IV. COMPACT BINARIES: QUASISTATIONARY EQUILIBRIA

A. Data sets and full solutions with helical symmetry

B. 1st law of thermodynamics for binary black holes and neutron stars; turning point instability; locating the ISCO

A. Data Sets and Full Solutions with Helical Symmetry

In the newtonian limit, because a binary system does not radiate, it is stationary in a rotating frame. Because radiation appears only in the 2 1/2 post-newtonian order (to order $(v/c)^5$), one computes radiation for most of the inspiral from a stationary post-newtonian orbit:

WAVES FROM THE INSPIRAL CAN BE FOUND FROM A POST-NEWTONIAN POINT-PARTICLE TREATMENT;

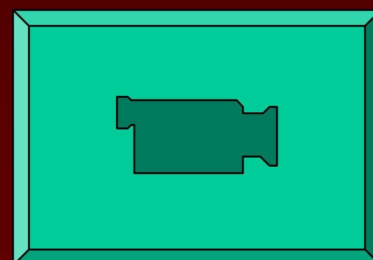
WAVES FROM AFTER LATE COALESCENCE CAN BE FOUND FROM A PERTURBATIVE TREATMENT OF THE FINAL BLACK HOLE;

LATE INSPIRAL THROUGH COALESCENCE WILL REQUIRE A NUMERICAL EVOLUTION OF THE FULL EINSTEIN EQUATIONS.

Codes running for a few orbits are now possible for neutron stars

e.g., coalescence of two neutron stars

Shibata-Uryu



Less than an orbit in current BH simulations

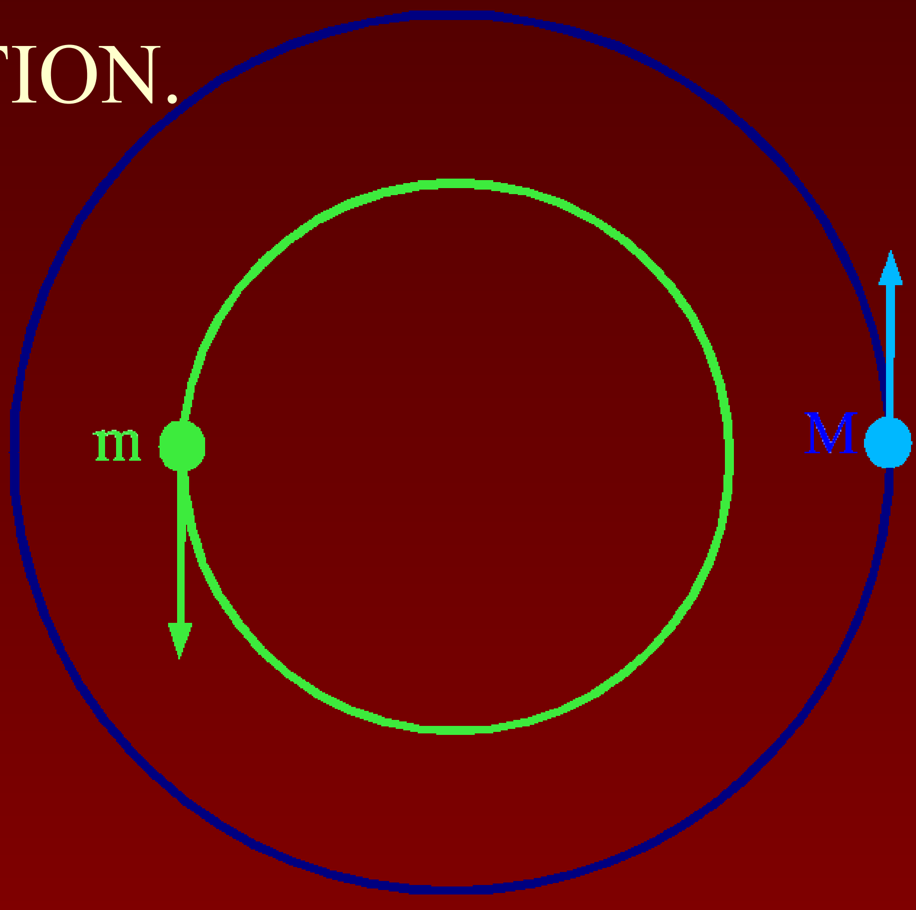
SHORT RUN-TIMES MAKE ACCURATE INITIAL DATA AND QUASISTATIONARY APPROXIMATIONS MORE VALUABLE

