

Critical Collapse of an Ultrarelativistic Fluid in the $\Gamma \rightarrow 1$ Limit

Martin Snajdr

Department of Physics and Astronomy
University of British Columbia
Vancouver, BC
`msnajdr@physics.ubc.ca`

APCTP, Seoul

March 22, 2006

Outline

- Introduction
- Hydrodynamics in GR
- Hydrodynamics in Spherical Symmetry
- Type II Critical Phenomena for Ultrarelativistic Fluids
- Continuously Self-similar Solutions
- Perturbations of the CSS Solutions
- Limiting Solutions and their Perturbations ($\Gamma \rightarrow 1$)
- Results of Numerical Calculations
- Summary

talk based on gr-qc/0508062, to appear in CQG

Hydrodynamics in GR

- the stress-energy tensor and mass flux of an ideal fluid is

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}$$

$$J^\mu = \rho_0 u^\mu$$

- the total energy density

$$\rho = \rho_0(1 + \epsilon)$$

- equations of motion of the fluid are derived from the conservation laws

$$T^{\mu\nu}_{;\nu} = 0$$

$$J^\mu_{;\mu} = 0$$

- additional equation – equation of state

$$P = P(\rho_0, \epsilon) = (\Gamma - 1)\rho_0\epsilon$$

- for **ultrarelativistic** fluid the internal energy dominates $\epsilon \gg 1 \Rightarrow \rho \approx \rho_0 \epsilon$

$$P = P(\rho) = (\Gamma - 1)\rho$$

Hydrodynamics in Spherical Symmetry

- we use spherical polar coordinates for the metric

$$ds^2 = -\alpha(t, r)^2 dt^2 + a(t, r)^2 dr^2 + r^2 d\Omega^2$$

- primitive variables

$$\mathbf{u} = (\rho, v)$$

- conservative variables

$$\mathbf{q} = (S, E)$$

- relation between them

$$\begin{aligned} S &= (\rho + P)W^2 v \\ E &= (\rho + P)W^2 - P \end{aligned}$$

- the velocity v and the Lorentz factor W are defined as

$$v = \frac{a u^r}{\alpha u^t}$$

$$W = \frac{1}{\sqrt{1 - v^2}}$$

- equations of motion for the fluid

$$\dot{S} + \frac{1}{r^2} \left[r^2 \frac{\alpha}{a} (Sv + P) \right]' = \Sigma$$

$$\dot{E} + \frac{1}{r^2} \left[r^2 \frac{\alpha}{a} S \right]' = 0$$

- the (elliptic) equations for the metric functions a, α

$$\frac{a'}{a} = a^2 \left(4\pi r E - \frac{m}{r^2} \right)$$

$$\frac{\alpha'}{\alpha} = a^2 \left(4\pi r (Sv + P) + \frac{m}{r^2} \right)$$

- the **mass function** is defined as

$$m(r, t) = \frac{r}{2} \left(1 - \frac{1}{a(t, r)^2} \right)$$

- in calculations we use new set of conservative variables

$$\Phi = E - S$$

$$\Pi = E + S$$

Type II Critical Phenomena for Ultrarelativistic Fluids

- discovered by Choptuik (93) in scalar field collapse, for fluid collapse investigated by Evans *etal* , Koike *etal*, Neilsen *etal*, Noble *etal*
- dynamical system with two possible outcomes - BH or empty ST
- initial data are controlled by one tunable parameter p
- end state of evolution
 - for $p > p^*$ BH
 - for $p < p^*$ empty ST (matter disperses to infinity)
- scaling relation for values of p “close” to the critical value p^*

$$M_{\text{BH}} = |p - p^*|^\gamma$$

- the scaling exponent is universal

Continuously Self-Similar Solutions

- the critical solutions happen to be continuously self-similar
- self-similar coordinates

$$x = \log \left(-\frac{r}{t} \right) \quad s = -\log(-t)$$

- self-similar variables

$$\begin{aligned} N &= \frac{\alpha}{ae^x} \\ A &= a^2 \\ w &= 4\pi r^2 a^2 \rho \\ v &= v \end{aligned}$$

- if we assume CSS solutions then EOM depend only on $x \Rightarrow$ set of ODEs

$$M(y)y' = f(y)$$

with $y = (N_{\text{ss}}(x), A_{\text{ss}}(x), w_{\text{ss}}(x), v_{\text{ss}}(x))^T$

- the ODEs can be solved subject to regularity condition at **sonic point** ($x = 0$)

$$\det(M(x = 0)) = 0$$

- the solution is then characterized by a single value $v(0)$
- $v(0)$ is tuned so that $v(x)$ remains bounded as $x \rightarrow -\infty$

Perturbations of the CSS Solutions

- the scaling exponent can be calculated using perturbation theory
- the ansatz is (H is one of $\{\log(N), \log(A), \log(w), v\}$)

$$H(x, s) = H_{\text{ss}}(x) + \epsilon h_{\text{var}}(x, s)$$

- we choose the eigenmodes of the form

$$h_{\text{var}}(x, s) = h_{\text{p}}(x)e^{\kappa s} \quad \kappa \in \mathbb{C}$$

- κ is not arbitrary — we require that v_{p} does not blow up as $x \rightarrow -\infty$
- the scaling exponent γ is related to the largest κ

$$\gamma = \frac{1}{\kappa}$$

Limiting CSS Solutions and their Perturbations ($\Gamma \rightarrow 1$)

- as $\Gamma \rightarrow 1$ we observe the following behaviour ($k = \sqrt{\Gamma - 1}$)

$$N_{\text{ss}} = \bar{N}_0 e^{-x} / k$$

$$A_{\text{ss}}(x) = 1 + \bar{A}(x) k^2$$

$$w_{\text{ss}}(x) = \bar{w}(x) k^2$$

$$v_{\text{ss}}(x) = \bar{v}(x) k$$

- for the perturbations we have

$$N_{\text{p}} = \bar{N}_{\text{p}}$$

$$A_{\text{p}} = \bar{A}_{\text{p}}$$

$$w_{\text{p}} = \bar{w}_{\text{p}} / k^2$$

$$v_{\text{p}} = \bar{v}_{\text{p}} / k$$

- we also observe the following dependence

$$\kappa = \bar{\kappa} + O(k^2)$$

- we can substitute the above expressions into the ODEs and calculate all the limiting (“barred”) quantities
- the equations for the limiting CSS solution are the equations of a **Newtonian theory**
- in particular we can calculate

