CRITICAL BEHAVIOUR IN GRAVITATIONAL COLLAPSE

- The Threshold of BH Formation: Overview
- Phenomenology & Properties of Critical Solutions
- Self-Similarity & Perturbation Theory
- SU(2) Einstein-Yang-Mills Collapse
  - Coordinate Systems & Black Hole Excision
  - Static Solutions
  - Type I & Type II Critical Solutions
  - Forming Colored Black Holes via Fine Tuning
- Open Questions & Ongoing Research

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The Threshold of Black Hole Formation

- Will focus on spherically-symmetric collapse—basic picture expected to extend to *generic* collapse, but needs detailed study

- Adopt some matter model (e.g. massless scalar field) and couple to gravitational field

- Consider parametrized families of collapse solutions—i.e. vary initial conditions and see what happens

- Demand that family "interpolate" between "flat" spacetimes and BH spacetimes (as $t \to \infty$)

  - $p < p^\star$ No BH forms (complete dispersal)
  - $p > p^\star$ BH forms

The Threshold of Black Hole Formation (cont.)

- Interpolating families characterized by *COMPETITION*

- Where can energy in system end ups?

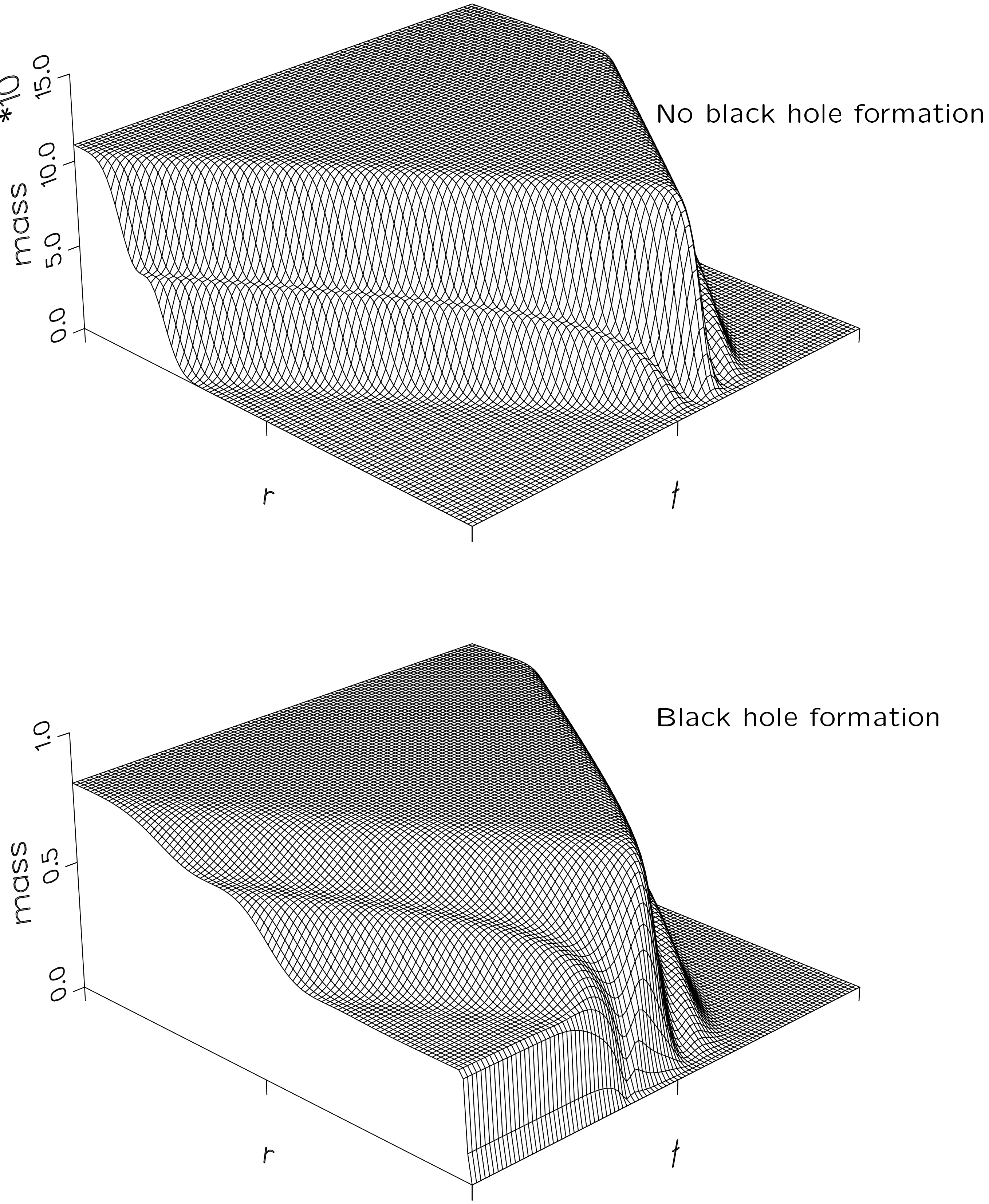
  - *K.E.* $\implies$ $r = \infty$

  - *GRAVITATIONAL P. E.:* $\implies$ $r < r_{BH}$

- BH formation "turns on" at threshold, or critical, parameter value, $p^\star$

Phenomena in near-threshold regime $\equiv$ *Critical Phenomena*

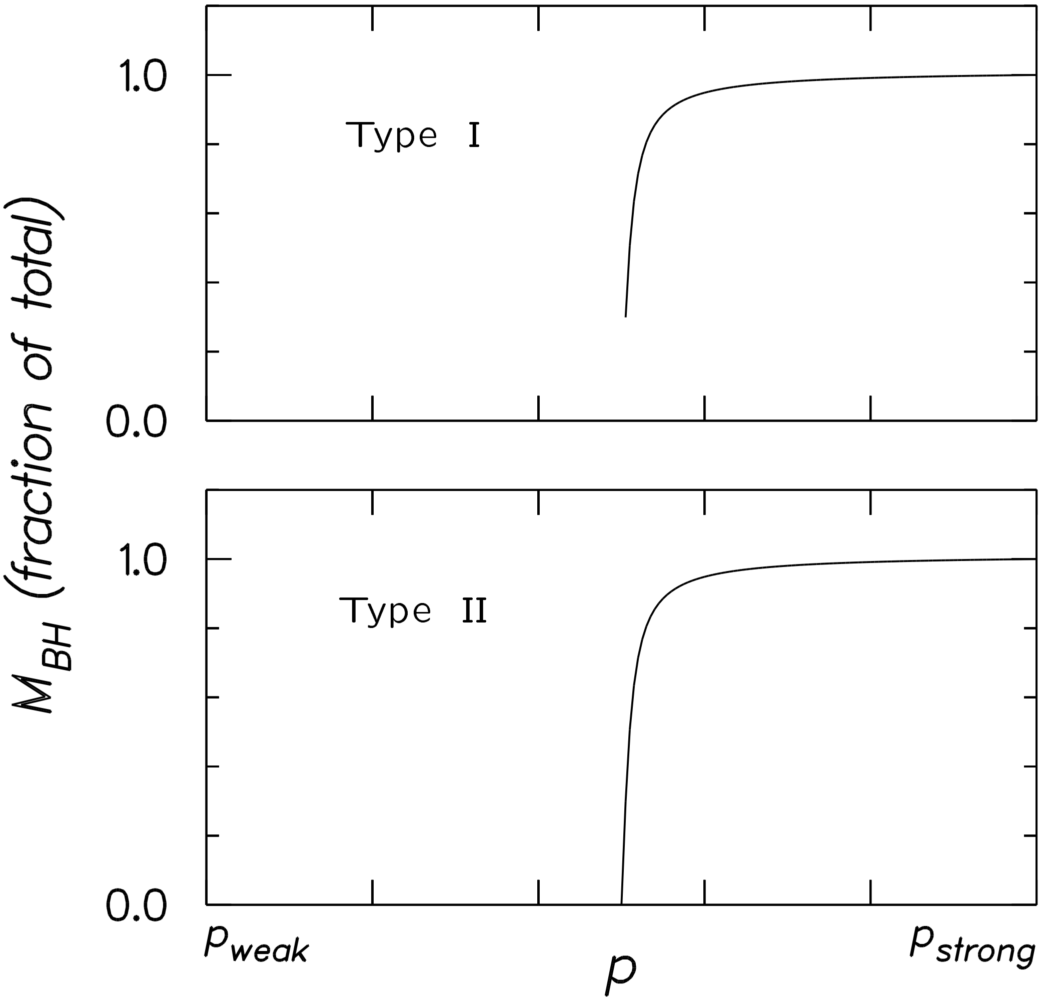
Massless Scalar Collapse: "Interpolating" Solutions



