Hypergeometric functions and Mahler measure

by

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Abstract

The (logarithmic) Mahler measure of an n-variable Laurent polynomial, $P(x_1, \ldots, x_n)$, is defined by

$$\mathrm{m}\left(P\right) = \int_0^1 \cdots \int_0^1 \log\left|P\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right)\right| \mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

Using experimental methods, David Boyd conjectured a large number of explicit relations between Mahler measures of polynomials and special values of different types of L-series. This thesis contains four papers which either prove or attempt to prove conjectures due to Boyd. The introductory chapter contains an overview of the contents of each manuscript.

Table of Contents

Statement of Co-Authorship

Chapter 3 of this thesis was written jointly with Matilde Lalín. Section 3.1 was prepared jointly, while the research and preparation of Section 3.2 is due to myself, and the research and preparation of Section 3.3 is due to Lalín. The project was initiated by my rediscovery of equation (3.1.6), and by my later proof of equation (3.1.7).

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Finally, I would like to thank my family for their unfailing support.

Dedication

I would like to dedicate this thesis to my mother for her love and support.

Chapter 1

Introductory Chapter

1.1 Introduction

Constants such as π and $\zeta(3)$ occupy a special place in the history of number theory. In the past, famous mathematicians including Newton, Euler, Cauchy and Ramanujan expended a great deal of energy searching for efficient ways to calculate these numbers. They were motivated by more than simple curiosity, as they lacked the calculators that we now take for granted. Research in this direction helped spur the development of a variety of important mathematical tools, including calculus, complex analysis, the theory of elliptic functions, and the theory of hypergeometric functions. This thesis falls under the final category, and can loosely be described as an effort to obtain new hypergeometric formulas for special values of L-functions.

The generalized hypergeometric function is usually defined in terms of a power series involving Pochhammer symbols (which are also known as generalized factorials). The Pochhammer symbol is defined by

$$(x)_n := \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)\dots(x+n-1) & \text{if } n \geq 1, \end{cases}$$

and the generalized hypergeometric function can be written as

$${}_pF_q\left(\substack{a_1,\dots,a_p\\ b_1,\dots,b_q}; x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{x^n}{n!}.$$

By specializing the a_i's and b_i's we can reduce many elementary and special functions to hypergeometric functions. Bessel functions of the second kind are probably the most important non-hypergeometric special functions.

It is a classical fact that hypergeometric functions satisfy interesting transformations: Dixon's, Whipple's and Saalschutz's theorems are all examples [2]. One consequence of this fact is that new series expansions are continually being discovered for familiar special constants. While Nicolas Mercator discovered the first infinite series for $\log(2)$ in 1668, we recently found a new identity via numerical searches:

$$ {}_6F_5\left(\substack{\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1,\frac{32}{15}\\ \frac{5}{2},2,2,2,\frac{17}{15}};\frac{1}{16}\right) \stackrel{?}{=} -\frac{2112}{17}+\frac{3072}{17}\log(2). \tag{1.1.1} $$

In equation (1.1.1), and throughout the rest of this thesis, we will use "$\stackrel{?}{=}$" to denote an unproven equality that holds to at least 60 decimal places. Equation (1.1.1) exemplifies many hypergeometric formulas, in that the simplicity of the identity obscures the reason why it exists at all. The Borwein brothers tackled a similar problem when they proved Ramanujan's seventeen formulas for $1/\pi$ (see [22], [9], and [15]). While the theory of elliptic functions ultimately underpins Ramanujan's claims, equation (1.1.1) is equivalent to an unproven conjecture about Mahler measures of elliptic curves.