Inaccurate data has elliptical orbits and spurious initial radiation. Although the spurious radiation quickly radiates away, an initially elliptical orbit takes an impractically large number of orbits to circularize.

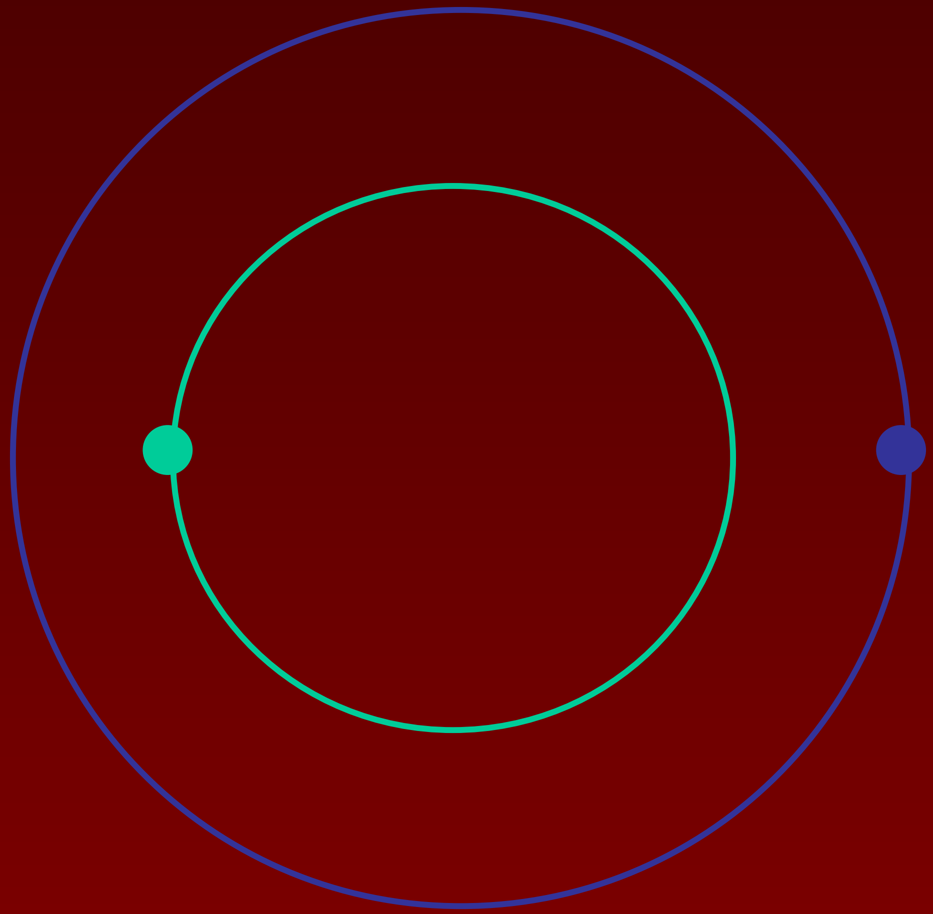
In constructing initial data sets (and quasistationary solutions that piece them together), one uses the fact that A BINARY SYSTEM WITH