Key Question

(D. Christodoulou, 1987)

What is the behaviour of the black hole mass at a critical point for a generic interpolating family?



Critical Solutions

- Given collapse model (matter content, couplings, symmetry), find one or more isolated critical solutions, $g^\star$, of coupled Einstein/matter equations

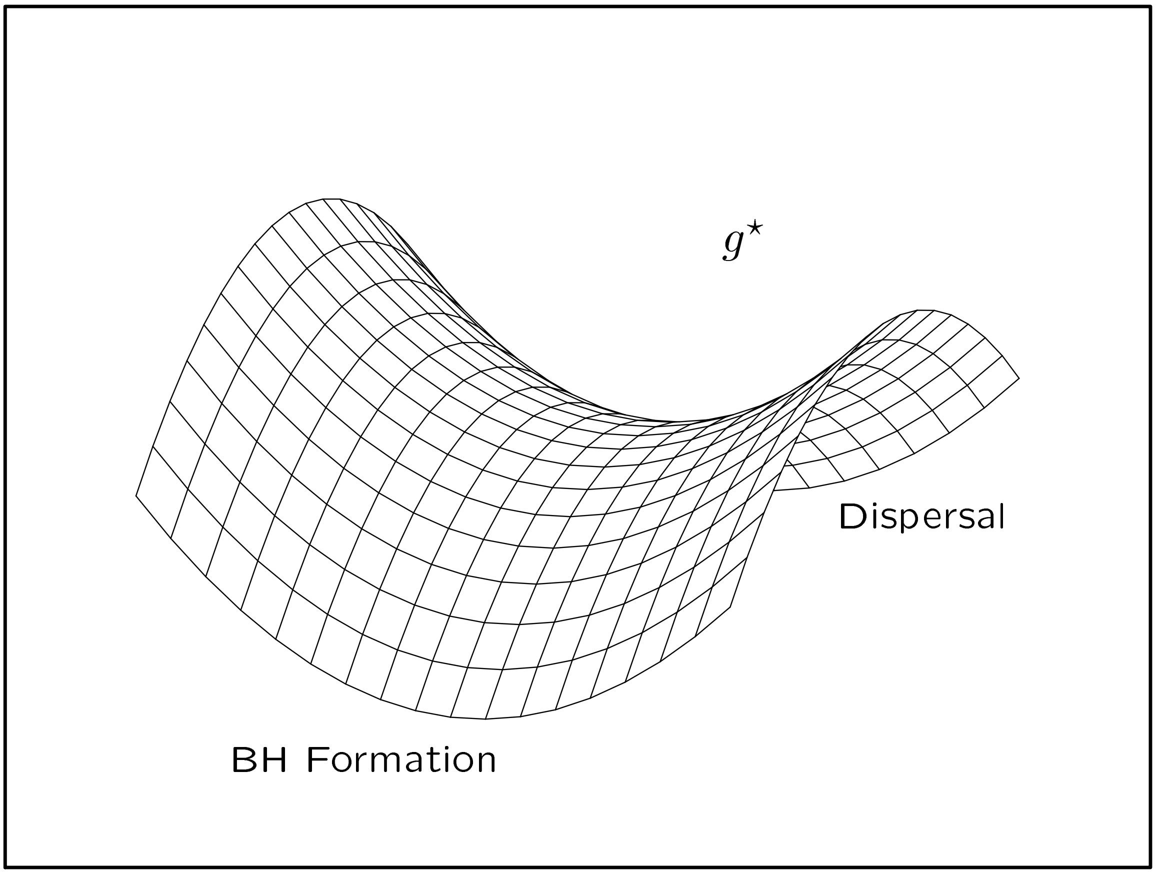
- Critical solutions are universal in the sense that they don't depend on the details of initial data $\Longrightarrow$ can be computed directly via appropriate *ansatz*

- Critical solutions are *UNSTABLE* by construction—lie on boundary between two *distinct*, stable end-states

- Consider perturbation theory about $g^\star$: Typically find precisely *one* unstable eigenmode—associated eigenvalue λ_1 has Re $\lambda_1 > 0$

(Evans & Coleman; Koike, Hara & Adachi; Maison; Eardley & Hirschmann; Gundlach)

Critical Solutions (cont.)



- Critical solutions are *intermediate attractors* (in language of **`Barenblatt`**)

- Tuning $p \to p^\star \quad \equiv \quad$ tuning away single unstable mode

Critical Phenomena—Type II Generic Features

- Arbitrarily small black holes possible (definition)
  - Sufficient fine-tuning of initial data can result in regions of spacetime with arbitrarily high curvature
  - Precisely critical solutions contain naked singularities

- Power-law scaling of black hole mass (and other dimension-full quantities)

