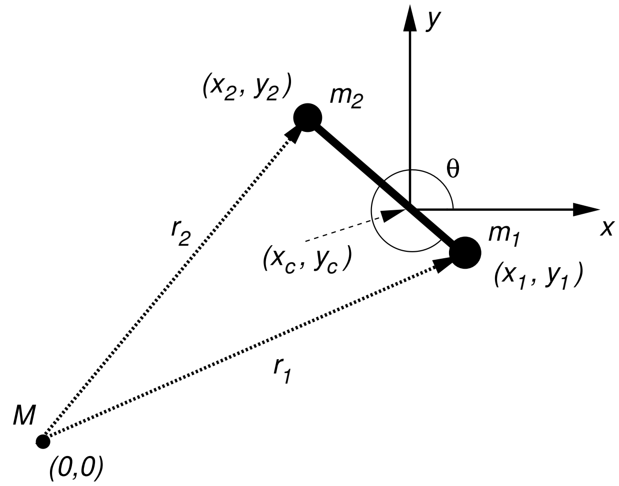
PHYS 410/555 Computational Physics: The Orbiting Dumbbell

(Following Giordano, *Computational Physics*, Section 4.6)

Background: With the exception of Hyperion, which is one of Saturn's satellites, all of the moons in the solar system are "spin-locked"; a moon which is spin-locked has a rotational frequency, ω about its own spin axes which is the same as its orbital frequency, Ω. The supposed mechanism by which the spin-locking comes about is somewhat involved; however the point is that Hyperion is somehow exceptional—study of its $\omega(t)$ suggests that it is tumbling *chaotically* in its orbit about Saturn, which is presumably due to both to its peculiar shape (like that of an egg), and the fact that it is in an elliptical orbit about Saturn.

To investigate the effects of a non-spherical distribution of mass on a satellite's spin as it orbits its parental body, we consider the model of an "orbiting dumbbell".



Consider two test masses, m_1, m_2 connected by a massless rigid rod of length d, in orbit about a mass, $M \gg m_1, m_2$ as shown in the figure above. Let $(x_i, y_i), i = 1,2$ be the coordinates of the two test masses, let (x_c, y_c) be the coordinates of the dumbbell's center of mass, and let θ be the angle the rod makes with the x-axis. Defining

$$\mu \equiv \frac{m_2}{m_1 + m_2}$$

the distances of the masses from the center of mass are

$$d_1 = \mu\, d \qquad d_2 = (1 - \mu) d$$

then

$$x_i = x_c \pm d_i \cos\theta \qquad y_i = y_c \pm d_i \sin\theta$$

The moment of inertia of the dumbbell about (x_c, y_c) is

$$I = m_1 {d_1}^2 + m_2 {d_2}^2 = \frac{m_1 {m_2}^2}{(m_1 + m_2)^2} d^2 + \frac{m_2 {m_1}^2}{(m_1 + m_2)^2} d^2 = \frac{m_1 m_2}{m_1 + m_2} d^2$$

The equations of motion for the body are

$$(m_1 + m_2)\, \mathbf{a_c} = (m_1 + m_2)\, \ddot{\mathbf{r}}_c = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$I\alpha = I\ddot{\theta} = \sum \tau = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_2 \times \mathbf{F}_2$$

where

$$\mathbf{F}_1 = -\frac{GMm_1}{{r_1}^3}\left[x_1, y_1\right]$$
$$\mathbf{F}_2 = -\frac{GMm_2}{{r_2}^3}\left[x_2, y_2\right]$$

are the gravitational forces acting on m_1 and m_2 respectively. The translational equations yield:

$$(m_1 + m_2)\,\ddot{x}_c = -GM\left(\frac{m_1}{{r_1}^3}x_1 + \frac{m_2}{{r_2}^3}x_2\right)$$
$$(m_1 + m_2)\,\ddot{y}_c = -GM\left(\frac{m_1}{{r_1}^3}y_1 + \frac{m_2}{{r_2}^3}y_2\right)$$

or

$$\ddot{x}_c = -GM\left(\frac{1-\mu}{{r_1}^3}x_1 + \frac{\mu}{{r_2}^3}x_2\right)$$
$$\ddot{y}_c = -GM\left(\frac{1-\mu}{{r_1}^3}y_1 + \frac{\mu}{{r_2}^3}y_2\right)$$

The rotational equation gives:

$$\begin{aligned}
I\ddot{\theta} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_2 \times \mathbf{F}_2 &= -GM\frac{m_1}{{r_1}^3}d_1\left(\cos\theta y_1 - \sin\theta x_1\right) + GM\frac{m_2}{{r_2}^3}d_2\left(\cos\theta y_2 - \sin\theta x_2\right) \\
&= -GM\frac{m_1}{{r_1}^3}d_1\left(\cos\theta y_c - \sin\theta x_c\right) + GM\frac{m_2}{{r_2}^3}d_2\left(\cos\theta y_c - \sin\theta x_c\right) \\
&= GM\left(\frac{m_2}{{r_2}^3}d_2 - \frac{m_1}{{r_1}^3}d_1\right)\left(\cos\theta y_c - \sin\theta x_c\right) \\
&= GM\frac{m_1 m_2}{m_1 + m_2}d\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)\left(\sin\theta x_c - \cos\theta y_c\right)
\end{aligned}$$

so

$$\ddot{\theta} = \frac{GM}{d}\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)\left(\sin\theta x_c - \cos\theta y_c\right)$$

Summarizing, we have:

$$\begin{aligned}
\ddot{x}_c &= -GM\left(\frac{1-\mu}{{r_1}^3}x_1 + \frac{\mu}{{r_2}^3}x_2\right) \\
\ddot{y}_c &= -GM\left(\frac{1-\mu}{{r_1}^3}y_1 + \frac{\mu}{{r_2}^3}y_2\right) \\
\ddot{\theta} &= \frac{GM}{d}\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)\left(\sin\theta x_c - \cos\theta y_c\right)
\end{aligned}$$

where

$$\begin{aligned}
\mu &\equiv \frac{m_2}{m_1 + m_2} \\
d_1 &= \mu\, d \\
d_2 &= (1-\mu)d \\
x_i &= x_c \pm d_i \cos\theta \\
y_i &= y_c \pm d_i \sin\theta \\
{r_i}^3 &= \left({x_i}^2 + {y_i}^2\right)^{3/2}
\end{aligned}$$

The total (conserved) energy of the system is

$$E_{\text{tot}} = T_{\text{trans}} + T_{\text{rot}} + V_{\text{grav}}$$

where

$$\begin{aligned}
T_{\text{trans}} &\equiv \frac{1}{2}\left(m_1 + m_2\right)\left(\dot{x}_c^2 + \dot{y}_c^2\right) \\
T_{\text{rot}} &\equiv \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\, d^2\, \dot{\theta}^2 \\
V_{\text{grav}} &\equiv -GM\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)
\end{aligned}$$