Lie derivatives

The equation

$$[u,v]^a = 0$$

has a geometric description that can be stated this way: v^a is dragged along with the motion of a fluid having velocity field u^a, and for small $\lambda, \lambda v^a$ behaves like an arrow embedded in the fluid, always connecting the same two fluid elements. One can make the description precise and extend the notion of Lie derivative

$$\pounds_u v^a \equiv [u,v]^a$$

to arbitrary tensor fields.

The idea will be that a vector field u^a generates a family of smooth maps of the spacetime to itself. Think of the field u^a generating the fluid motion and of the maps—call them ψ_τ—moving any point P a proper distance τ up the fluid worldline through P. The motion of the fluid is described by the family of maps ψ_τ: Under ψ_τ, each spacetime point $P$is mapped

to the place where the fluid element at P has moved after the proper time τ.
More generally, under any family of smooth maps (diffeos), ψ_t, of M into itself, the orbit of each point P is a curve $t \to \psi_t(P)$.
The vector field u^a tangent to the family of curves is said to *generate* the family of diffeos.
Let us now make the connection with Lie derivatives, making precise the notion that v^a is Lie derived by u^a $([u,v]^a = 0)$ if the curve $c(\lambda)$ to which v^a is tangent is dragged along by the fluid motion. This can be stated in terms of the diffeos generated by u^a. A curve $c\colon \mathbb{R} \to M$ is mapped by a diffeo ψ to a new curve
$\psi c = \psi \circ c$
The vector v^a tangent to c at $P = c(0)$ is mapped by ψ to the vector ψv^a tangent to ψc at $\psi(P)$
A vector field v^a is in this way mapped by ψ to a vector field ψv^a with $\psi v^a|_{\psi(P)}$ tangent to ψc, where c is any curve through p with tangent $v^a|_p$. A vector field v^a is Lie derived by u^a if $\psi_\lambda v^a = v^a$,

that is, if v^a at $\psi_\lambda(P)$ is obtained from $v^a(P)$ by the map ψ_λ.
In terms of the components in a chart, ψv^i is given by

$$(\psi v)^i(\psi(P)) = \frac{d}{d\lambda}\psi^i(c(\lambda)|_{\lambda=0} = \frac{\partial\psi^i}{\partial x^j}\frac{dc^j}{d\lambda}|_{\lambda=0} = \frac{\partial\psi^i}{\partial x^j}v^j(P)$$

$$(\psi v)^i(P) = \frac{\partial\psi^i}{\partial x^j}v^j[\psi^{-1}(P)].$$

Now

$$\psi_\tau v^a = v^a \Longleftrightarrow \frac{d}{d\tau}\psi_\tau v^a = 0$$

and

$$\begin{aligned}
\frac{d}{d\tau}\left(\psi_\tau v^i|_P\right)\Big|_{\tau=0} &= \frac{d}{d\tau}\left[\frac{\partial\psi^i_\tau}{\partial x^j}v^j\left(\psi^{-1}_\tau(P)\right)\right]_{\tau=0} \\
&= \frac{\partial}{\partial x^j}\left(\frac{d\psi^i_\tau}{d\tau}\right)_{\tau=0} v^j(P) + \frac{\partial\psi^i_\tau}{\partial x^j}\bigg|_{\tau=0}\frac{d}{d\tau}v^j\left(\psi^{-1}_\tau(P)\right).
\end{aligned}$$

$$\psi^i_\tau(P) = c^i_P(\tau) \Longrightarrow \frac{d}{d\tau}\psi^i_\tau(P) = \frac{d}{d\tau}c^i_p(\tau) = u^i(P).$$

From

$$\frac{d}{d\tau} v^i \left(\psi_\tau^{-1}(P)\right) = \frac{\partial v^i}{\partial x^j} \frac{d\psi^j_{-\tau}(P)}{d\tau} = -u^j \partial_j v^i,$$

and

$$\left.\frac{\partial \psi_\tau^i}{\partial x^j}\right|_{\tau=0} = \delta^i_j,$$

we have

$$\begin{aligned} \frac{d}{d\tau} \psi_\tau v^i|_{\tau=0} &= \partial_j u^i v^j - u^j \partial_j v^i \\ &= -[u,v]^i. \end{aligned}$$

Finally

$$[u,v]^a = -\frac{d}{d\tau} \psi_\tau v^a|_{\tau=0}, \tag{2.81}$$

and we have shown that $[u,v]^a = 0$ is simply the infinitesimal version of $\psi_\tau v^a = v^a$, the statement that v^a is dragged along by the diffeos generated by u^a. Since $[u,v]^a = -[v,u]^a, u^a$ Lie-derives v^a if and only if v^a Lie-derives u^a.

The Lie derivative can be extended to arbitrary tensor fields in the following way. First extend the action of diffeos to tensors:

$$\psi(v^a w^b) = \psi v^a \psi w^b \text{ gives } \psi T^{a\cdots b} \tag{2.82}$$

$$\psi f = f \circ \psi^{-1} : \text{ a function } f \text{ on } M \text{ about } P \text{ gives a function } \psi f \text{ about } \psi(P \tag{2.83}$$

$$\psi f|_{\psi(P)} = f|_p \text{ or } \psi f|_p = f|_{\psi^{-1}(P)} \text{ or } \psi f = f \circ \psi^{-1}.$$