This thesis stems from the observation that Mahler identities are intimately tied to formulas for generalized hypergeometric functions. The (logarithmic) Mahler measure of an n-variable Laurent polynomial, $P(x_1,\ldots,x_n)$, is defined by

$$ \mathrm{m}(P) := \int_0^1 \cdots \int_0^1 \log\left|P\left(e^{2\pi\mathrm{i}\theta_1},\ldots,e^{2\pi\mathrm{i}\theta_n}\right)\right| \mathrm{d}\theta_1\ldots\mathrm{d}\theta_n. $$

Boyd conjectured a large number of interesting relations between values of L-series of elliptic curves and Mahler measures of polynomials in [10]. For example, he showed that the following conjecture (which Denninger also considered [17]):

$$ \mathrm{m}\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right) \stackrel{?}{=} \frac{15}{4\pi^2}L\left(E,2\right), \tag{1.1.2} $$

with E having conductor 15, holds to at least 60 decimal places. Since the majority of Boyd's conjectures translate into hypergeometric identities, this gives rise to an entirely new class of *closed form evaluations* of the generalized hypergeometric function. The L-series of an elliptic curve is a fundamental arithmetic quantity, hence identities like equation (1.1.2) should be regarded with the same interest as most formulas for π or $\zeta(3)$.

1.2 Mahler measure background

The field of Mahler measure encompasses the study of both single-variable, and multi-variable polynomials. The first area is closely related to the study of heights of polynomials. Jensen's formula provides the operative result:

$$\mathrm{m}\left(P(x)\right) = \sum_{\substack{|\alpha|>1 \\ P(\alpha)=0}} \log|\alpha|,$$

by relating Mahler measures of monic polynomials to their zeros. Single-variable Mahler measures often appear in studies of Salem and Pisot numbers, and are especially relevant for Lehmer's problem:

Lehmer's Problem: Does there exist a monic, non-cyclotomic polynomial $P(x) \in \mathbb{Z}[x]$, with $P(0) \neq 0$, such that $\mathrm{m}(P(x)) < \mathrm{m}\left(P_0(x)\right) = \log\left(1.17\ldots\right)$, where $P_0(x) = 1 - x + x^3 - x^4 + x^5 - x^6 + x^7 - x^9 + x^{10}$?

In general, the Boyd-Lawton theorem shows that multi-variable Mahler measures arise as limit points of sequences of single-variable Mahler measures [11]. For example, in the two-dimensional case we have

$$\lim_{n\to\infty} \mathrm{m}\left(P\left(x, x^n\right)\right) = \mathrm{m}\left(P(x,y)\right). \tag{1.2.3}$$

Unfortunately, it is unreasonable to expect a single result like Jensen's formula to describe all of the interesting identities for multi-variable Mahler measures. Formulas exist which involve special values of polylogarithms, elliptic dilogarithms, Riemann and Dedekind zeta functions, Dirichlet and elliptic curve L-functions, and even L-series of $K3$ hypersurfaces. In general, it is usually far easier to prove identities between Mahler measures, than to actually relate them to L-functions [23].

1.3 An overview of the manuscripts in this thesis

Despite the fact that no one has proved (1.1.2), several special cases of Boyd's conjectures have been rigorously established. Rodriguez-Villegas demonstrated that Boyd-like identities hold for elliptic curves with complex multiplication [24], and Brunault proved several of Boyd's conjectures for elliptic curves with prime conductors [13]. This thesis contains four papers aimed at establishing results related to Boyd's conjectures (see [26], [25], [19], and [27]). In the remainder of this section we will briefly outline some of the main theorems in those manuscripts.

Lattice sums and Mahler measures

While a variety of procedures exist for relating Mahler measures to values of the Riemann zeta function (see [18] or [12]), the situation is more complicated for L-functions of elliptic curves. Rodriguez-Villegas proved several of Boyd's conjectures for CM elliptic curves; however, his proofs depended upon Deuring's theorem, which only applies in fortuitous situations. In this paper, we have used the modularity theorem to obtain a variety of formulas relating L-functions to lattice sums. This enabled us to translate many of Boyd's conjectures into equivalent, although still unproven, relations between lattice sums and hypergeometric functions. For example, equation (1.1.2) is equivalent to the following formula:

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{(1/16)^{2n+1}}{2n+1} \stackrel{?}{=} \frac{540}{\pi^2}\sum_{\substack{n_i=-\infty\\ i\in\{1,2,3,4\}}}^{\infty}\frac{(-1)^{n_1+n_2+n_3+n_4}}{\left((6n_1-1)^2+3(6n_2-1)^2+5(6n_3-1)^2+15(6n_4-1)^2\right)^2}. \tag{1.3.4}$$

Despite the fact that we were unable to prove (1.3.4), we successfully used our ideas to recover Rodriguez-Villegas's results, and several corollaries. For instance, if ϕ equals the golden ratio, then

$$\frac{3456}{\sqrt{15}}\sum_{\substack{n_i=-\infty\\ i\in\{1,2,3,4\}}}^{\infty}\frac{(-1)^{n_1+n_2+n_3+n_4}}{\left[(6n_1+1)^2+(6n_2+1)^2+(6n_3+1)^2+\frac{1}{5}(6n_4+1)^2\right]^2} = \frac{C_1}{\sqrt[3]{\phi}}\,{}_3F_2\left({\frac{1}{3},\frac{1}{3},\frac{1}{3} \atop \frac{2}{3},\frac{4}{3}};\frac{1}{\phi}\right)+\frac{C_2}{\sqrt[3]{\phi^2}}\,{}_3F_2\left({\frac{2}{3},\frac{2}{3},\frac{2}{3} \atop \frac{4}{3},\frac{5}{3}};\frac{1}{\phi}\right), \tag{1.3.5}$$

where $C_1 = 2\sqrt[3]{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{6}\right)$, and $C_2 = 3\sqrt{3}\Gamma^3\left(\frac{2}{3}\right)$. The paper concludes with a brief discussion of elliptic dilogarithms and higher dimensional lattice sums.

Functional equations for Mahler measures

Matilde Lalín and I proved functional equations for Mahler measures of elliptic curves in [19]. For example, we proved that the following identity holds for $|\alpha| < 1$:

$$\begin{aligned}\mathrm{m}\left(\frac{4}{\alpha^2}+x+\frac{1}{x}+y+\frac{1}{y}\right) =&\mathrm{m}\left(2\alpha+\frac{2}{\alpha}+x+\frac{1}{x}+y+\frac{1}{y}\right)\\ &+\mathrm{m}\left(2\mathrm{i}\alpha+\frac{2}{\mathrm{i}\alpha}+x+\frac{1}{x}+y+\frac{1}{y}\right).\end{aligned}$$

Our theorems enabled us to express $\mathrm{m}\left(2+x+x^{-1}+y+y^{-1}\right)$ and $\mathrm{m}\left(8+x+x^{-1}+y+y^{-1}\right)$ in terms of $L(E,2)$, where E has conductor 24 (subject to a claim of Rodriguez-Villegas). Our proofs depended upon Ramanujan's theory of modular equations to alternative bases [6], as well as formulas for the Rogers-Ramanujan continued fraction [3], and q-series results of Stienstra [28] and Verrill [29]. As a final corollary, we obtained new ${}_2F_1$ transformations which Maier also studied [20].

New ${}_5F_4$ transformations and Mahler measures

Often it is quite difficult to identify Mahler measures as hypergeometric functions. This is true for both two-variable and three-variable Mahler measures. In [27] I used modular equations to equate several three variable Mahler measures to linear combinations of ${}_5F_4$ functions. For example, for $|\alpha|$ sufficiently large:

$$\begin{aligned}&\mathrm{m}\left(3\alpha+\frac{3}{\alpha}+x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}\right)\\ &\quad=-\frac{1}{20}\sum_{n=1}^{\infty}\frac{1}{n}\frac{(4n)!}{n!^4}\left(\frac{\alpha^3}{3\left(3+\alpha^2\right)^2}\right)^{2n}-\frac{3}{20}\sum_{n=1}^{\infty}\frac{1}{n}\frac{(4n)!}{n!^4}\left(\frac{\alpha^{-3}}{3\left(3+\alpha^{-2}\right)^2}\right)^{2n}\\ &\qquad+\frac{1}{5}\log\left(9\alpha^3\left(3+\alpha^2\right)\left(3+\alpha^{-2}\right)^3\right).\end{aligned}$$