$$P << \frac{E}{\dot{E}}$$

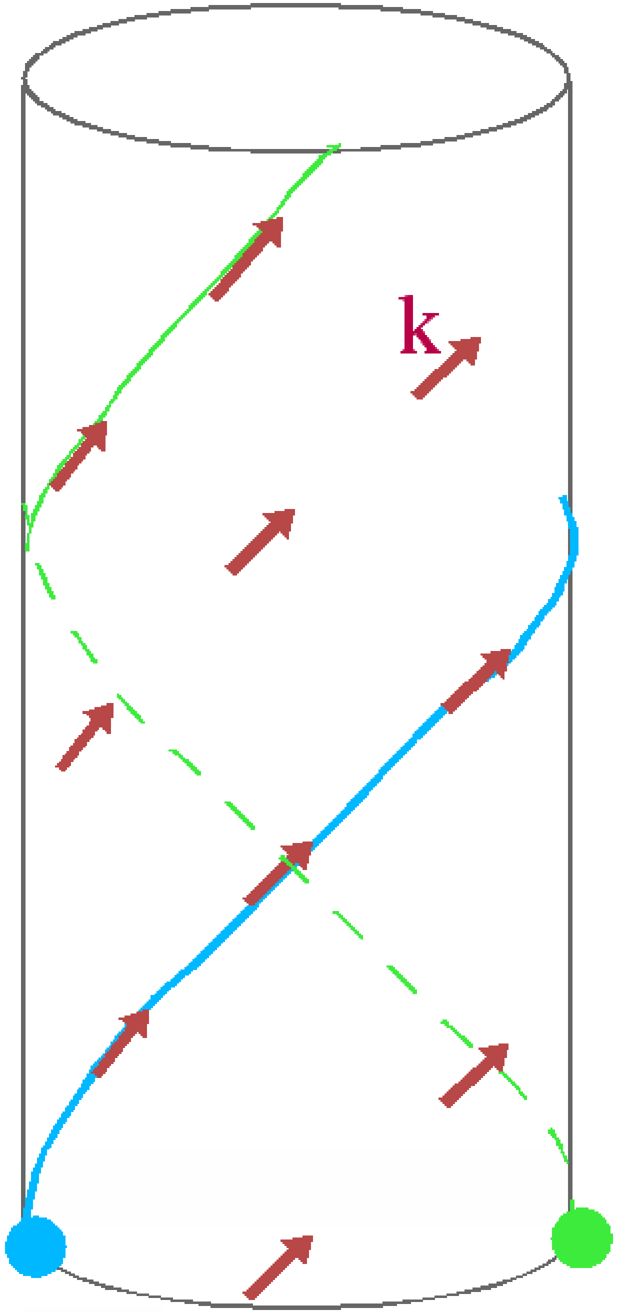
HAS LITTLE RADIATION.