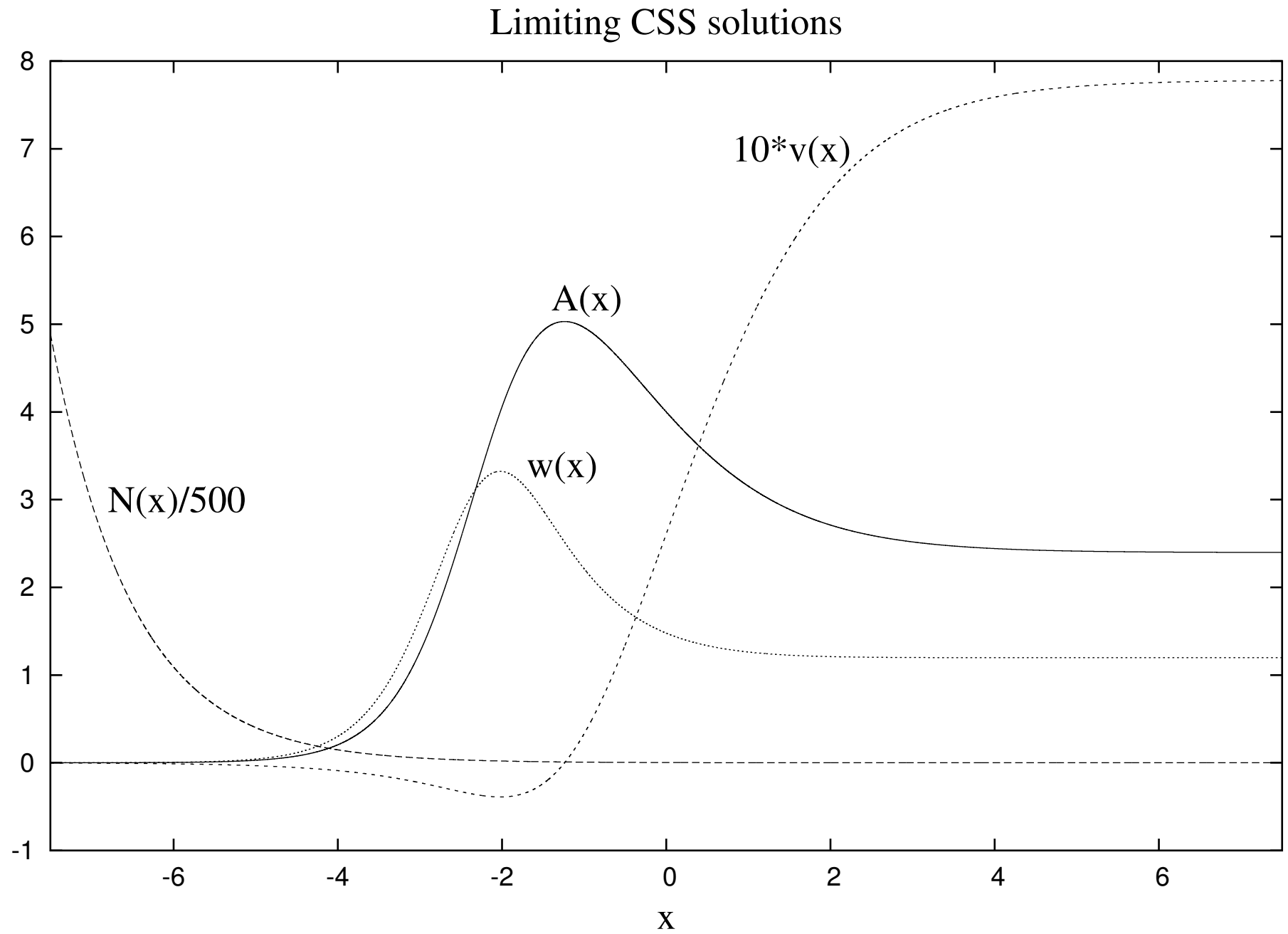
$$\bar{\kappa} = \lim_{k \rightarrow 0} \kappa(k)$$

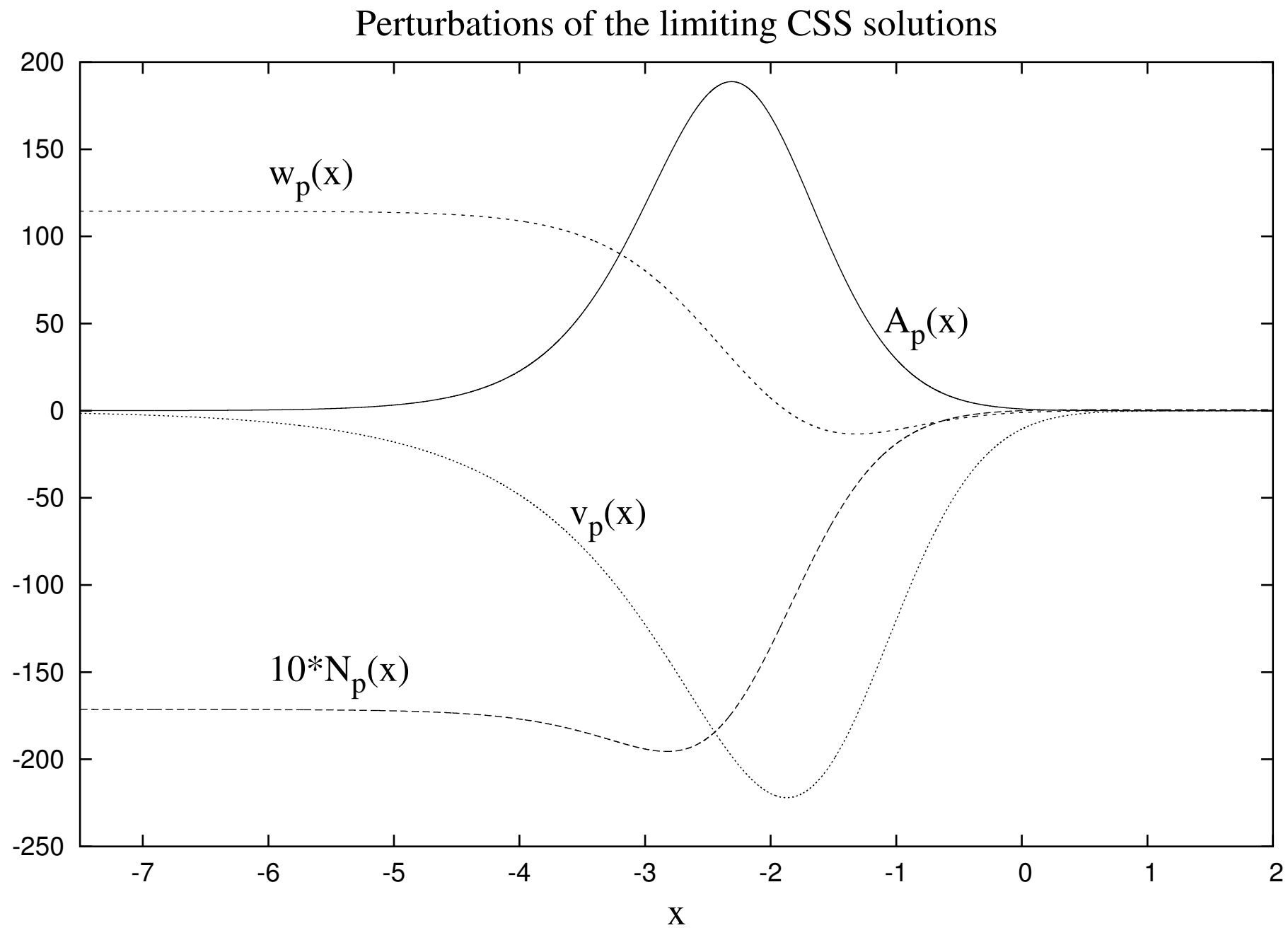
- this in turn allows us to calculate the limiting value of the scaling exponent