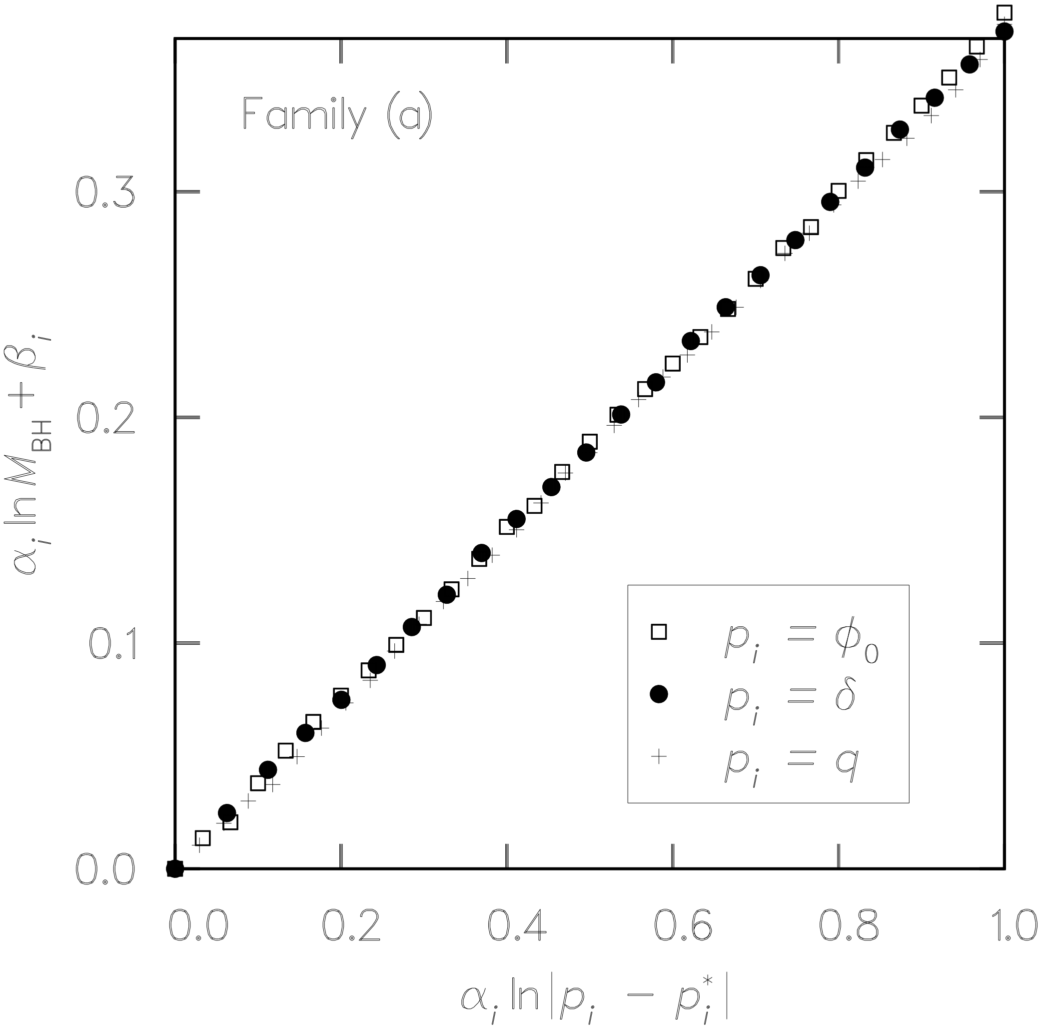
$$M_{BH} \sim c_f \, |p - p^\star|^{\gamma_m}$$

  where $\gamma_m = (\mathrm{Re}\ \lambda_1)^{-1}$

- Scaling behaviour of critical solution
  - Discrete self-similarity (DSS)
    - Massless scalar, gravitational, SU(2) Yang-Mills
    - Characterized by rescaling exponent: Δ_m
  - Continuous self-similarity (CSS)
    - Perfect fluid, mutiple-scalar systems

Type II Black Hole Mass Scaling (Scalar Collapse)

$$\phi\,(r,0) = \phi_0\, r^3 \,\mathrm{exp}\left(-\left(\left(r - r_0\right)/\delta\right)^q\right)$$



- Scaling provides remarkably good fit even when $> 90\%$ of pulse energy ends up in black hole $(\gamma_m \sim 0.37)$

Critical Phenomena—Type I Generic Features

- Smallest possible black hole has finite mass (definition)

- Critical solution is *static* or *periodic*

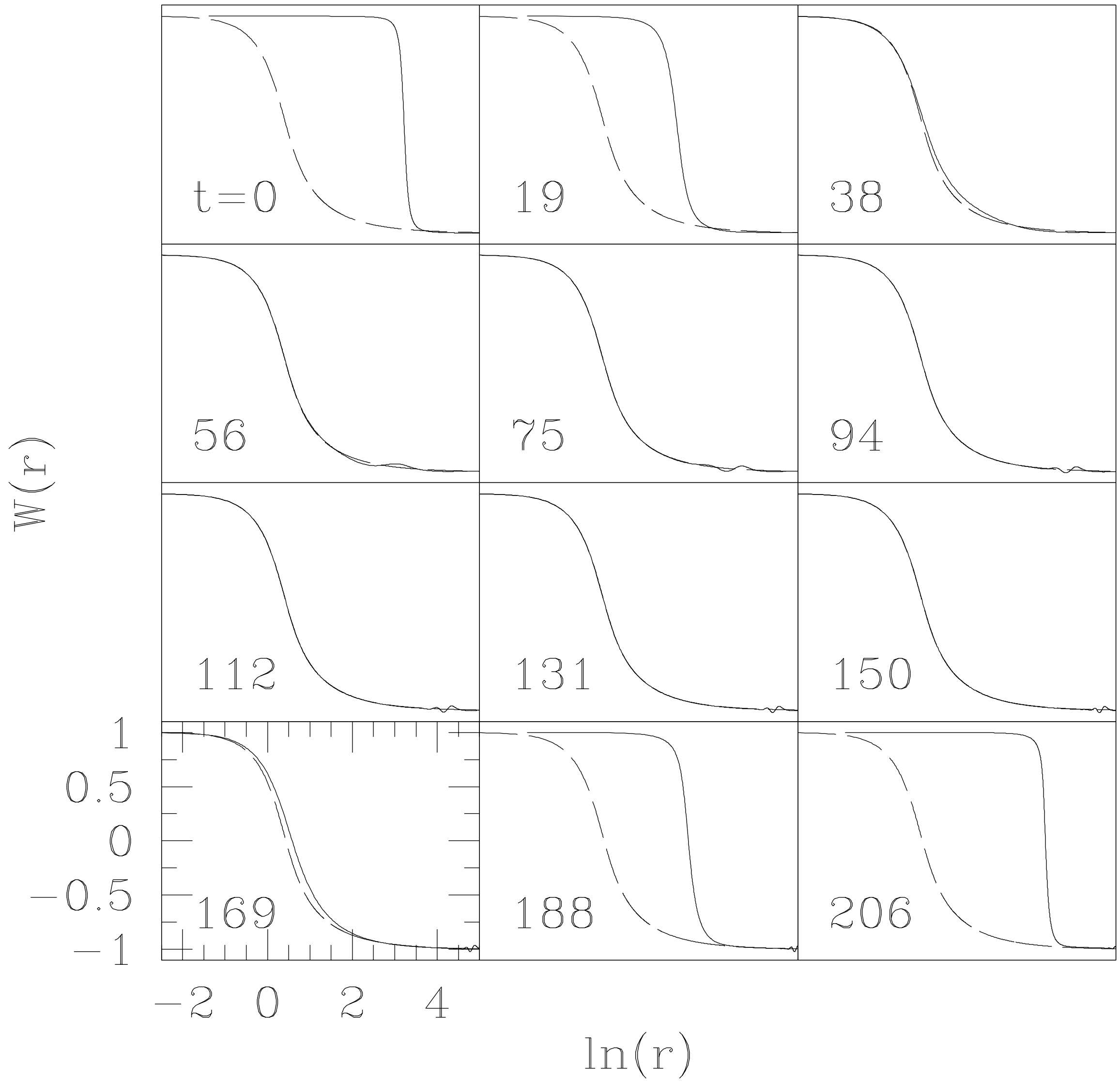
- Scaling law for *lifetime*, τ, of near-critical solution

$$\tau \sim -\sigma_m \ln |p - p^\star|$$

where $\sigma_m = \left(\mathrm{Re}\ \lambda_1\right)^{-1}$

- Observed in collapse of SU(2) YM field and massive scalar field(s), and expected for "non-ultra-relativistic" fluid collapse

