ is the (spatial) 3-dimensional Laplacian operator.[5] Since

$$\Delta U = \vec{\nabla} \cdot \vec{E} = \rho_Q \ , \tag{36}$$

we recognize our constraint equation as the usual Poisson equation. As expected, it is of elliptic type. Once we have solved differential equation (36) for U, we can determine $\vec{E}$ up to a transverse component $\vec{E}^T$ as

$$\vec{E} = \vec{E}^L + \vec{E}^T = \vec{\nabla} U + \vec{E}^T \ . \tag{37}$$

The electromagnetic initial value problem has thus been reduced to a single equation (36). Now, while the longitudinal component $\vec{E}^L$ is explicitly involved in the solution of U, the transverse component $\vec{E}^T$ (which represents the radiative degrees of freedom) is freely specifiable. We could just as well choose it to be vanishing at initial time. Thus, given charge density ρ_Q, we can solve for U via (36). Having found U, we can then find $\vec{E}^L$ via (33). Meanwhile, we have freedom to specify any reasonable (i.e., smooth and differentiable) initial values for any remaining variables.

We now consider initial data constraint (36) in polar areal coordinates. First, we calculate the conserved charge density ρ_Q as seen by an observer with four velocity $v^\nu = n^\nu = (1/\alpha, 0, 0, 0)$. From the definition of j_Q^ν (i.e., equation(13)), we obtain

$$\begin{aligned} \rho_Q &\equiv -n^\mu j_{Q\mu} \\ &= -\frac{1}{\alpha}\left(j_{Qt}\right) \\ &= -\frac{1}{\alpha}\left[ie(\Phi^* \dot{\Phi} - \Phi \dot{\Phi}^*) + 2e^2 \Phi \Phi^* A_t\right] \\ &= -\frac{ie}{a}\left(\Phi^* \Pi_\Phi - \Phi \Pi_\Phi^*\right) \ . \end{aligned} \tag{38}$$

The procedure for solving the initial data for an observer with 4-velocity $v^\nu = n^\nu = (1/\alpha, 0, 0, 0)$ can then be written as follows:

(i) specify e, a, Φ, Φ^*, Π_Φ, and Π_Φ^*;

(ii) determine ρ_Q from the specified quantities using

$$\rho_Q = -\frac{ie}{a}\left(\Phi^* \Pi_\Phi - \Phi \Pi_\Phi^*\right); \tag{39}$$

(iii) solve for U using

$$\left(\frac{1}{a^2}\right)\left[\frac{2}{r}\left(\partial_r U\right) - \frac{1}{a}\left(\partial_r a\right)\left(\partial_r U\right) + \left(\partial_r \partial_r U\right)\right] = \Delta U(r,t) = \rho_Q \tag{40}$$

[5]The 3-dimensional Laplacian in polar areal coordinates acting on $U(r,t)$ is

$$\begin{aligned} \Delta U(r,t) &= \left(\frac{1}{a^2}\right)\left[\frac{2}{r}\left(\partial_r U\right) - \frac{1}{a}\left(\partial_r a\right)\left(\partial_r U\right) + \left(\partial_r \partial_r U\right)\right] \\ &= \left(\frac{1}{a}\right)\frac{1}{r^2}\partial_r\left[\frac{r^2}{a}\left(\partial_r U\right)\right] \ . \end{aligned} \tag{35}$$

Furthermore, $\rho_Q = -\frac{1}{\alpha}\left(j_{Qt}\right)$.

(iv) solve for $\vec{E}^L$ from the solution of U using

$$\vec{E}^L = \vec{\nabla} U = \left(\frac{1}{a^2} \left(\partial_r U \right), 0, 0 \right) \ ; \tag{41}$$

(v) specify a solution for $\vec{E}^T$ that satisfies $\vec{\nabla} \cdot \vec{E}^T = 0$ (e.g., $\vec{E}^T = \vec{0}$);

(vi) reconstruct the solution for $\vec{E}$ using

$$\vec{E} = \vec{E}^L + \vec{E}^T \ ; \tag{42}$$

(vii) solve for Π_{A_r} using the solution for $\vec{E}$ along with

$$\begin{aligned} \vec{E} &= \left(\frac{1}{\alpha a^2} \left(A_{tr} - \dot{A}_r \right), 0, 0 \right) \\ &= \left(-\frac{1}{a} \left(\Pi_{A_r} \right), 0, 0 \right) \end{aligned} \tag{43}$$

for the observer with four velocity $v^\mu = n^\mu = (1/\alpha, 0, 0, 0)$;