$$\bar{\gamma} = \lim_{k \rightarrow 0} \gamma(k) = \frac{1}{\bar{\kappa}}$$

Results of the critical exponents calculations

k^2	κ	γ_{ss}	γ_{fit}	error(%)
10^{-2}	8.748687152	0.1143028643	0.1148	0.4
10^{-3}	9.386603219	0.1065348110	0.1071	0.5
10^{-4}	9.455924881	0.1057538012	0.1062	0.4
10^{-5}	9.462917038	0.1056756596	0.1062	0.5
10^{-6}	9.463616859	0.1056678451	0.1064	0.7
0	9.463694624	0.1056669768		

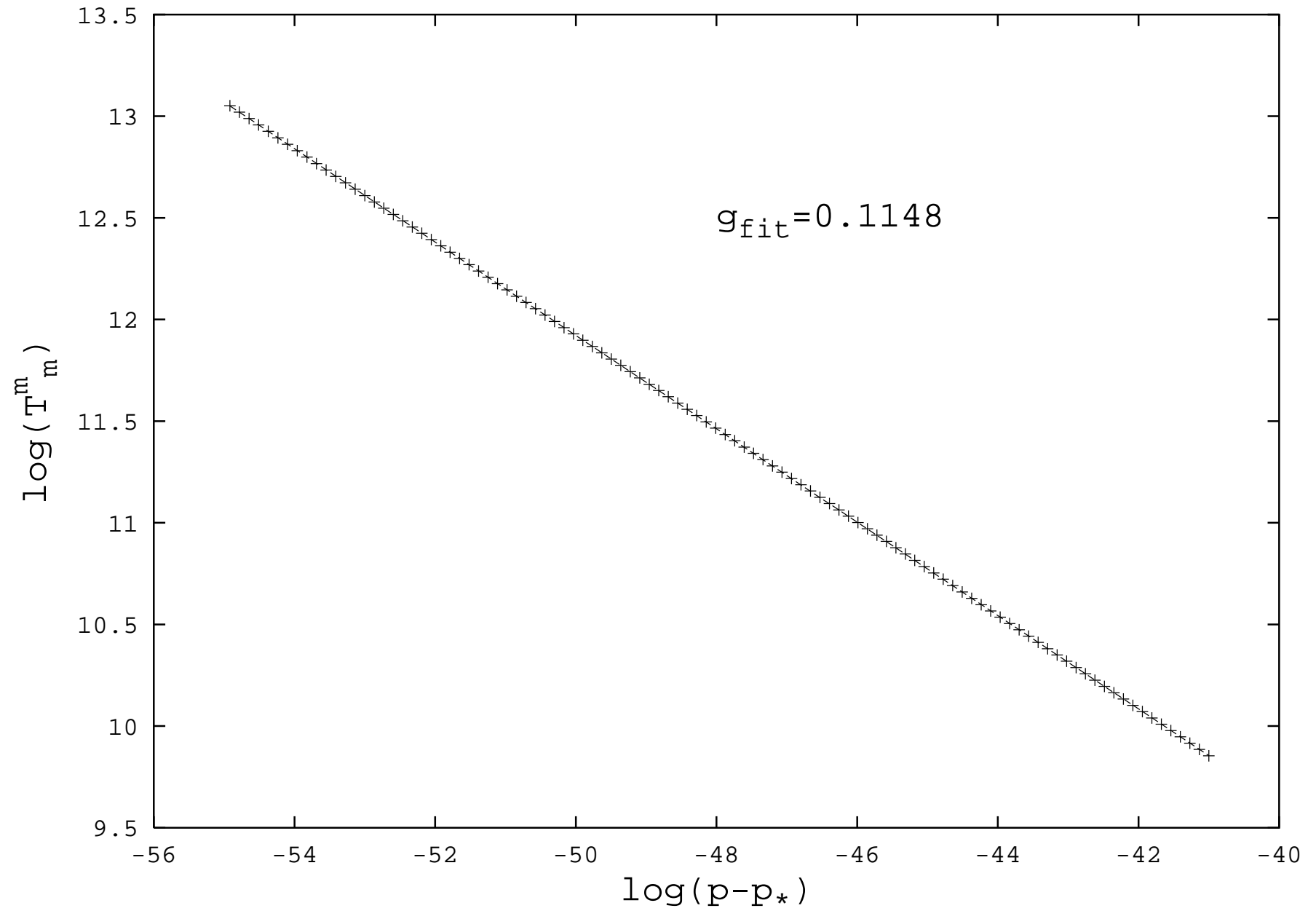


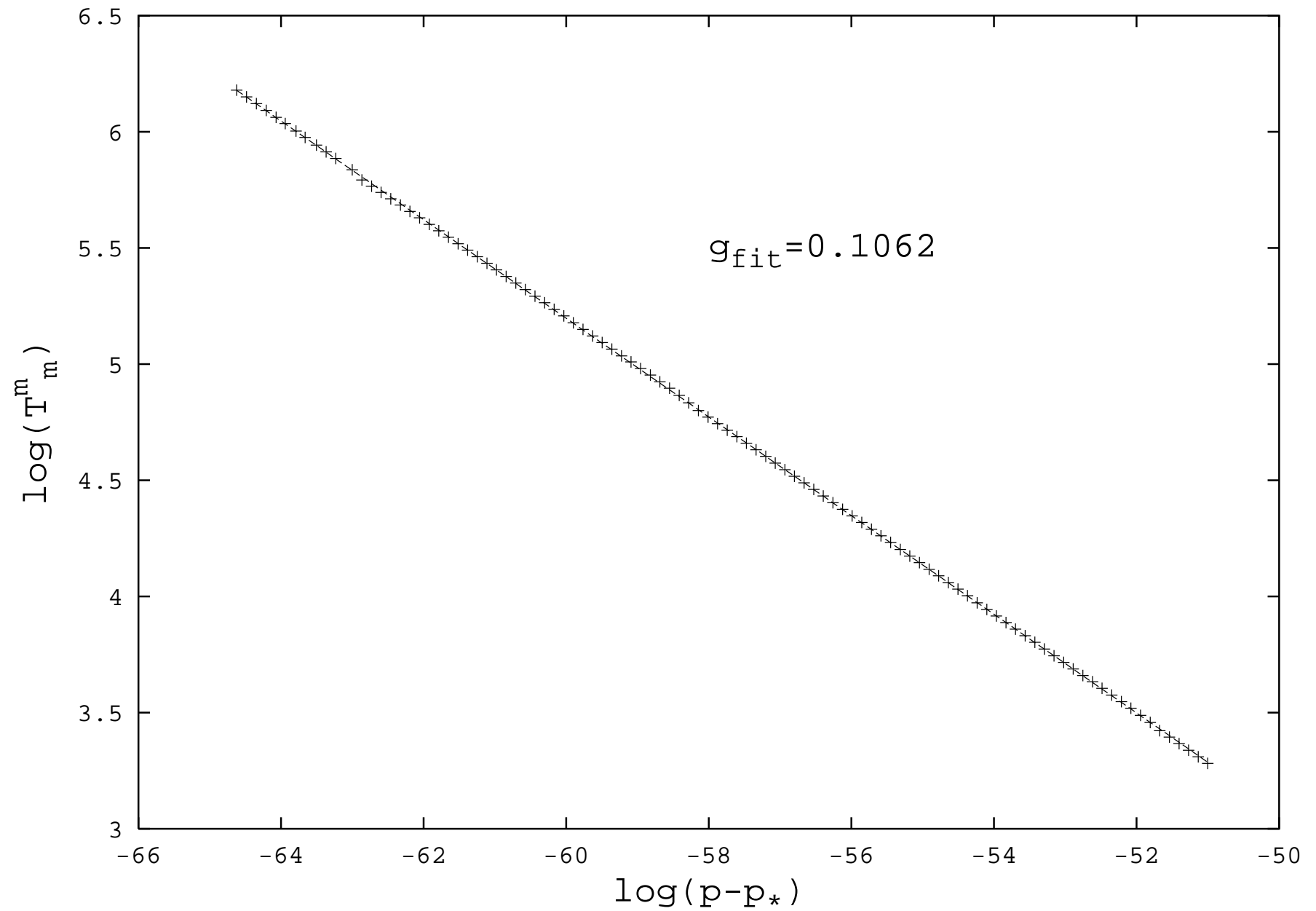


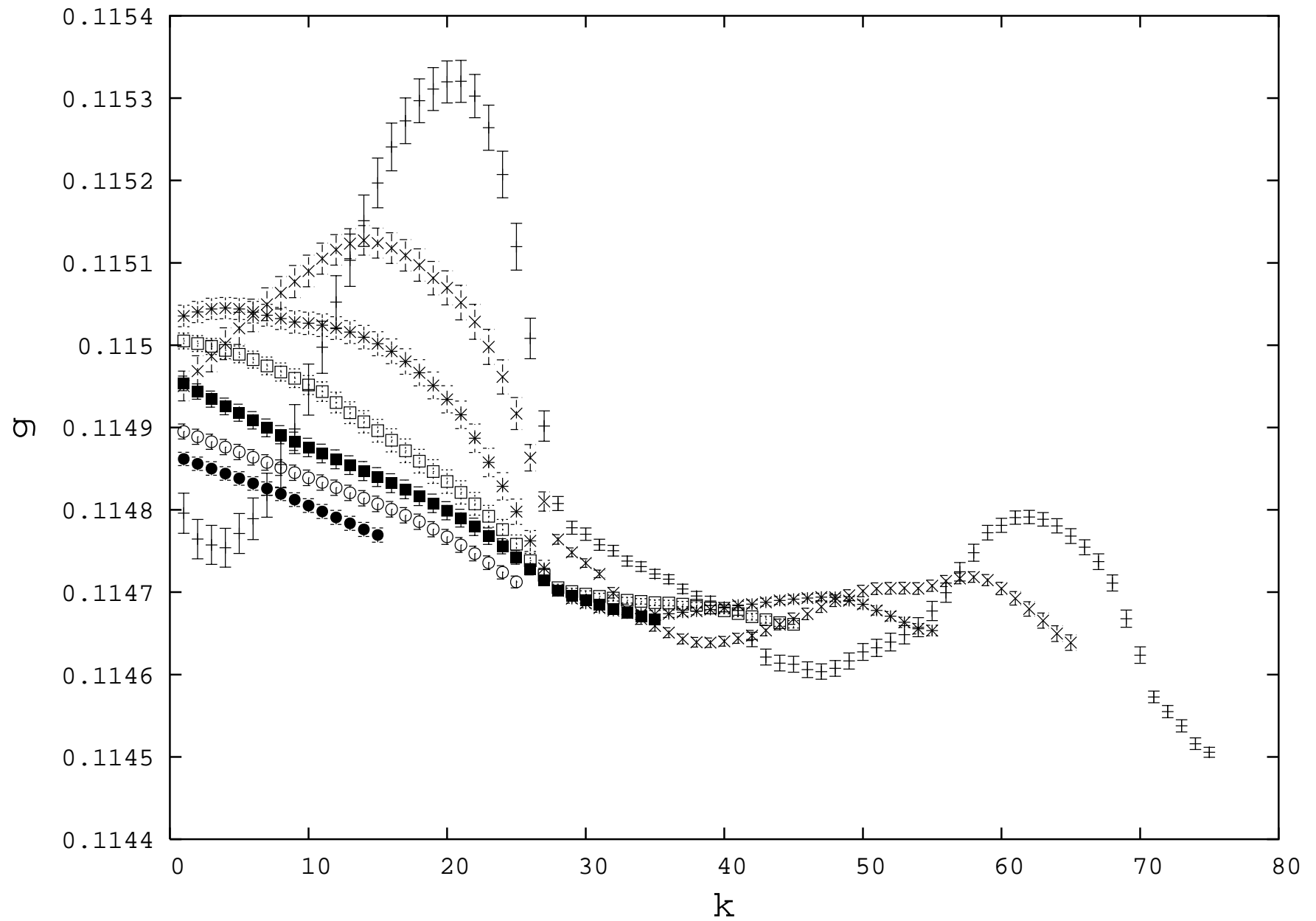
Results of Numerical Calculations

- main features:
 - high resolution shock capturing method used
 - **adaptive mesh refinement (AMR)** necessary to capture dynamics on continuously decreasing length scales
 - **quadruple precision** used to tune p up to **30 significant digits**
 - no “floor” in vacuum regions
- initial data — Gaussian with $p \equiv \rho(0)$
- simulations performed for $k^2 = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$
- critical exponent calculated from **subcritical** runs
- the order parameter is the maximum of $T^\mu_\mu = 3P - \rho$