Approximate it for a few orbits by a SPACETIME STATIONARY IN A ROTATING FRAME

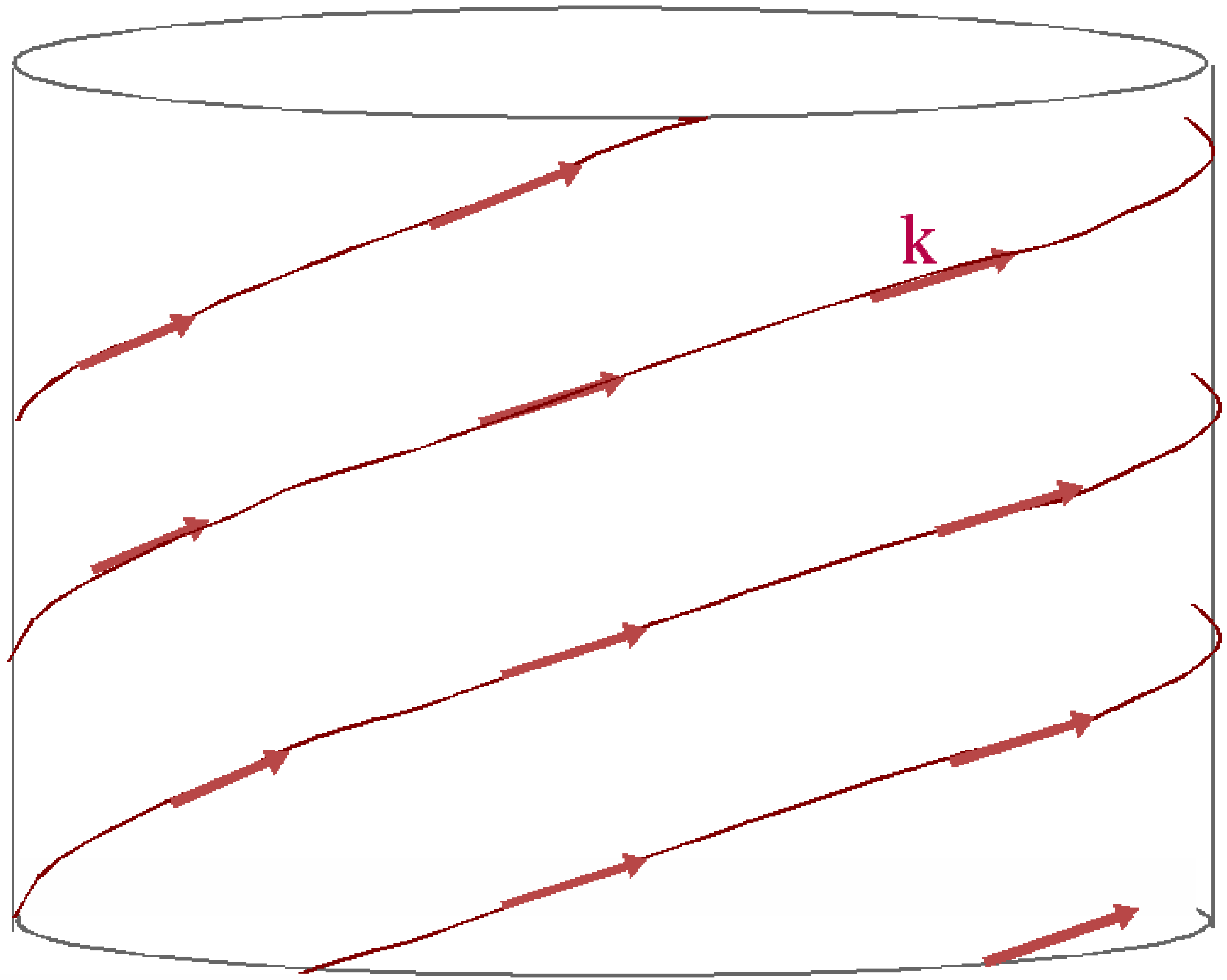


Time translations in a rotating frame are generated by a helical Killing vector k^α. The Killing vector is timelike near the binary system, but is spacelike outside a cylinder on which a corotating observer moves with speed c.



A STATIONARY BINARY SPACETIME HAS A SYMMETRY VECTOR k.

SEEN FROM AN INERTIAL FRAME, k IS HELICAL


k

QUASISTATIONARY APPROXIMATION: REPRESENT SYSTEM BY SEQUENCE OF SUCH SPACETIMES.

Blackburn & Detweiler   Wilson,Mathews,Marronetti

Price,Krivan,Whelan,Beetle,Landry   Uryu,Eriguchi,Shibata

Baumgarte, Cook, Scheel, Shapiro & Teukolsky,   Pfeiffer   Baker

Bonazzola, Gourgoulhon, Marck, Grandclement, Taniguchi

(Miller   Matzner et al   Diener)

Although k^α is spacelike outside a large cylinder, one can write the 3+1 split as usual in the form $k^\alpha = \alpha n^\alpha + \beta^\alpha$,

$$ds^2 = (-\alpha^2 + \beta_j \beta^j) dt^2 + 2\beta_j dx^j dt + \gamma_{ij} dx^i dx^j, \tag{4.1}$$

where $g_{\alpha\beta}$, α, β^a), and γ_{ab} are the 4D metric, lapse function, shift vector, and 3D spatial metric of the intitial hypersurface Σ. The extrinsic curvature K_{ab} can be written as

$$\gamma_{ab} = g_{ab} + n_a n_a, \tag{4.2}$$

$$K_{ab} = -\gamma_a^{\ \alpha} \gamma_b^{\ \beta} \nabla_\alpha n_\beta, \tag{4.3}$$

where ∇_α is the covariant derivative with respect to $g_{\alpha\beta}$. Quasiequilibrium models are helically symmetric spacetimes in which five field equations and the equation of hydrostatic equilibrium are solved for five independent equations. Because a solution to the full Einstein equations has six independent equations, the solutions so far have been to a truncated set, in which the spatial metric is conformally flat (Isenberg-Wilson-Mathews). The five equations are