This particular identity allowed me to translate several of Bertin's formulas for L-series $K3$ hypersurfaces into explicit hypergeometric formulas [8]. As a corollary to my hypergeometric transformations, I deduced several new formulas for $1/\pi$ involving Domb numbers, including:

$$\frac{2}{\pi}=\sum_{n=0}^{\infty}(-1)^n\frac{(3n+1)}{32^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2.$$

Similar identities for $1/\pi$ and $1/\pi^2$ have been studied in [14], [30], and [31]. Zudilin summarized a variety of related results in [32].

Trigonometric integrals and Mahler measures

While it is difficult to speculate on the nature of corollaries that might emerge from a proof of (1.1.2), I will point to the fourth paper in this thesis for an example. Several years ago Boyd conjectured an identity expressing a Mahler measure in terms of $\zeta(3)/\pi^2$. In particular, he calculated that

$$\frac{14}{5\pi^2}\zeta(3) = \mathrm{m}\left((1+x)+(1-x)(y+z)\right). \tag{1.3.6}$$

Condon first established this formula using contour integration [16]. After considering Condon's proof, I realized that (1.3.6) followed from an identity for the ${}_4F_3$ hypergeometric function, and I proceeded to discover a proof based upon a new series transformation [25]:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3\binom{2n}{n}}\left(\frac{4r}{1-r^2}\right)^{2n+1}& \\ =&\frac{1}{2}\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_3(1-r)+4\mathrm{Li}_3\left(\frac{r}{1+r}\right)-4\zeta(3)\\ &-\log\left(\frac{1+r}{1-r}\right)\mathrm{Li}_2\left(r^2\right)-\frac{2\pi^2}{3}\log(1-r)-\frac{2}{3}\log^3(1+r)\\ &+2\log(r)\log^2(1-r),\end{aligned}$$

which holds for appropriate values of r. As usual, the polylogarithm is defined by

$$\mathrm{Li}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}.$$

Transformations for sums involving binomial coefficients have attracted interest since Apéry proved the irrationality of $\zeta(2)$ and $\zeta(3)$ using such identities (see [1], [4], or [5]). As a corollary to my hypergeometric transformation I also proved eight new Mahler measure formulas. Thus, the solutions of numerically conjectured problems often lead to unexpected and interesting corollaries.

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Chapter 2

Lattice sums and Mahler measures

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2.1 Introduction

In this paper we will prove a number of formulas relating the special values of L-series of elliptic curves to hypergeometric functions. This paper was partially inspired by the work of Boyd and Rodriguez-Villegas. Recall that Boyd used numerical methods to conjecture a large number of formulas relating the L-series of elliptic curves to special values of Mahler's measure. The first example of such an identity was due to Deninger, who hypothesized that

$$\mathrm{m}\left(1+y+y^{-1}+z+z^{-1}\right) \stackrel{?}{=} \frac{15}{4\pi^2}L(E,2), \tag{2.1.1}$$

where E is a conductor 15 elliptic curve. Although we will not offer a proof of Deninger's formula in this paper, we will provide a new method for establishing results due to Rodriguez-Villegas, and we are hopeful that our method will eventually apply to formulas such as (2.1.1). The essential result that we will require is the modularity theorem, which supplants Deuring's theorem in Rodriguez-Villegas's work.