Covectors:

$$\nabla_a f|_p \to \nabla_a \left(f \circ \psi^{-1}\right)\big|_{\psi(P)} \tag{2.84}$$

so that the covector field $\nabla_a f$ is dragged by ψ to $\psi \nabla_a f$,

$$\psi \nabla_a f = \nabla_a(\psi f) = \nabla_a(f \circ \psi^{-1}),$$

and, writing a general covector in the form

$$\sigma_a = \sigma_i \nabla_a x^i,$$

we have

$$\psi\sigma_a = \sigma_i \circ \psi^{-1} \nabla_a (x^i \circ \psi^{-1}) \tag{2.85}$$

Components:

$$(\psi T)^{i \cdots j} = \frac{\partial \psi^i}{\partial x^k} \cdots \frac{\partial \psi^j}{\partial x^l} T^{k \cdots l} \tag{2.86}$$

$$(\psi \omega)_{i \cdots j} = \frac{\partial \psi^{-1k}}{\partial x^i} \cdots \frac{\partial \psi^{-1l}}{\partial x^j} \omega_{k \cdots l} \tag{2.87}$$

The Lie derivative of a tensor $T^{a \cdots b}{}_{c \cdots d}$ with respect to a vector field u^a is then defined by

$$\pounds_u T^{a \cdots b}{}_{c \cdots d} = -\frac{d}{d\tau} \psi_\tau T^{a \cdots b}{}_{c \cdots d} \Big|_{\tau=0} \tag{2.88}$$

For a covector, for example,

$$\begin{aligned}
\pounds_u \sigma_i &= -\frac{d}{d\tau}(\psi_\tau \sigma)_i \\
\pounds_u \sigma_i|_P &= -\frac{d}{d\tau}\left[\left.\frac{\partial \psi_\tau^{-1j}}{\partial x^i}\right|_P \sigma_j\left(\psi_\tau^{-1}(P)\right)\right] = \left(\frac{\partial}{\partial x^i}u^j\right)\sigma_j(P) + \frac{\partial \sigma_j}{\partial x^k}u^k(P) \\
\Longrightarrow \pounds_u \sigma_a &= \sigma_b \nabla_a u^b + u^b \nabla_b \sigma_a \qquad (2.89)
\end{aligned}$$

In general,

$$\begin{aligned}
\pounds_u T^{a\cdots b}{}_{c\cdots d} = u^e \nabla_e T^{a\cdots b}{}_{c\cdots d} &- T^{e\cdots b}{}_{c\cdots d}\nabla_e u^a - \cdots - T^{a\cdots c}{}_{c\cdots d}\nabla_e u^b \\
&+ T^{a\cdots b}{}_{e\cdots d}\nabla_c u^e + \cdots + T^{a\cdots b}{}_{c\cdots e}\nabla_d u^e . \qquad (2.90)
\end{aligned}$$

Exercise: Obtain eq. (2.90) for $\pounds_u$ algebraically from the following axioms:

(1) $\pounds_u f = u^b \nabla_b f$

(2) $\pounds_u v^a = u^b \nabla_b v^a - v^b \nabla_b u^a$

(3) Leibniz: $\pounds_u \left(S^{\cdots}{}_{\cdots} T^{\cdots}{}_{\cdots}\right) = S^{\cdots}{}_{\cdots} \pounds_u T^{\cdots}{}_{\cdots} + \left(\pounds_u S^{\cdots}{}_{\cdots}\right) T^{\cdots}{}_{\cdots}$

Note that $\pounds_u T^{a\cdots b}{}_{c\cdots d}$ is independent of connection ∇_a; in a chart it involves no $\Gamma^i{}_{jk}$'s:

$$
\begin{aligned}
& u^m\nabla_m T^{i\cdots j}{}_{k\cdots l} - T^{m\cdots j}{}_{k\cdots l}\nabla_m u^i - \cdots - T^{i\cdots m}{}_{k\cdots l}\nabla_m u_j \\
&\qquad + T^{i\cdots j}{}_{m\cdots l}\nabla_k u^m + \cdots + T^{i\cdots j}{}_{k\cdots m}\nabla_l u^m \\
= {}& u^m\left(\partial_m T^{i\cdots j}{}_{k\cdots l} + \underline{\Gamma^i{}_{nm}T^{n\cdots j}{}_{k\cdots l}} + \cdots + \underline{\Gamma^j{}_{nm}T^{i\cdots n}{}_{k\cdots l}} \right.\\
&\qquad \left. - \underline{\Gamma^n_{km}T^{i\cdots j}{}_{n\cdots l}} - \cdots - \underline{\Gamma^n{}_{lm}T^{i\cdots j}{}_{k\cdots n}}\right)\\
& - T^{m\cdots j}{}_{k\cdots k}\partial_m u^i - \underline{T^{m\cdots j}{}_{k\cdots l}\Gamma^i{}_{nm}u^n} - \cdots\\
& - T^{i\cdots m}{}_{k\cdots l}\partial_m u^j - \underline{T^{i\cdots m}{}_{k\cdots l}\Gamma^j{}_{nm}u^n}\\
& + T^{i\cdots j}{}_{m\cdots l}\partial_k u^m + \underline{T^{i\cdots j}{}_{m\cdots l}\Gamma^m{}_{nk}u^n} + \cdots\\
& + T^{i\cdots j}{}_{k\cdots m}\partial_l u^m + \underline{T^{i\cdots j}{}_{k\cdots m}\Gamma^m{}_{nl}u^n}\\
= {}& u^m\partial_m T^{i\cdots j}{}_{k\cdots l} - T^{m\cdots j}{}_{k\cdots l}\partial_m u^i - \cdots - T^{i\cdots m}{}_{k\cdots l}\partial_m u^j\\
&\qquad + T^{i\cdots j}{}_{m\cdots l}\partial_k u^m + \cdots + T^{i\cdots j}{}_{k\cdots m}\partial_l u^m.
\end{aligned}
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