Type I Near-Critical Evolution (Einstein-Yang-Mills)



- Intermediate attractor (critical solution) is *static*
- Lifetime: $\tau \sim -\sigma_m \ln|p - p^\star|$

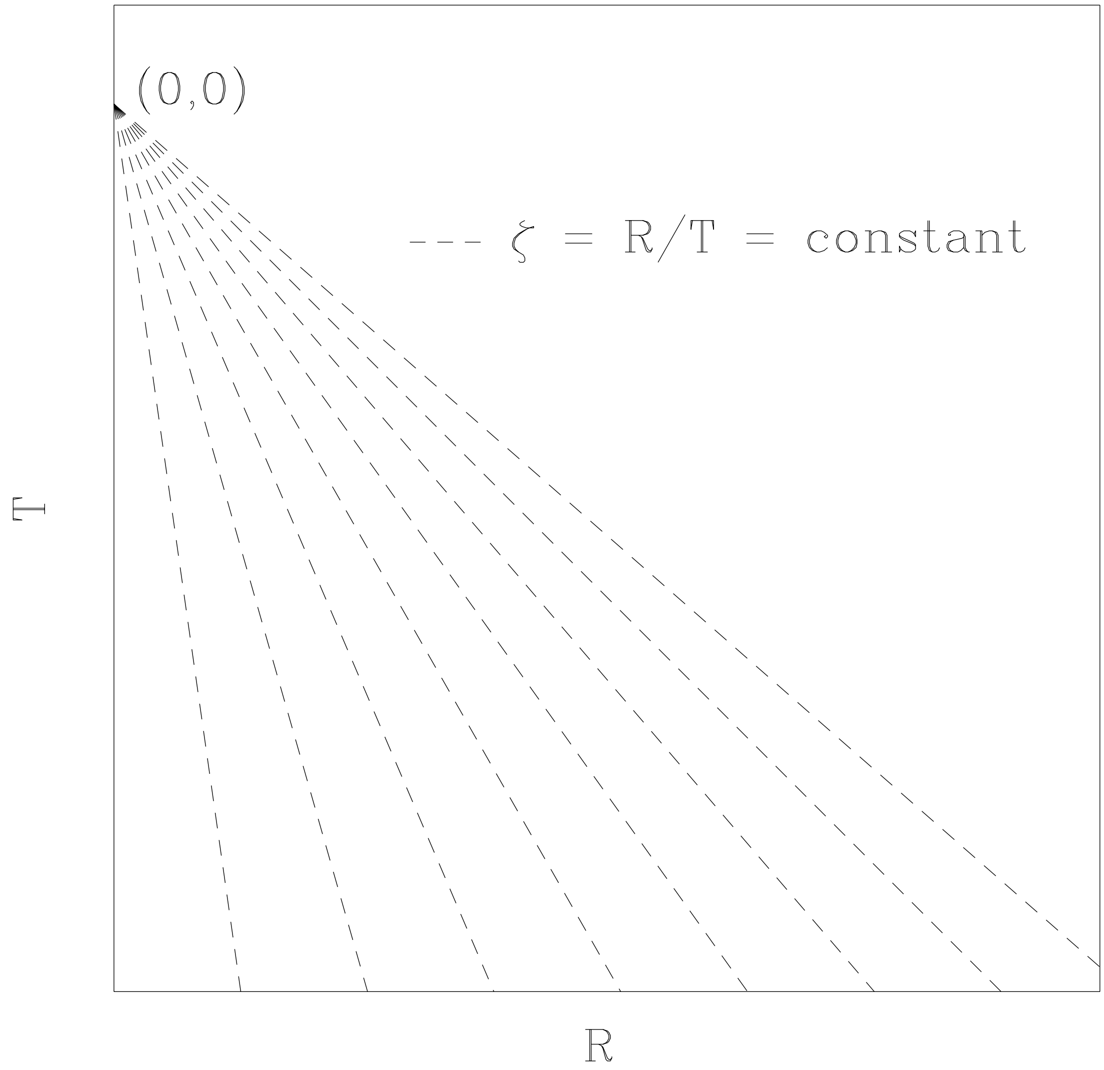
Critical Solutions—Extended "Definition"

One-mode unstable solution $\equiv$ *critical solution*

$g^\star$ will *not* necessarily sit at a No-BH—BH threshold

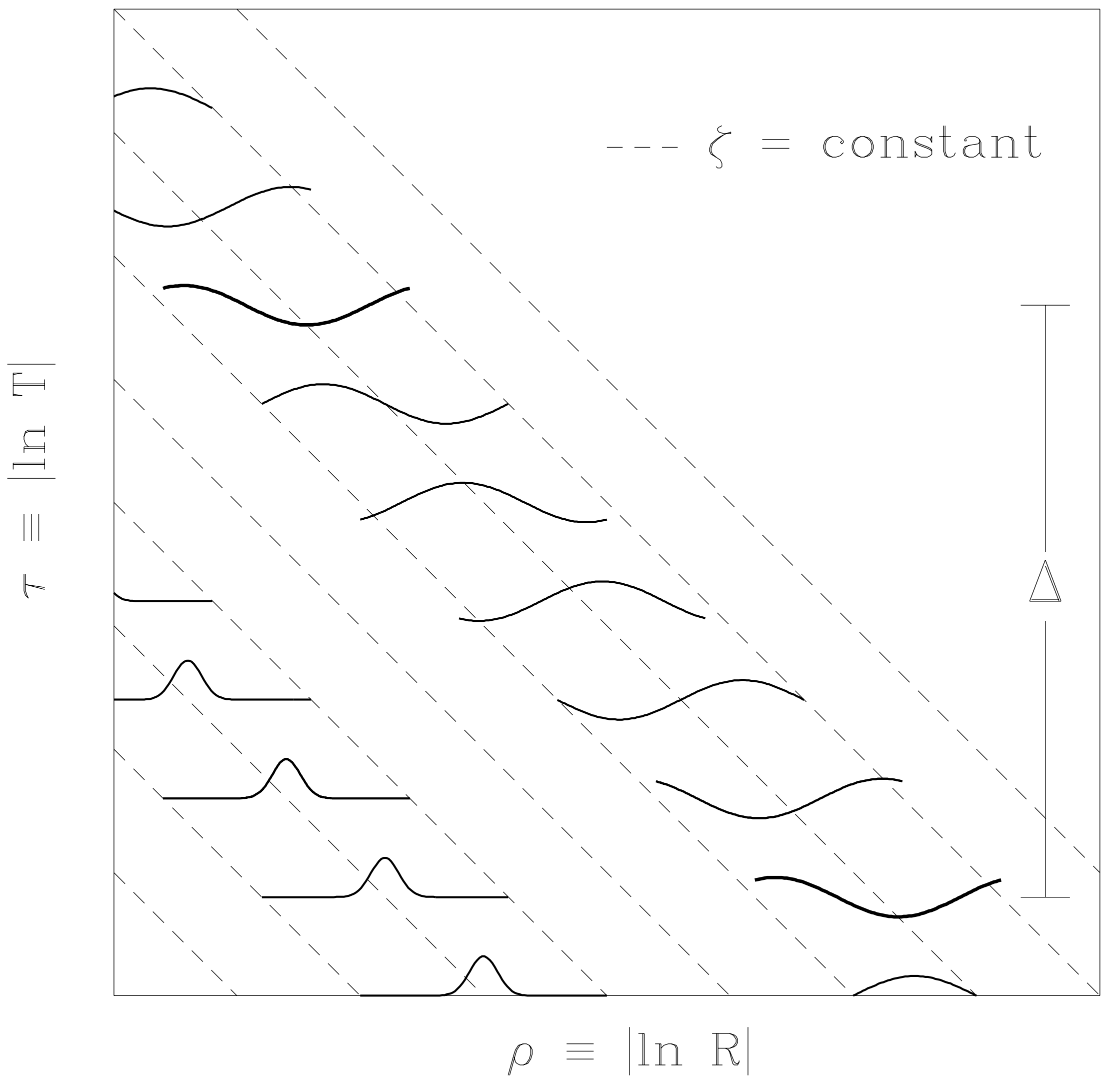
- In particular, now have example where $g^\star$ sits on the threshold between *two distinct black hole states* ($g^\star \equiv$ "hairy BH")