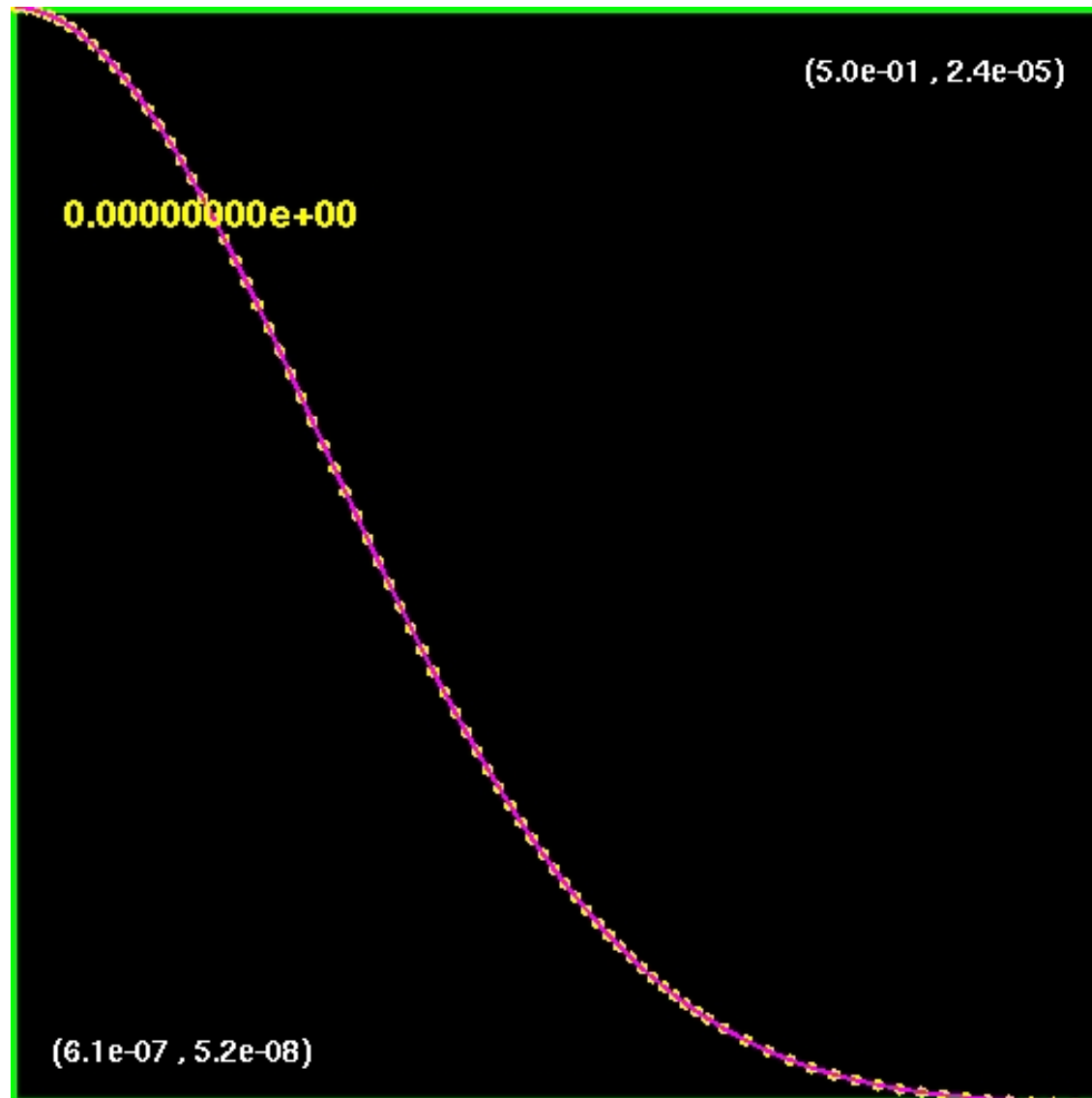
$$\max(T^\mu_\mu) = |p - p^*|^{-2\gamma}$$