Let us briefly recall that to any elliptic curve E with conductor N, we can associate an L-series

$$\begin{aligned} L(E,s) &:= \prod_{p\nmid N}\frac{1}{\left(1-\frac{a_p}{p^s}+\frac{p}{p^{2s}}\right)}\prod_{p\mid N}\frac{1}{\left(1-\frac{a_p}{p^s}\right)} \\ &= \sum_{n=1}^{\infty}\frac{a_n}{n^s}, \end{aligned} \tag{2.1.2}$$

[1] A version of this paper has been submitted for publication. Rogers, M. D. Hypergeometric formulas for lattice sums and Mahler measures.

which converges for $\mathrm{Re}\,(s) > \frac{3}{2}$. If E has good reduction at p, then $p+1-a_p$ equals the number of integral points on E modulo p. The modularity theorem shows that $L(E,s)$ has a meromorphic continuation to the entire complex plane, and that the sum

$$g(e^{2\pi \mathrm{i}\tau}) = \sum_{n=1}^{\infty} a_n e^{2\pi \mathrm{i} n\tau},$$

is a weight two modular form on $\Gamma_0(N)$ with respect to τ. In general, if $g(q) = \sum_{n=1}^{\infty} a_n q^n$ is an arbitrary power series, then we will commit a slight abuse of notation and write

$$L(g,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Since a_n is not necessarily multiplicative with respect to n, it follows that $L(g,s)$ may or may not have an Euler product. However, if $g(q)$ corresponds to an elliptic curve E, then we will always have $L(g,s) = L(E,s)$ by the modularity theorem.

In this paper we will illustrate that the Mahler measure identities of Boyd and Deninger belong to a larger class of formulas for the Mellin transforms of weight-two modular forms. In particular, we can often derive identities between such modular forms and Mahler measures, irrespective of whether or not the modular form is associated to an elliptic curve. For instance, we can prove identities such as

$$\mathrm{m}\left(4\mathrm{i}\left(8+3\sqrt{7}\right) + y + y^{-1} + z + z^{-1}\right) = \frac{16\sqrt{7}}{\pi^2} \sum_{n=1}^{\infty} \frac{a_n}{n^2}, \tag{2.1.3}$$

where

$$g(q) := \sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} \frac{(1-q^{8n})^3 (1-q^{28n})^2}{(1-q^{56n})},$$

is a weight-two cusp form on $\Gamma_0(56)$. A cursory computation reveals the coefficients of $g(q)$ are not multiplicative, since $a_{77} = 14$, but $a_7 = a_{11} = 0$. Therefore, while $g(q)$ can not arise from an elliptic curve, we will also look at many eta products which do possess arithmetic interpretations. Martin and Ono established an exhaustive list of eta quotients associated with elliptic curves in [49].

Many of the Mahler measure identities in this paper can be related to generalized hypergeometric functions. Recall that the generalized hypergeometric function is defined by

$$ {}_pF_q\left({a_1,\ldots,a_p \atop b_1,\ldots,b_q};x\right) := 1+\sum_{n=1}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{x^n}{n!}, $$

where $(z)_n = z(z+1)\ldots(z+n-1)$. One consequence of this fact is that we can obtain series acceleration formulas for values of certain L-series. Notice that equation (2.1.3) can be transformed into

$$ \frac{16\sqrt{7}}{\pi^2}\sum_{n=1}^{\infty}\frac{a_n}{n^2} = \log\left(32+12\sqrt{7}\right)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\binom{2n}{n}^2\frac{1}{\left(32+12\sqrt{7}\right)^{2n}}. \tag{2.1.4} $$

Given the appearance of hypergeometric functions, it should not be surprising that Ramanujan's work will enter into our proofs.

2.1.1 Boyd's conjectures and four-dimensional lattice sums

In this section we will summarize a variety of explicit formulas relating four-dimensional lattice sums to Mahler measures of polynomials. Most of these results are only conjectures, although numerical calculations can be used to verify them to any degree of accuracy. Our first step will be to invoke the modularity theorem to find explicit formulas for L-functions of elliptic curves with conductors $N \in \{11,14,15,20,24,27,32,36\}$.