Self-Similarity: Continuous and Discrete



- $T \equiv$ central proper time (measured from "accumulation event")
- $R \equiv$ physical (e.g. areal) radius
- Similarity variable: $\zeta \equiv R/T$
- (Naked) singularity at $(0,0)$

Self-Similarity: Continuous and Discrete



- Continuously self-similar (CSS):

$$g^\star(\zeta,\tau) = g^\star(\zeta,\tau') \qquad \tau,\ \tau' \quad \text{arbitrary}$$

- Discretely self-similar (DSS):

$$g^\star(\zeta,\tau) = g^\star(\zeta,\tau+n\Delta) \qquad n = 0, \pm 1, \pm 2, \cdots$$

Δ—*model-dependent "echoing exponent"*

Perturbation Theory

(Evans & Coleman, Koike et al, Maison
Eardley & Hirschmann, Gundlach)

- For given model (matter content, couplings, symmetry), find one or more isolated critical (threshold) solutions, $g^\star$ via dynamic evolution of 1-parameter families

- $g^\star$ can generally be constructed "directly" from ansatz which reflects particular self-similarity exhibited by solution

- Consider perturbation theory using $g^\star$ as background—find eigenmodes, $f_i(\zeta)$, eigenvalues, λ_i, then as $p \to p^\star$

$$\delta g(\zeta, \rho\,;\, p) \;\equiv\; g(\zeta, \rho\,;\, p) - g^\star(\zeta, \rho) = \sum_{n=1}^{\infty} C_i(p) \exp\left(\lambda_i \rho\right)\; f_i(\zeta)$$

where the $C_i(p)$ are constants (expansion coefficients).

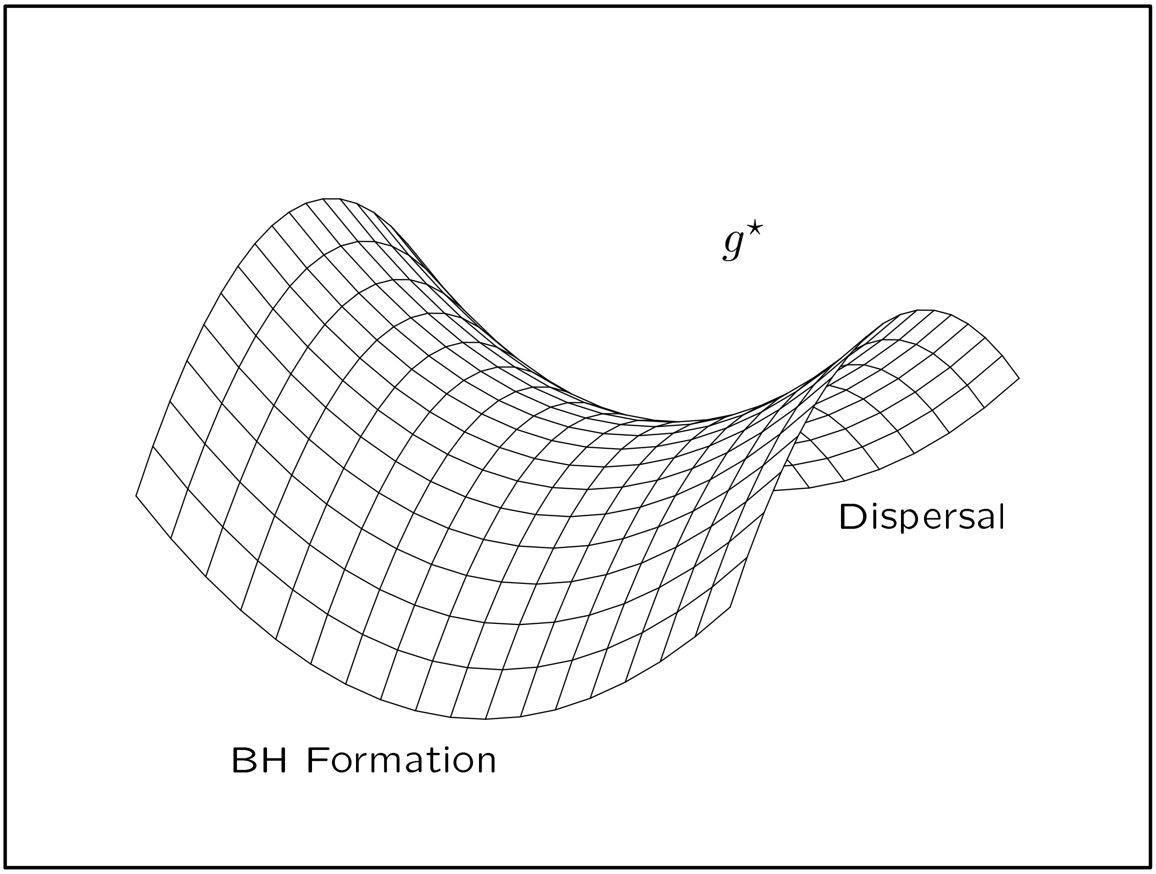
Perturbation Theory (cont.)

- Observed critical behaviour strongly suggests that *only one eigenmode is unstable:*

Re $\lambda_1 > 0$ single unstable mode

Re $\lambda_i < 0, \quad i = 2, 3, \cdots$ all other modes damp as $p \to p^\star$

(Assumes no marginally stable/relevant modes, Re $\lambda_i = 0$)



Perturbation Theory (cont.)