Fitted data for $k^2 = 0.01$ 

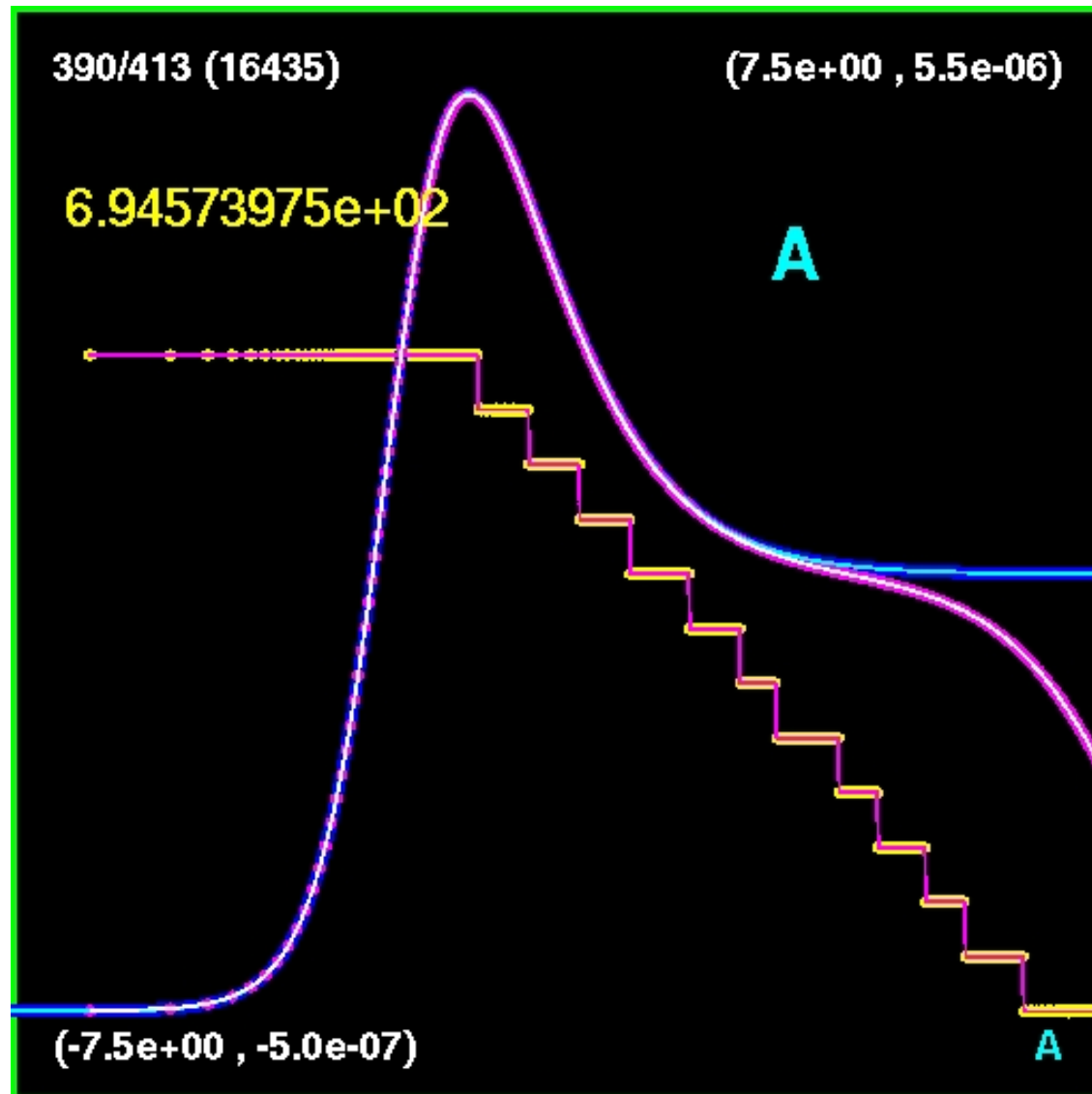
Fitted data for $k^2 = 10^{-5}$ 

Windowed fits of data for $k^2 = 0.01$ 

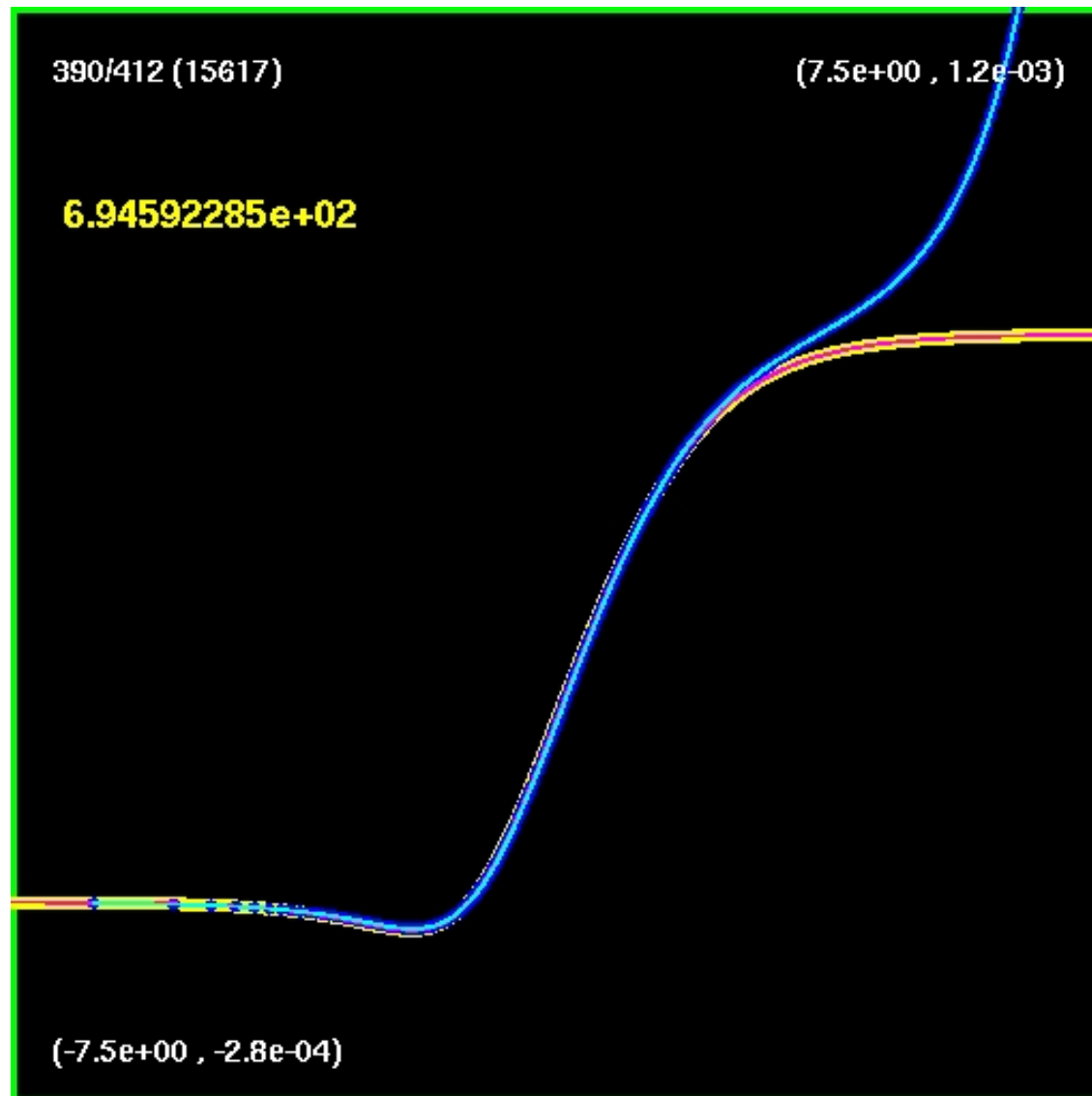