- Hamiltonian constraint
- Momentum constraint (3 components)
- Spatial trace of the Einstein equation: $\gamma^{\alpha\beta}(G_{\alpha\beta} - 8\pi T^{\alpha\beta}) = 0$.

They have the form

$$\Delta_{\text{flat}}\psi = -2\pi E\psi^5 - \frac{\psi^5}{8}\tilde{A}_{ab}\tilde{A}^{ab},$$

$$\Delta_{\text{flat}}\tilde{\beta}_b + \frac{1}{3}\nabla_b\nabla_k\tilde{\beta}^k + \nabla^a \ln\left(\frac{\psi^6}{\alpha}\right)(L\beta)_{ab} = 16\pi\alpha J_b, \tag{4.4}$$

$$\Delta_{\text{flat}}(\alpha\psi) = 2\pi\alpha\psi^5(E+2S) + \frac{7}{8}\alpha\psi^5\tilde{A}_{ab}\tilde{A}^{ab}, \tag{4.5}$$

(4.6)

where

$$(L\beta)_{ab} = \nabla_a\tilde{\beta}_b + \nabla_b\tilde{\beta}_a - \frac{2}{3}\eta_{ab}\nabla_b\tilde{\beta}^b, \tag{4.7}$$

$$\tilde{A}_{ab} = \frac{1}{2\alpha}(L\beta)_{ab}, \tag{4.8}$$

We can always write the 4-velocity in the 3+1 form

$$u^\alpha = u^t(k^\alpha + v^\alpha),$$

with $v^\alpha n_\alpha = 0$. For irrotational flow (a good approximation), $hv_\alpha = \nabla_\alpha \Psi$. The equation of hydrostatic equilibrium again has a first integral (for a barotropic EOS), namely

$$\frac{h}{u^t} + hu_a v^a = \mathcal{E}. \qquad (4.9)$$

The iteration proceeds exactly as in the case of a rotating star for a barotropic EOS. One is now solving six equations (the integrated equation of hydrostatic equilibrium and five components of the field equation) for the enthalpy h, the lapse α, the 3 components of the shift β^a, and the conformal factor ψ. Again one specifies a velocity field - here by specifying the velocity potential Ψ.

1. Start with a guessed solution.
   Each of the 5 field equations can be written with a linear operator on the left side that is or includes a flat laplacian, and with the nonlinear terms on the right. Solve the equations by regarding the right side of each equation as a source.
2. Update h from the first integral of the equation of hydrostatic equilibrium, and use the EOS to find P, ε.
3. Find the new surface of the stars.
4. Use the updated ε, P and the updated potentials to recompute the right-hand sides of the field equations.
5. $\equiv$ 1.

When only 5 equations are solved, one does not find the potentials accurately enough to enforce $F = ma$ - to correctly obtain the right value of Ω for a given separation r between the stars (or black holes).

In the exact theory, one can formally consider a binary system that is again exactly stationary by requiring equal amounts of ingoing and outgoing radiation. A unique solution for two opposite charges in electromagnetism was constructed by Schild, with field given by the 1/2-advanced + 1/2 retarded Green's function.