- Assuming Re $\lambda_1 = \lambda_1$, then as $p \to p^\star$

$$\delta g(\zeta, \rho\,;\, p) \;=\; C_1(p)\ \text{exp}\,(\lambda_1 \rho)\ f_1(\zeta)$$

$$\sim\; C_1'(p^\star) \quad (p - p^\star)\,\text{exp}\,(\lambda_1 \rho) \quad f_1(\zeta)$$

- Independent of scale, black hole formation (or "peel-off" to dispersal—differs only in sign of $(p - p^\star)$) is signaled by

$$(p - p^\star)\ \text{exp}\,(\,\lambda_1\,\rho(p)\,)\;=\;K\;=\;\text{constant}$$

and the black hole will form on the scale, $\rho(p)$:

$$\text{log}\,M_{BH}(p)\;=\;\text{log}(\frac{R_{BH}(p)}{2})\;=\;\rho(p) + k$$

Perturbation Theory (cont.)

- Scale of BH formation

$$\log M_{BH}(p) \;=\; \log(\frac{R_{BH}(p)}{2}) \;=\; \rho(p) + k$$

- Then

$$\exp\left(\log\left(p - p^{\star}\right)\right)\, \exp\left(\lambda_1 \log M_{BH}(p)\right) = K'$$

which is just the scaling law

$$M_{BH}(p) = c_f |p - p^{\star}|^{\gamma}$$

with

$$\gamma = \frac{1}{\mathrm{Re}\ \lambda_1}$$

Perturbation Theory (cont.)

- This picture of critical solutions as "intermediate attractors" (**Barenblatt**) with 1-dimensional unstable manifolds ("minimally unstable solutions") also explains observed universality of critical phenomena

- Validated for many spherically symmetric models, including
  - Perfect fluid ($p \propto \rho$): **Koike et al, Maison, Evans**
  - Axion-dilaton: **Eardley & Hirschmann, Hamade et al**
  - Non-linear σ: **Eardley & Hirschmann**
  - Massless scalar (DSS): **Gundlach**
  - SU(2) Einstein-YM: **Gundlach**
  - Scalar electrodynamics: **Gundlach, Martin-Garcia** (additional scaling law for charge)

Perturbation Theory (cont.)

- Extensions

  - General analysis predicts "wiggle" in scaling laws for DSS transitions (**Gundlach; Hod & Piran**)

  - Other dimension-full quantities (e.g. curvature scalars) scale near criticality (**Garfinkle & Duncan**)

  - Further analogues to statistical mechanical critical phenomena via study of dominant *decaying* modes (**Gundlach via Goldenfeld**)

  - Studies of non-spherical perturbations strongly suggest that at least some of the Type II solutions will persist when spherical symmetry is relaxed (following axisymmetric vacuum collapse by **Abrahams & Evans**)

    - Perfect fluid (**Gundlach**)

    - Massless scalar (**Gundlach, Martin-Garcia, Garfinkle**)

    - Predicts scaling of angular momentum (**Gundlach**)

Type II Critical Solutions–Summary

Model	d	γ	Δ
Massless Scalar*	2	0.37	3.4
SU(2) EYM	2	0.20	0.74
Perfect Fluid*	2	varies	∞
Non-Linear σ	2	varies	3.4, ∞
Gravitational Waves	3	0.36	0.6

* evidence from perturbation theory that critical solution is stable to small deviations from spherical symmetry

SU(2) Einstein-Yang-Mills Collapse

- Action for Einstein-Yang-Mills (EYM) theory:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{g^2}F^a_{\mu\nu}F^{a\mu\nu}\right]$$

- Einstein equations:

$$\begin{aligned}\frac{1}{16\pi G}G_{\mu\nu} =& T_{\mu\nu} \\ =& \frac{1}{g^2}(2F^a_{\mu\lambda}F^{a}{}_{\nu}{}^{\lambda} - \frac{1}{2}g_{\mu\nu}F^a_{\alpha\beta}F^{a\alpha\beta})\end{aligned}$$

- Yang-Mills field equations:

$$D_\mu F^{a\mu\nu} = 0$$

SU(2) EYM in Spherical Symmetry: Magnetic Ansatz

- Two kinds of charge density: electric & magnetic

- Purely-magnetic + "hedge-hog" *ansatz* self-consistent with spherical symmetry (static and dynamic)

  - Yang-Mills dynamics completely described by single function:

$$W(r,t)$$

    (specific component of YM connection)

  - Effective Lagrangian

$$\mathcal{L}_W \;\sim\; \frac{1}{r^2}\left(\, \nabla_\mu W\, \nabla^\mu W \;+\; \frac{\left(1-W^2\right)^2}{2r^2}\,\right)$$

  - Regularity at $r=0$ & finite mass as $r\to\infty$

$$W(0,t)=\pm 1 \qquad\qquad W(\infty,t)=\pm 1$$

Coordinate Systems and Black Hole Excision

- **Traditional Approach:** (DeWitt et al)
  - Use coordinate freedom (usually time slicing), to "freeze out" evolution in vicinity of physical singularity
  - Coordinate singularities appear *inevitable*. Simulations still crash on a dynamical time scale

- **Excising Approach:** **(Unruh, Thornburg, Seidel/Suen, here)**
  - No need to evolve interior of BHs—so simply excise from domain
  - Track apparent horizon or interior thereof to ensure excised region remains strictly inside BHs
  - Boundary of excised region is super-characteristic: *no* special boundary conditions required, EOM apply up to and including boundary of computational domain

- Not all coordinate choices are compatible with excising
  - Good choices exist for 1D.

Coordinate Systems and BH Excision (cont.)

I) Polar/Areal Coordinates

- **slicing condition:** $K = K^r{}_r$ (polar slicing)
- **spatial coordinate condition:** areal coordinates

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$$

- Natural generalization of Schwarzschild coordinates: *cannot* penetrate apparent horizons, thus cannot use with excision.
- Minimizes algebraic complexity of Einstein equations, easy to design stable finite-differencing schemes

II) Maximal/Areal Coordinates

- **slicing condition:** $K = 0$ (maximal slicing)
- **spatial coordinate condition:** areal coordinates

$$ds^2 = \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2a^2\beta\, dt dr + a^2 dr^2 + r^2 d\Omega^2$$

- Coordinates *CAN* penetrate apparent horizons, can use with excision.

Static Solutions: Regular Configurations

(Bartnik and Mckinnon (1988))