Evolution of the density profile ρ



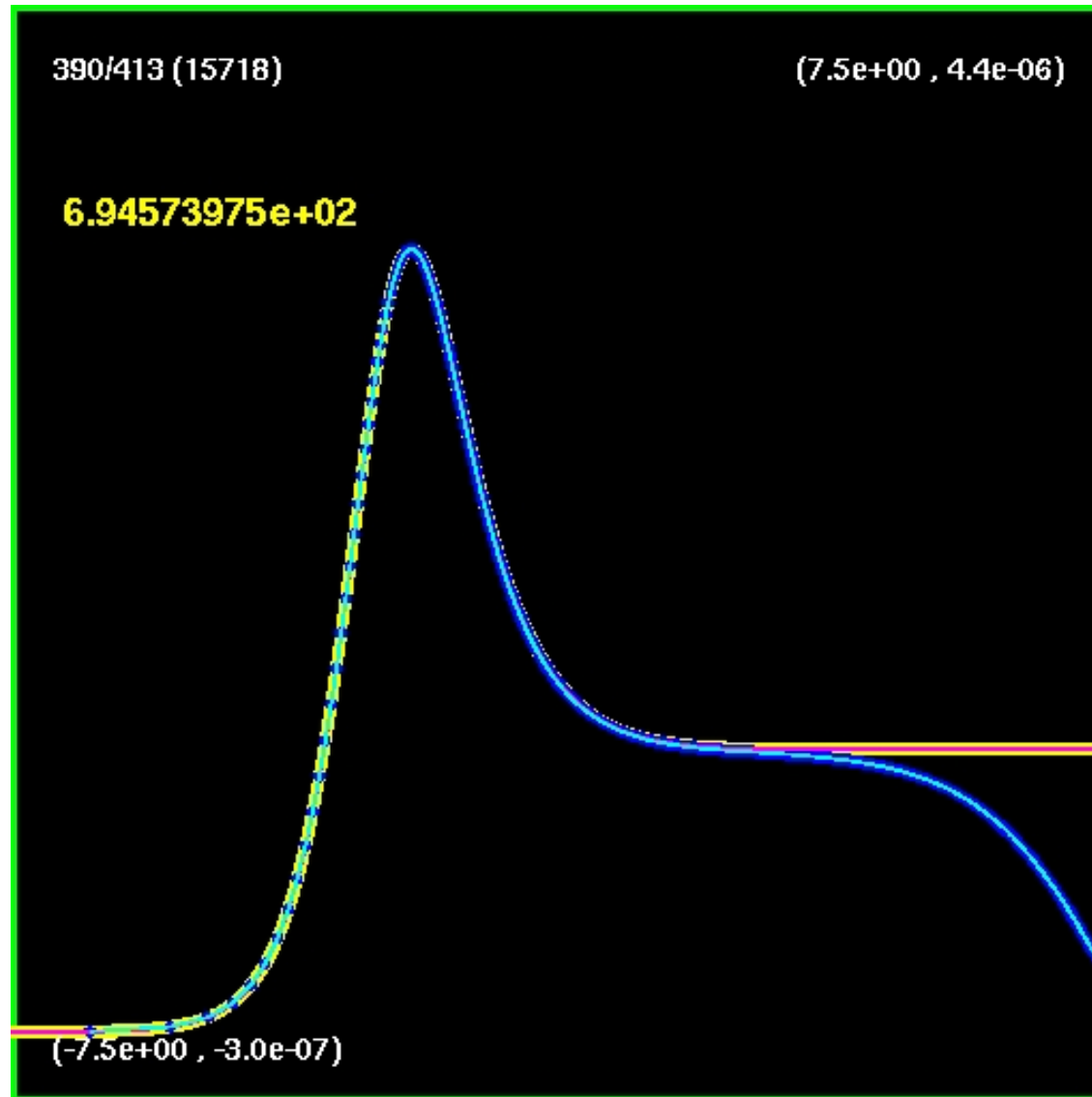
Evolution of the $a^2 - 1$ and grid hierarchy in self-similar coordinates