The energy density agrees with that of the retarded solution to within a few percent out to several wavelengths. The difference arises only when one goes out far enough that the change in energy density from the changing orbital radius is important. For the inspiral phase, the helically symmetric solution is thus accurate well into the wave zone. Even when the system is near the ISCO, helical symmetry gives a fair approximation to the outgoing-wave energy density to a distance of a few wavelengths. An exact helically symmmetric solution is not asymptotically flat, because the energy radiated at all past times is present on a spacelike hypersurface. At a distance of a few wavelengths (larger than the present grid size) the energy is dominated by the mass of the binary system, and the solution appears to be asymptotically flat. Only at distances larger than about $10^4 M$ is the energy in the radiation field comparable to the mass of the binary system.

If, instead of obtaining initial data by solving a truncated set of equations, one models a binary system as an exact solution stationary in a rotating frame, the error is

$O(\Delta E/E)^2$:

$$r_{true}(\Omega) = r(\Omega)\,[1+(\Delta E/E)^2]$$

Expect

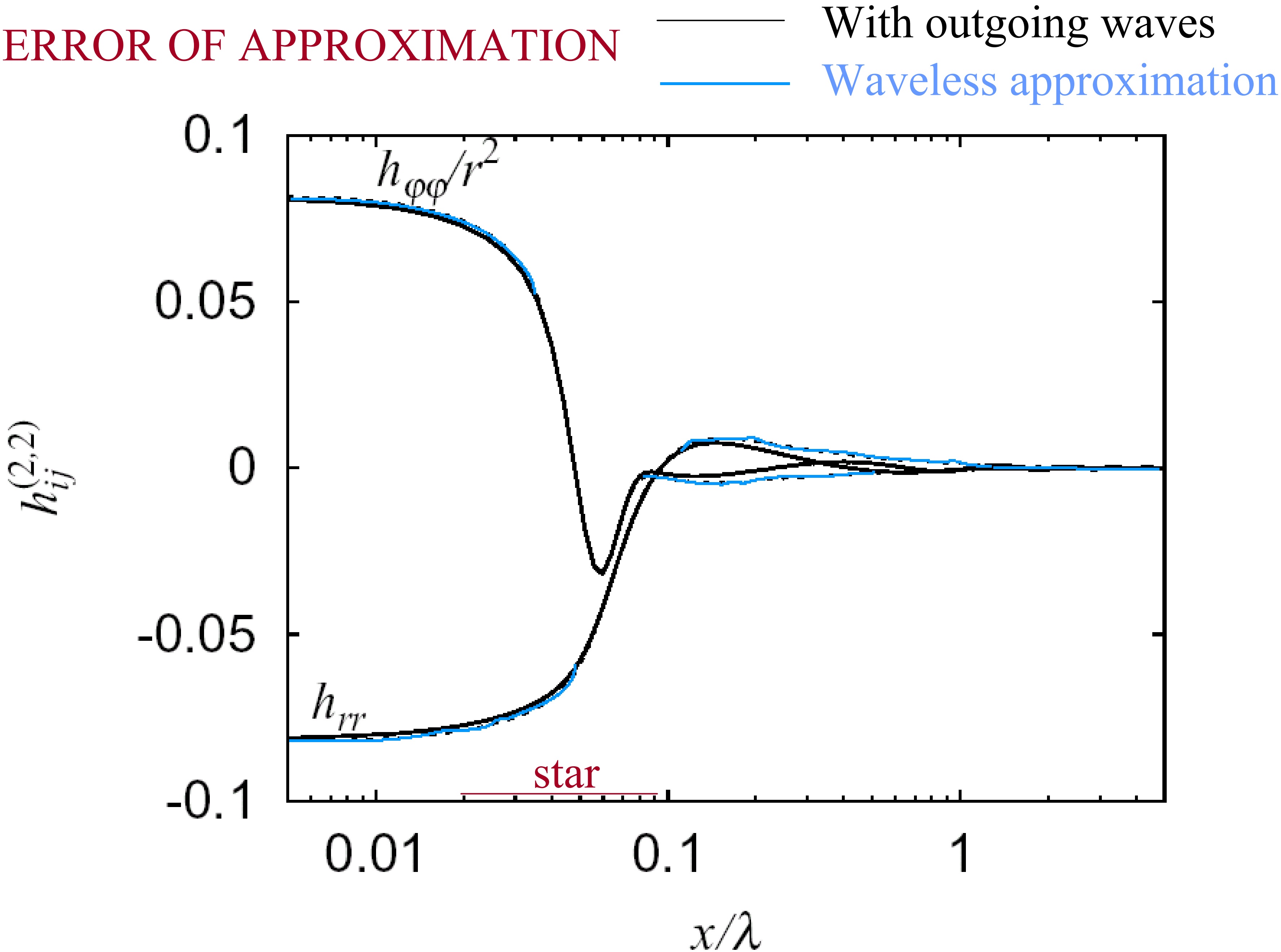
REDUCED ECCENTRICITY

SOMEWHAT REDUCED SPURIOUS RADIATION