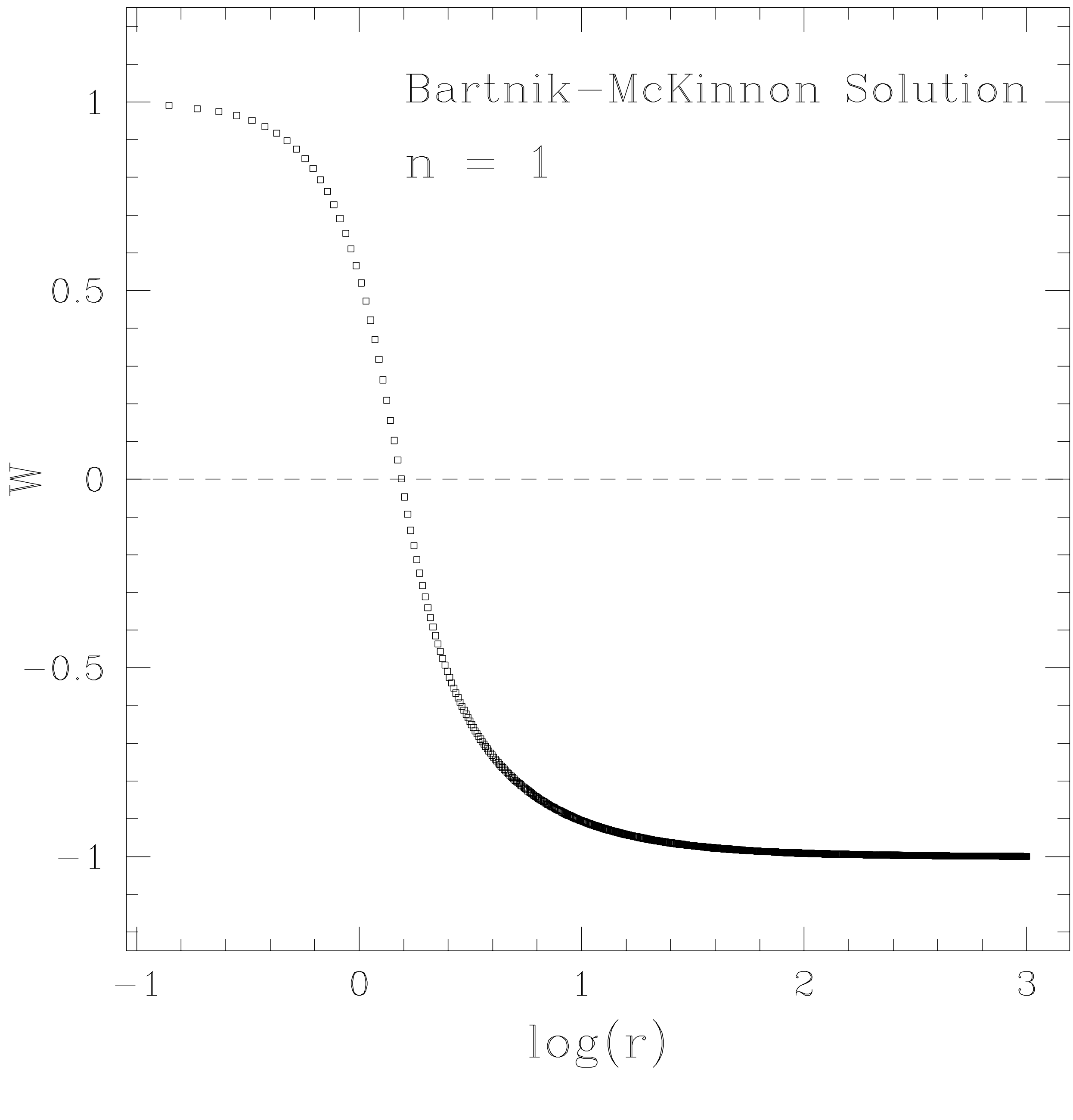
- Looked for static solutions in purely-magnetic model

- Found countable infinity of regular (at $r = 0$) solutions which can be labeled by the number, n, of zero-crossings of $W(r)$

- Somewhat surprising result since neither component theory (Einstein, Yang-Mills) admits regular static solutions: heuristically, non-linearity of YM field provides effective self-repulsion to "balance" gravity; non-trivial vacuum structure plays important role

- Was soon shown that all solutions, $W_n(r)$, are unstable with precisely n unstable modes in the context of the magnetic ansatz, and $2n$ unstable modes in the context of the more general description

$n = 1$ **Bartnik-Mckinnon Solution**



Static Solutions: Colored Black Holes

(Bizon (1990); Volkov and Gal'tsov (1990))

- Same assumptions which lead to Bartnik/Mckinnon solutions, but this time spacetime has a horizon rather than regular center

- Again find countably infinite families of solutions (again labeled by number of zero-crossings of $W(r)$), but now there is a *continuum* of such families, rather than just one

- Each family characterized by mass of black hole (radius of horizon) which effectively becomes a freely specifiable parameter

- Solutions violate the letter (if not the spirit) of the no-hair theorems—are black holes dressed with non-trivial YM configurations (YM "hair") a.k.a. *colored black holes*

- Also shown to be unstable with n or $2n$ unstable modes, depending on whether one works in the purely magnetic, or general framework

BH Threshold Critical Behaviour (magnetic ansatz)
`(Choptuik, Bizon and Chmaj (1996))`

- Studied dynamics of collapsing YM field, in polar/areal coordinates; found two distinct types of black-hole threshold behaviour:
  - **Type I behaviour**: As $p \to p^\star$, dynamics asymptotes to $n = 1$ Bartnik-Mckinnon solution, with "lifetime" of configuration given by

    $$\tau \approx -\sigma \ln|p - p^\star|$$

    where $\sigma \approx 0.552$ is the reciprocal Lyapunov exponent of the single unstable mode for the $n = 1$ configuration.

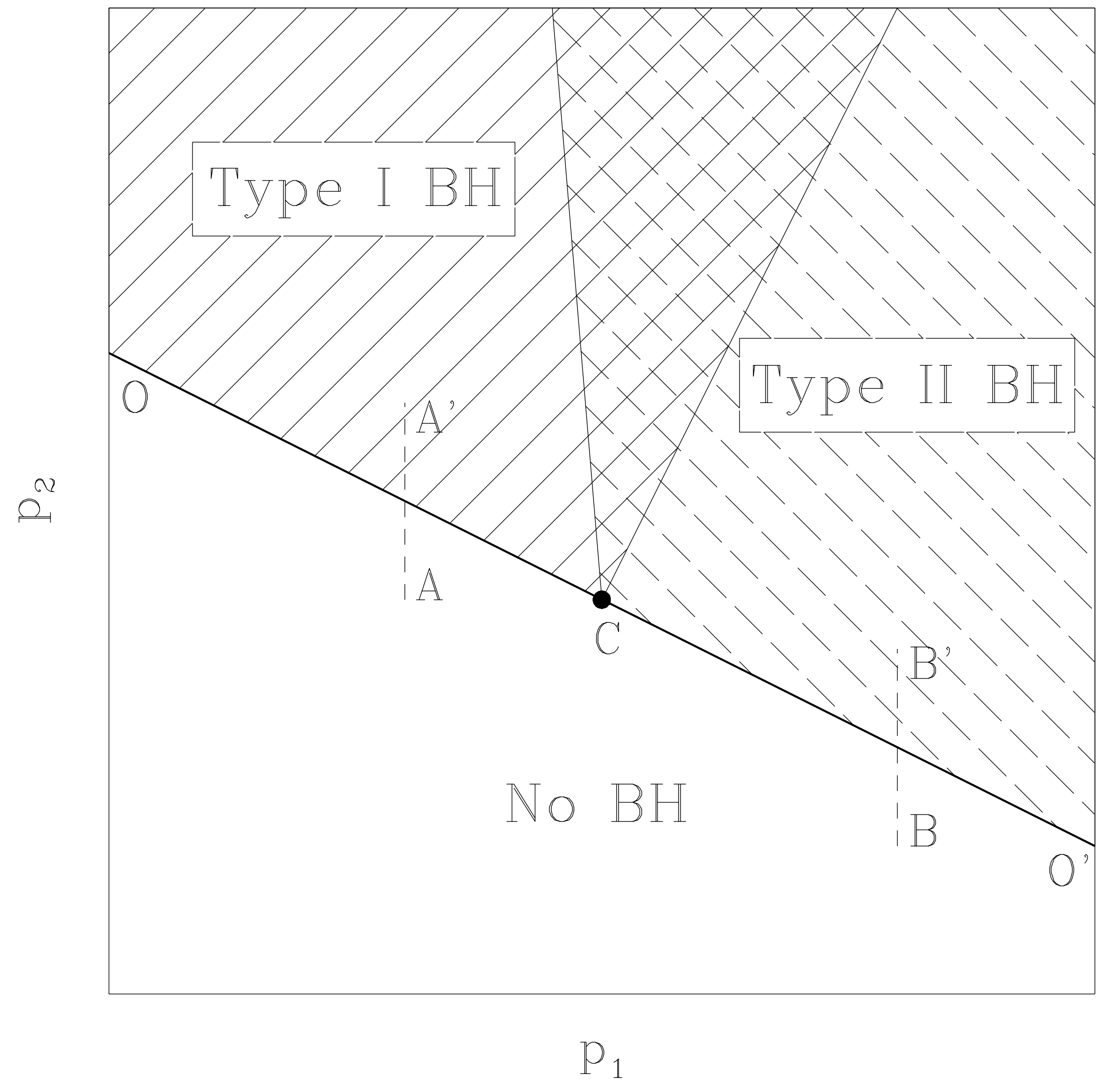
  - **Type II behaviour**: As $p \to p^\star$, dynamics asymptotes to discretely self-similar solution; black hole masses satisfy a scaling law