Evolution of the velocity profile v



Evolution of w

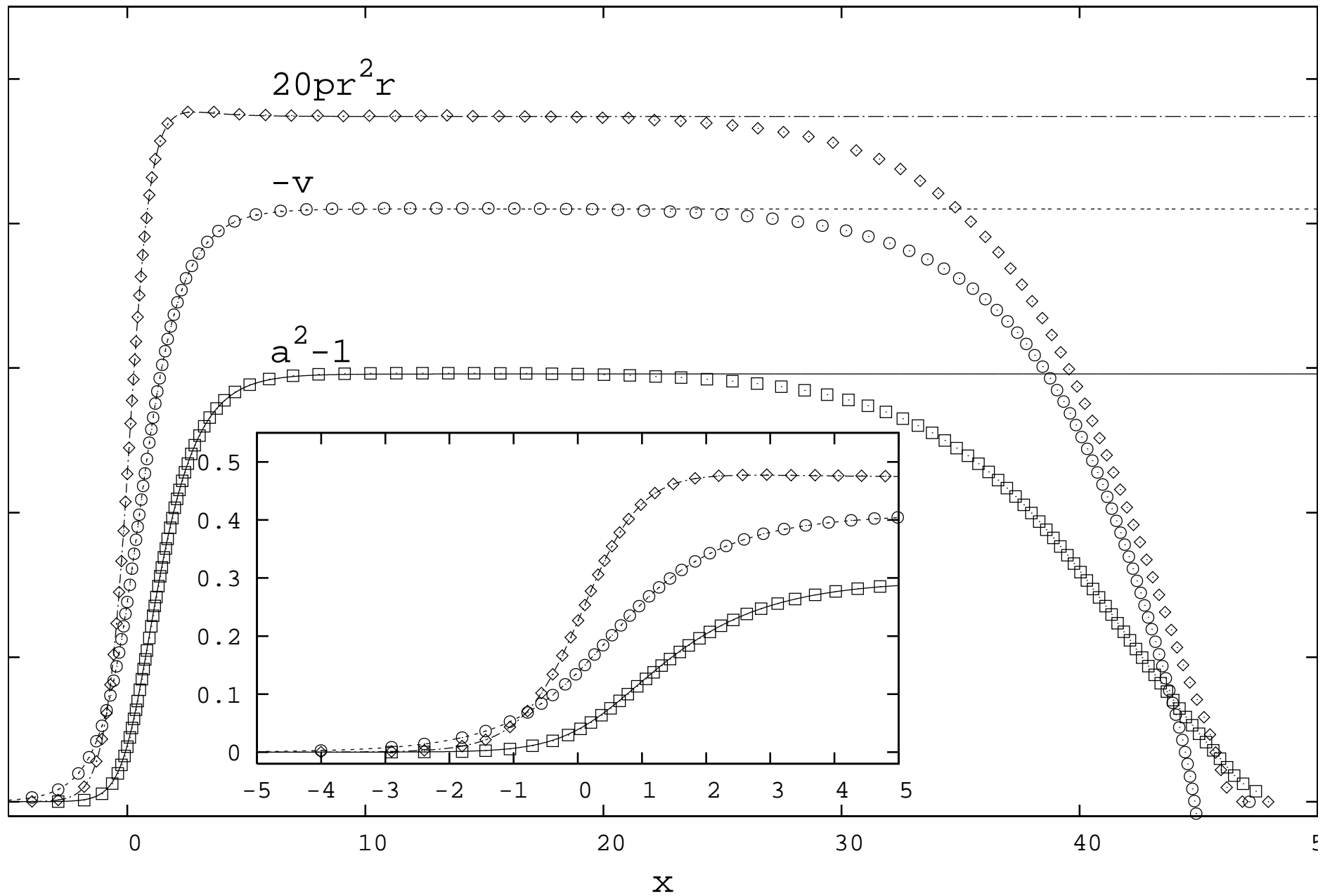


Supercritical collapse

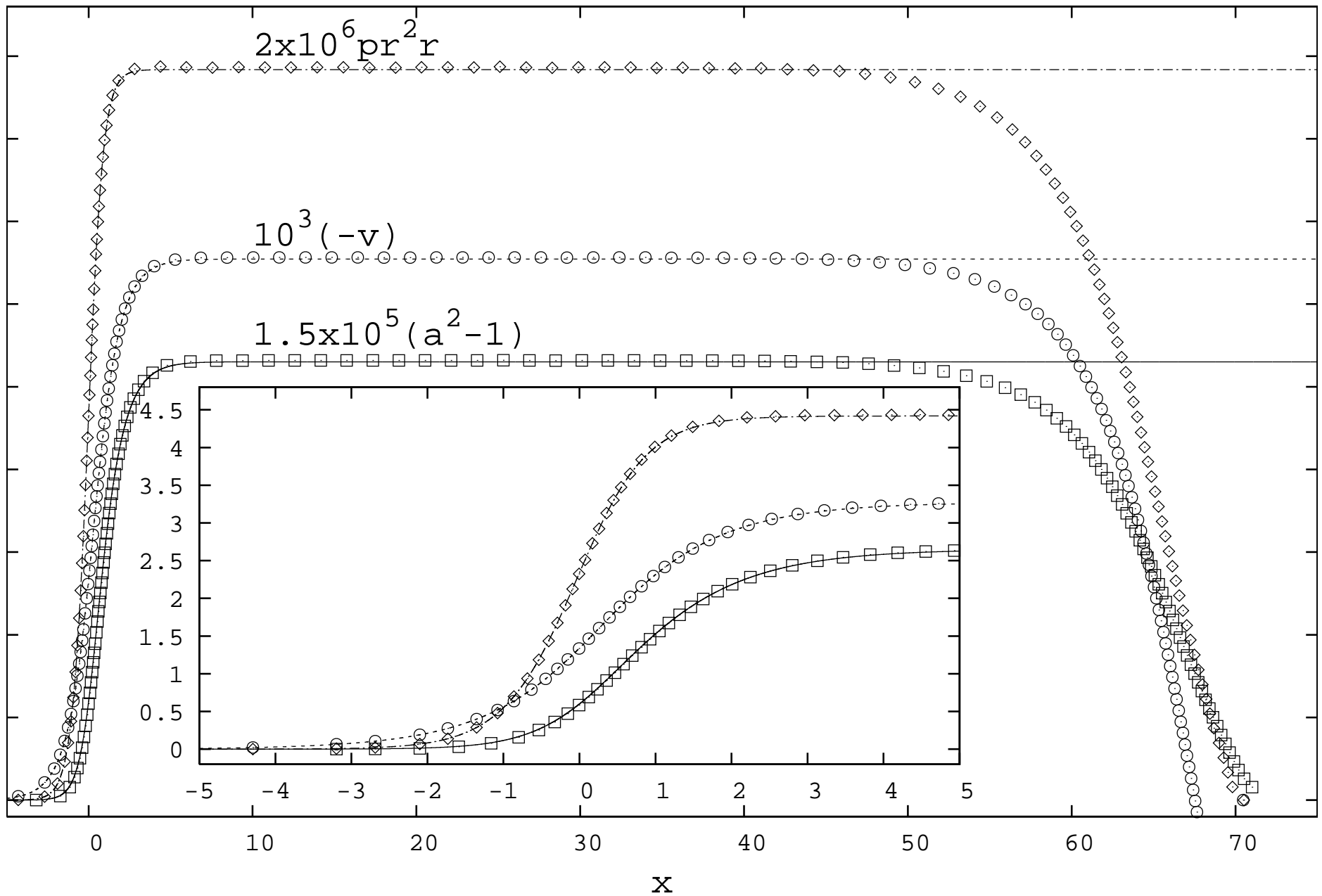
- Harada and Maeda (PRD **63**, 2001) suggested that the universal attractor for collapse of an ultrarelativistic fluid with small $\Gamma - 1$ is **not a black hole** but a general relativistic Larson-Penston solution (GRLP)
- GRLP is a “pure collapse” self-similar solution (not asymptotically flat)
- Ori and Piran (PRD **42**, 1990) showed that the GRLP exists only for $\Gamma - 1 < 0.036 \pm 0.002$ and contains naked singularity for $\Gamma - 1 < 0.0105$
- the critical exponents were calculated from subcritical solutions because no signs of black hole formation were observed
- the AMR code with quadruple precision is an ideal tool to test the hypothesis
- tests were performed for $\Gamma - 1 = 0.01$ and $\Gamma - 1 = 10^{-6}$
- generic initial data were taken (no fine tuning)

- for $\Gamma - 1 = 10^{-6}$ the refinement level reached 100 and the central density reached 10^{54} ($\Delta r = 10^{-32}$)
- for $\Gamma - 1 = 10^{-6}$ the refinement level reached 65 and the central density reached 10^{38} ($\Delta r = 10^{-22}$)
- control supercritical run was performed for $\Gamma - 1 = 0.02$ and we observed $2m/r$ approaching 1, i.e. the formation of a black hole

Comparison of the supercritical numerical solution and the GRLP solution for $\Gamma - 1 = 0.01$



Comparison of the supercritical numerical solution and the GRLP solution for $\Gamma - 1 = 10^{-6}$



Summary

- we obtained the critical solutions and critical exponents for small values of $\Gamma - 1$ both analytically and numerically
- the numerical solutions agree very well with the analytical calculations
- we calculated the CSS limiting solution (which is the Newtonian limit) and its perturbations
- we found the limiting value of the scaling exponent

$$\lim_{k \rightarrow 0} \gamma(k) = 0.1056669768$$

- our calculations seem to confirm the hypothesis of the formation of **generic naked singularities** in the collapse of matter with the equation of state $P = (\Gamma - 1)\rho$ for values of $\Gamma - 1 \leq 0.01$