(viii) freely specify A_r;

(ix) solve for Π_{A_t} using the Lorentz gauge condition

$$\Pi_{A_t} = \frac{1}{r^2} \partial_r \left[r^2 \frac{\alpha}{a} \left(A_r \right) \right] \ . \tag{44}$$

4 The Einstein and Total Lagrangians

The Einstein Lagrangian is

$$\mathcal{L}_\mathcal{G} = \sqrt{-g} R \ . \tag{45}$$

The total Lagrangian is given by minimal coupling the matter Lagrangian to the geometry via

$$\mathcal{L} = \mathcal{L}_\mathcal{G} + \alpha_\mathcal{M} \mathcal{L}_\mathcal{M} \ , \tag{46}$$

where coupling constant $\alpha_\mathcal{M}$ will have to be chosen to give the correct normalization for the stress energy tensor of the matter field.

5 The Einstein Field Equations

The Einstein field equations are obtained by variation of the action

$$\mathcal{S}[g_{\mu\nu}, \Phi, \Phi^*, A_\mu] = \int \mathcal{L} d^4 x \tag{47}$$

with respect to $g_{\mu\nu}$. They are

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \ , \tag{48}$$

where the stress energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{\alpha_\mathcal{M}}{8\pi} \left(-\frac{\partial \mathcal{L}_\mathcal{M}}{\partial g^{\mu\nu}} + \frac{1}{2} g_{\mu\nu} \mathcal{L}_\mathcal{M} \right) \ . \tag{49}$$

And, later on, we will have to set $\alpha_\mathcal{M} = 8\pi$ in order to retrieve the Einstein-Maxwell equations from (48), (49) and matter Lagrangian (1).

6 Spacetime Decomposition and Equations of Evolution in Spherical Symmetry

In the standard 3+1 Arnowitt-Deser-Misner (ADM) decomposition of spacetime, we view the spacetime manifold as a foliation of spacelike hypersurfaces through a sequence of times. The equations which govern will the temporal evolution of our spacelike hypersurfaces are the *Hamiltonian constraint*, the *slicing condition*, and the *momentum constraint*. To write these equations out explicitly, we introduce the variables

$$\rho = n^\mu n^\nu T_{\mu\nu} \ , \qquad (50)$$

$$j_i = -n_\mu T^\mu_{\ i} \ , \qquad (51)$$

$$S^i_{\ j} = \gamma^{ik} T_{kj} \ , \qquad (52)$$

where $n^\mu = (1/\alpha, 0, 0, 0)$ is again our future-directed spatial hypersurface normal vector, and γ_{ij} is the spatial 3-metric associated with our spacetime. Its components are

$$\gamma_{ij} = \mathrm{diag}(a^2, r^2, r^2 \sin^2\theta) \ . \qquad (53)$$

In a loose sense, ρ, j_i, and S_{ij} are often referred to as the *energy density*, *momentum density*, and *stress tensor* of our general relativistic system. For our spherically symmetric polar areal metric, and under these definitions, the general form of the Hamiltonian constraint (used to update metric component a) is

$$\frac{\partial_r a}{a} + \frac{1}{2r}\left(a^2 - 1\right) = 4\pi r a^2 \rho \ , \qquad (54)$$

the general form of the slicing condition (used to update α) is

$$\frac{\partial_r a}{a} - \frac{1}{2r}\left(a^2 - 1\right) = 4\pi r a^2 S^r_{\ r} \ , \qquad (55)$$

while the momentum constraint (used to check the evolution of a) becomes

$$\dot{a} = -\alpha a K^r_{\ r} \ , \qquad (56)$$

with $K^r_{\ r}$ given by

$$K^r_{\ r} = 4\pi r j_r \ . \qquad (57)$$

We now explicitly determine the equations which evolve our spacetime. Using (1) and (49) we obtain