    $$\ln M_{BH} \approx \gamma \ln|p - p^\star|$$

    where $\gamma \approx 0.20$ is the reciprocal Lyapunov exponent of the single unstable mode for the DSS "intermediate attractor".

- Found that suitable *two* parameter families of initial data could generate both types of behaviour, and that transition from Type I to Type II was characterized by co-existence of the "phases".

Phase-Space Schematic for Magnetic EYM Collapse
(**Choptuik, Bizon and Chmaj (1996)**)



- The two types of threshold behaviour co-exist at C.

- Similar picture also established for *massive* scalar collapse (**Brady, Chambers & Goncalves (1997)**)

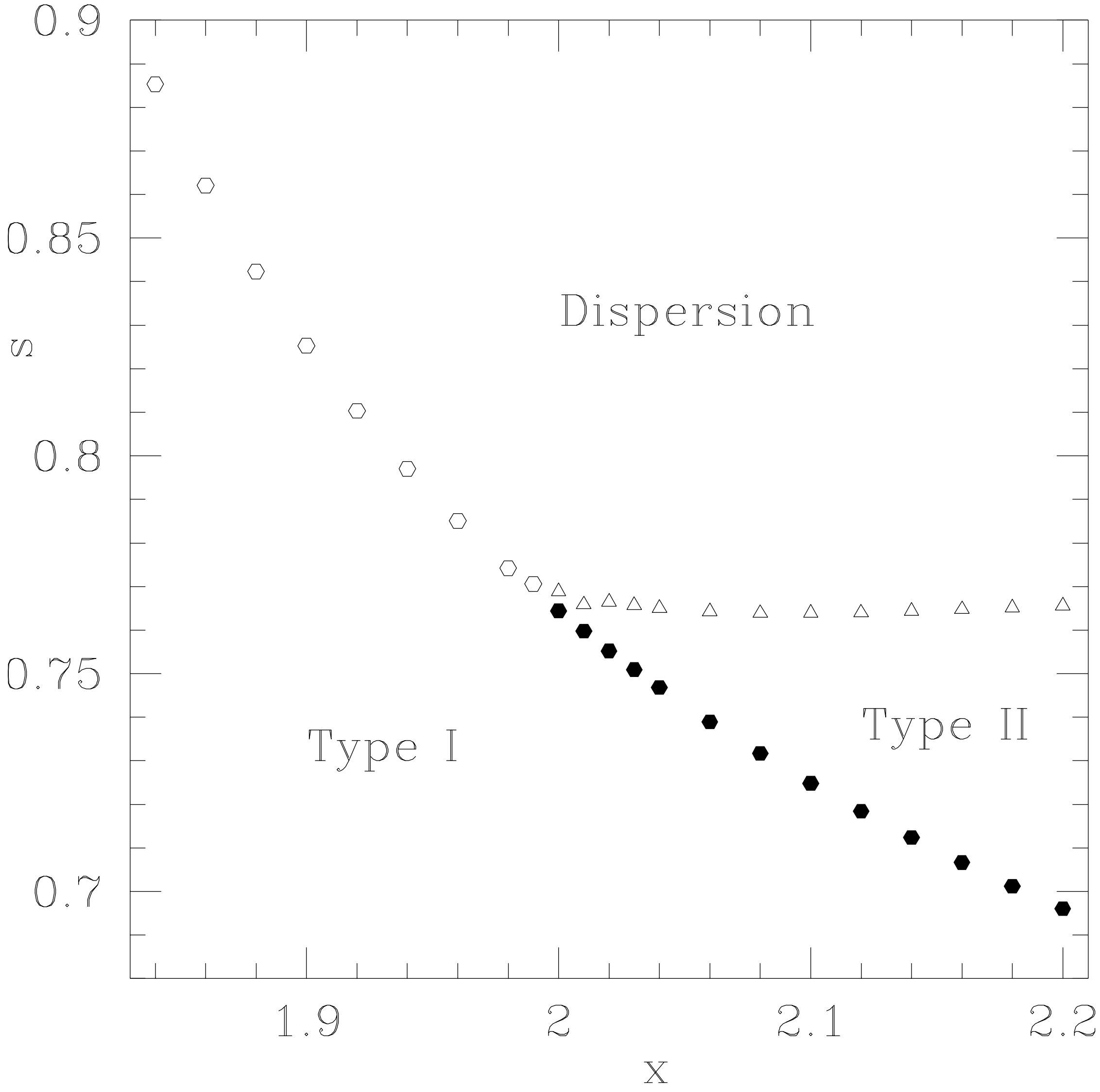
Forming Colored ("Hairy") BHs via Fine Tuning

(**Choptuik, Hirschmann and Marsa** (gr-qc/9903081))

- Still work in magnetic ansatz, but now use maximal/areal coordinates and black-hole excision techniques so that BH regime can be efficiently studied

- Find new type of critical behaviour: as $p \to p^\star$, dynamics asymptotes to an $n = 1$ colored black hole, lifetime is again related to the reciprocal Lyapunov exponent of the single unstable mode (**Bizon & Chmaj (gr-qc/9906070)**)

New Phase-Space Schematic for Magnetic EYM Collapse



Open Questions & Ongoing Research

- Relaxing spherical symmetry
  - Axisymmetry
    (with **Hirschmann & Liebling** and **Garfinkle**)
  - Full 3D
    (could come along relatively quickly—ideal test application for 3D AMR)
- Other matter models
- Quantum effects
  - Do Type II solutions *demand* to be quantized?
  - What does the semi-classical approximation really say about critical behaviour
  - Does BH threshold game make sense in context of contemporary QG theories (i.e. strings, quantum geometry), and if so, what are predictions?
- Investigation/exploitation of phenomenology/analytic approaches from condensed matter theory (**Goldenfeld, Gundlach**)