Shibata and Uryu have recently solved the full set of equations, in a *waveless* approximation in which time derivatives of the extrinsic curvature are artificially dropped.

Work in progress by Price, Beetle, Bromley; Friedman, Uryu, Tsokaros seeks to solve the full equations without truncation for a helically symmetric spacetime.

ERROR OF APPROXIMATION



Shibata-Uryu

B. First Law of Thermodynamics for Binary Black Holes and Neutron Stars; Turning Point Instability; Locating the ISCO

One can use the solutions constructed in this way to estimate the location of the innermost stable circular orbit. One can prove (JF, Uryu, Shibata) that the Bardeen-Carter-Hawking 1st law of thermodynamics (for stationary axisymmetric black holes and rotating perfect fluid configurations) can be generalized to binary systems with a single Killing vector - in particular to helically symmetric binary systems: With

$$dM_0 := \rho u^\alpha dS_\alpha, \qquad dS := s dM_0, \qquad dC_\alpha := h u_\alpha dM_0,$$

$$\bar{T} := \frac{T}{u^t}, \qquad \bar{h} := \frac{h + v_b u^b}{u^t},$$

we have

$$dQ = \int_\Sigma [\bar{T}\Delta dS + \mathcal{E}\Delta dM_0 + v^\alpha \Delta dC_\alpha] + \sum_i \frac{1}{8\pi}\kappa_i \Delta A_i, \qquad (4.10)$$

For an asymptotically flat spacetime, $dQ = dM - \Omega dJ$. With two neutron stars and a change that conserves entropy and vorticity,