$$\begin{aligned} T_{\mu\nu} = \frac{\alpha_{\mathcal{M}}}{8\pi} \Bigg\{ & \frac{1}{2}\left(\nabla_\mu\Phi\nabla_\nu\Phi^* + \nabla_\nu\Phi\nabla_\mu\Phi^*\right) \\ & -\frac{1}{2} ie \left[\left(\Phi\nabla_\nu\Phi^* - \Phi^*\nabla_\nu\Phi\right) A_\mu + \left(\Phi\nabla_\mu\Phi^* - \Phi^*\nabla_\mu\Phi\right) A_\nu\right] \\ & + e^2 \Phi\Phi^* A_\mu A_\nu + \frac{1}{2} F_{\mu\gamma} F_{\nu\delta} g^{\gamma\delta} \\ & + \frac{1}{2} g_{\mu\nu}\left[-(\nabla_\gamma\Phi - ieA_\gamma\Phi)(\nabla^\gamma\Phi^* + ieA^\gamma\Phi^*) - \frac{1}{4}F^{\gamma\delta}F_{\gamma\delta} - m_\Phi^{\ 2}\Phi^*\Phi\right]\Bigg\} \ , \end{aligned} \qquad (58)$$

where $\alpha_{\mathcal{M}} = 8\pi$. Then taking the appropriate contractions with n^μ we find

$$\begin{aligned} \rho = & \frac{1}{2}\left(\frac{1}{a^2}\right)\left[\Pi_\Phi \Pi_\Phi^* + \left(\Phi_r - ie\Phi A_r\right)\left(\Phi_r^* + ie\Phi^* A_r\right)\right] \\ & + \frac{1}{4}\left(\Pi_{A_r}\right)^2 + \frac{1}{2} m_\Phi^{\ 2} \Phi\Phi^* \ , \end{aligned} \qquad (59)$$

$$j_r = -\frac{1}{2}\left(\frac{1}{a}\right)\left[\left(\Phi_r^* + ie\Phi^* A_r\right)\Pi_\Phi + \left(\Phi_r - ie\Phi A_r\right)\Pi_\Phi^*\right] \ , \tag{60}$$

and

$$\begin{aligned} S^r{}_r &= \frac{1}{2}\left(\frac{1}{a^2}\right)\left[\Pi_\Phi \Pi_\Phi^* + \left(\Phi_r - ie\Phi A_r\right)\left(\Phi_r^* + ie\Phi^* A_r\right)\right] \\ &\quad -\frac{1}{4}\left(\Pi_{A_r}\right)^2 - \frac{1}{2}{m_\Phi}^2 \Phi\Phi^* \ . \end{aligned} \tag{61}$$

Then, substituting (60) into (57) and (56) yields

$$\dot{a} = 2\pi r\alpha\left[\left(\Phi_r^* + ie\Phi^* A_r\right)\Pi_\Phi + \left(\Phi_r - ie\Phi A_r\right)\Pi_\Phi^*\right] \ , \tag{62}$$

substituting (59) into (54) yields

$$\begin{aligned} \frac{\partial_r a}{a} + \frac{1}{2r}\left(a^2 - 1\right) &= 4\pi r a^2 \left\{ \frac{1}{2}\left(\frac{1}{a^2}\right)\left[\Pi_\Phi \Pi_\Phi^* + \left(\Phi_r - ie\Phi A_r\right)\left(\Phi_r^* + ie\Phi^* A_r\right)\right] \right. \\ &\quad \left. + \frac{1}{4}\left(\Pi_{A_r}\right)^2 + \frac{1}{2}{m_\Phi}^2 \Phi\Phi^* \right\} \ , \end{aligned} \tag{63}$$

and substituting (61) into (55) yields

$$\begin{aligned} \frac{\partial_r \alpha}{\alpha} - \frac{1}{2r}\left(a^2 - 1\right) &= 4\pi r a^2 \left\{ \frac{1}{2}\left(\frac{1}{a^2}\right)\left[\Pi_\Phi \Pi_\Phi^* + \left(\Phi_r - ie\Phi A_r\right)\left(\Phi_r^* + ie\Phi^* A_r\right)\right] \right. \\ &\quad \left. - \frac{1}{4}\left(\Pi_{A_r}\right)^2 - \frac{1}{2}{m_\Phi}^2 \Phi\Phi^* \right\} \ . \end{aligned} \tag{64}$$

References

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime*, (Cambridge University Press, Cambridge, 1995).

[2] J. W. York and T. Piran, *The Initial Value Problem and Beyond*, in *Spacetime and Geometry: The Alfred Schild Lectures*, ed. by R. A. Matzner and L. C. Shepley, (University of Texas Press, Austin, 1982).