$$dM = \Omega dJ + \mathcal{E} dM_0$$

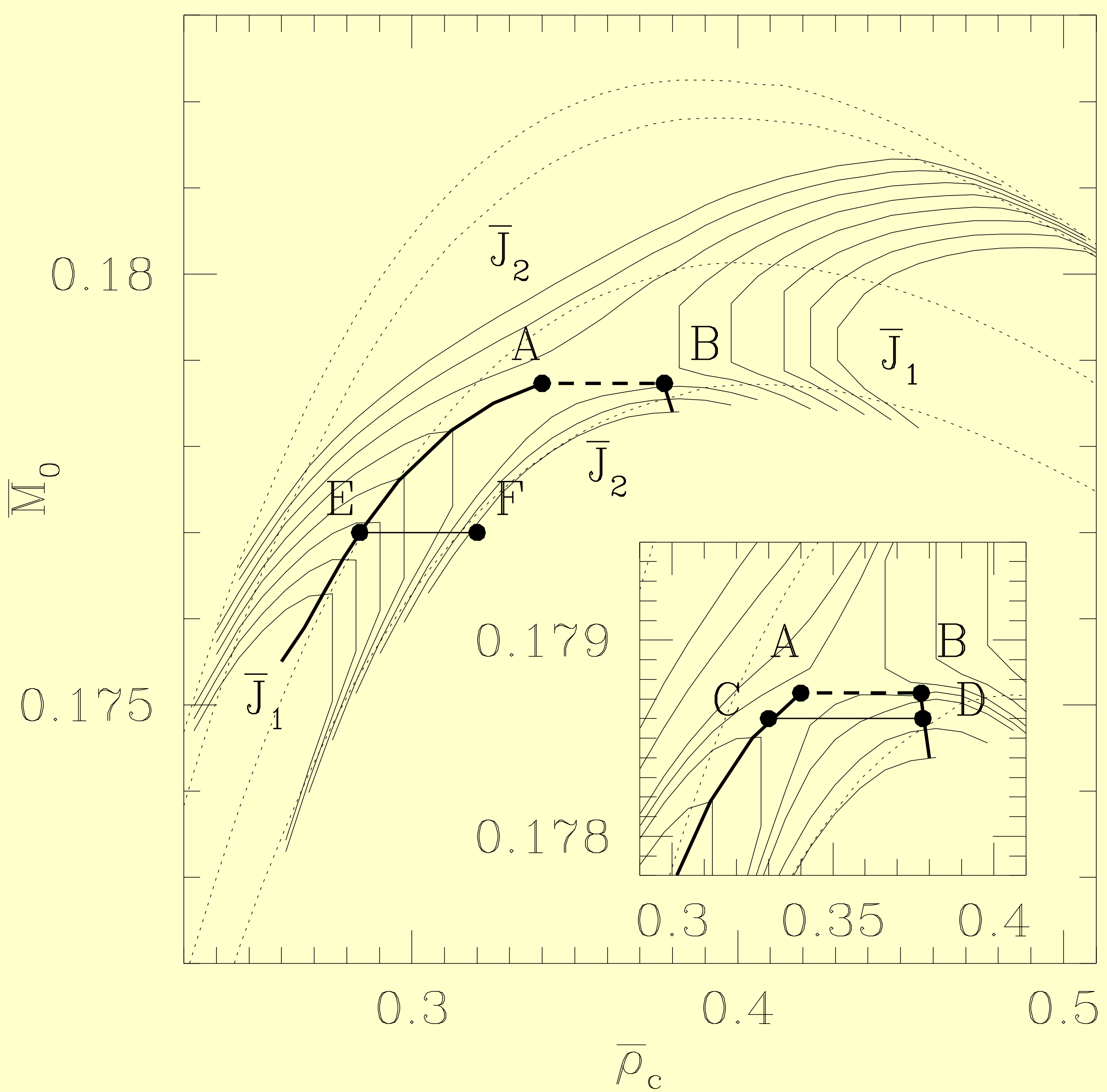
Turning-point stability

(Sorkin, JF, Ipser; Baumgarte, Cook,Scheel, Shapiro, Teukolsky)

When M turns over for fixed J, one one side of the turning point M will be larger than on the other, for the same M_0. The side on which M is smaller is more tightly bound - the stable side. The other side is unstable. (The argument for the instability does not imply that one can reach the lower energy state by a dynamical evolution; the angular momentum distribution might need to be redistributed by viscosity to allow transition to the lower energy state. But the fact that there is a lower energy state with the same vorticity suggests that we have identified the point of dynamical instability.)

In the figure below (Baumgarte, Cook, Scheel, Shapiro, Teukolsky) thin solid lines of constant J are plotted on a graph of rest mass vs central density. A thick solid curve marks the onset of orbital instability, the set of models with maximum rest mass M_0 (and maximum mass M) along a sequence of models with constant rest mass M_0. The segment FE is a sequence of quasiequilibrium corotating binaries. As the separation decreases, the angular velocity of the orbit increases, and so does the angular velocity of each star, because the models here are corotating. Consequently, lower density models along EF are models with smaller separation. The innermost stable orbit along FE is at point E.

At much larger central density, the stars are widely separated, and another set of mass-maxima along the constant J curves (e.g., point B in the inset) mark the the onset of instability to collapse of the individual stars. The turning point method identifies both instabilities.



0.18
0.175
$\bar{M}_0$
0.3
0.4
0.5
$\bar{\rho}_c$
$\bar{J}_2$
$\bar{J}_1$
A
B
E
F
$\bar{J}_2$
$\bar{J}_1$
0.179
0.178
0.3
0.35
0.4
A
B
C
D