Definition 2.1.1. *Let us define $F(b,c)$ by*

$$ F(b,c) := (1+b)^2(1+c)^2\sum_{\substack{n_i=-\infty \\ i\in\{1,2,3,4\}}}^{\infty}\frac{(-1)^{n_1+n_2+n_3+n_4}}{\left[(6n_1+1)^2+b(6n_2+1)^2+c(6n_3+1)^2+bc(6n_4+1)^2\right]^2}. $$

It turns out that $F(b,c)$ can be used to numerically calculate a variety of L-values. The following theorem is an easy consequence of the modularity theorem:

Theorem 2.1.2. *Suppose that E_N is an elliptic curve of conductor N in an appropriate isogeny class, then*

$$ L(E_N,2) = F(b,c) \tag{2.1.5} $$

for the following values of N and (b, c):

N	(b,c)
11	$(1,11)$
14	$(2,7)$
15	$(3,5)$
20	$(1,5)$
24	$(2,3)$
27	$(1,3)$
32	$(1,2)$
36	$(1,1)$

Proof. We are interested in cases where cusp forms of elliptic curves equal the product of four eta functions. An exhaustive list of all such cusp forms is provided in [49]. By inspection of that list, the eta product associated with E_N will have the form

$$g(q) := q\prod_{n=1}^{\infty}\left(1-q^{An}\right)\left(1-q^{Abn}\right)\left(1-q^{Acn}\right)\left(1-q^{Abcn}\right),$$

where $(1+b)(1+c)A = 24$. If we recall Euler's pentagonal number theorem

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2},$$

this becomes

$$g(q) = \sum_{\substack{n_i=-\infty \\ i\in\{1,2,3,4\}}}^{\infty}(-1)^{n_1+n_2+n_3+n_4} q^{\frac{A(6n_1+1)^2+Ab(6n_2+1)^2+Ac(6n_3+1)^2+Abc(6n_4+1)^2}{24}},$$

and it follows immediately that

$$L(E_N,2) = \frac{24^2}{A^2}\sum_{\substack{n_i=-\infty \\ i\in 1,2,3,4}}^{\infty}\frac{(-1)^{n_1+n_2+n_3+n_4}}{\left[(6n_1+1)^2+b(6n_2+1)^2+c(6n_3+1)^2+bc(6n_4+1)^2\right]^2}.$$

Since $(1+b)(1+c) = 24/A$, the theorem follows. □

Since we now have expressed several different L-values in terms of $F(b,c)$, it seems logical to list all of the known Mahler measures which reduce to values of that function.

Definition 2.1.3. *Let us fix the following notation:*

$$m(k) := \mathrm{m}\left(k + y + y^{-1} + z + z^{-1}\right), \tag{2.1.6}$$
$$n(k) := \mathrm{m}\left(y^3 + z^3 + 1 - kyz\right), \tag{2.1.7}$$
$$g(k) := \mathrm{m}\left((1+y)(1+z)(y+z) - kyz\right), \tag{2.1.8}$$
$$r(k) := \mathrm{m}\left((1+y)(1+z)(1+y+z) - kyz\right). \tag{2.1.9}$$

For convenience we have slightly altered the definitions of $n(k)$, $g(k)$ and $r(k)$ that appeared in [48]. All of the following examples were either extracted from Boyd's paper [41], or were deduced by combining Boyd's conjectures with functional equations in [48]. While Boyd's minimal Weierstrass models often do not coincide with the minimal Weierstrauss models in [49], the elliptic curves are presumably isogenous, and the following results are all numerically true:

$$n(3\sqrt[3]{2}) = \frac{27}{2\pi^2} F(1,1) \tag{2.1.10}$$
$$g(2) = \frac{9}{2\pi^2} F(1,1) \tag{2.1.11}$$
$$g(-4) = \frac{18}{\pi^2} F(1,1) \tag{2.1.12}$$
$$m(4\mathrm{i}) = \frac{16}{\pi^2} F(1,2) \tag{2.1.13}$$
$$m(2\sqrt{2}) = \frac{8}{\pi^2} F(1,2) \tag{2.1.14}$$
$$n(-6) = \frac{81}{4\pi^2} F(1,3) \tag{2.1.15}$$
$$n(\sqrt[3]{2}) \stackrel{?}{=} \frac{25}{6\pi^2} F(1,5) \tag{2.1.16}$$
$$n(\sqrt[3]{32}) \stackrel{?}{=} \frac{40}{3\pi^2} F(1,5) \tag{2.1.17}$$
$$g(-2) \stackrel{?}{=} \frac{15}{\pi^2} F(1,5) \tag{2.1.18}$$
$$g(4) \stackrel{?}{=} \frac{10}{\pi^2} F(1,5) \tag{2.1.19}$$
$$r(-1) = \frac{77}{4\pi^2} F(1,11) \tag{2.1.20}$$
$$m(2) \stackrel{?}{=} \frac{6}{\pi^2} F(2,3) \tag{2.1.21}$$
$$m(8) \stackrel{?}{=} \frac{24}{\pi^2} F(2,3) \tag{2.1.22}$$

$$m(3\sqrt{2}) \stackrel{?}{=} \frac{15}{\pi^2} F(2,3) \tag{2.1.23}$$

$$m(\mathrm{i}\sqrt{2}) \stackrel{?}{=} \frac{9}{\pi^2} F(2,3) \tag{2.1.24}$$

$$n(-1) \stackrel{?}{=} \frac{7}{\pi^2} F(2,7) \tag{2.1.25}$$

$$n(5) \stackrel{?}{=} \frac{49}{2\pi^2} F(2,7) \tag{2.1.26}$$

$$g(1) \stackrel{?}{=} \frac{7}{2\pi^2} F(2,7) \tag{2.1.27}$$

$$g(7) \stackrel{?}{=} \frac{21}{\pi^2} F(2,7) \tag{2.1.28}$$

$$g(-8) \stackrel{?}{=} \frac{35}{\pi^2} F(2,7) \tag{2.1.29}$$

$$m(1) \stackrel{?}{=} \frac{15}{4\pi^2} F(3,5) \tag{2.1.30}$$

$$m(3\mathrm{i}) \stackrel{?}{=} \frac{75}{4\pi^2} F(3,5) \tag{2.1.31}$$

$$m(5) \stackrel{?}{=} \frac{45}{2\pi^2} F(3,5) \tag{2.1.32}$$

$$m(16) \stackrel{?}{=} \frac{165}{4\pi^2} F(3,5) \tag{2.1.33}$$

All of the results involving $F(1,1)$, $F(1,2)$, and $F(1,3)$ can be deduced from Rodriguez-Villegas's paper [51]. In particular, those Mahler measures can be written in terms of two-dimensional Eisenstein-Kronecker series, and then the results follow from Deuring's theorem.

2.1.2 Summary of hypergeometric formulas for $F(b,c)$

In general, we believe that $F(b,c)$ can always be written in terms of integrals of hypergeometric functions, regardless of the values of b and c. In this subsection we will translate almost all of the known Mahler measures for $F(b,c)$ into hypergeometric functions. In Corollary 2.2.8 we will also prove that similar expressions exist for both $F(2,2)$ and $F(1,4)$, even though those sums are apparently unrelated to the theory of elliptic curves.

Theorem 2.1.4. *We can express $m(k)$, $n(k)$, and $g(k)$ in terms of generalized hypergeometric functions for most values of k:*

$$m(k) = Re\left(\log(k) - \frac{2}{k^2}\, {}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2}; \frac{16}{k^2}\right)\right), \tag{2.1.34}$$

$$n(k) = Re\left(\log(k) - \frac{2}{k^3}\,{}_4F_3\left({{\frac{4}{3},\frac{5}{3},1,1}\atop{2,2,2}};\frac{27}{k^3}\right)\right), \tag{2.1.35}$$

$$\begin{aligned} g(k) =& Re\left(\log\left(\frac{(4+k)(k-2)^4}{k^2}\right) - \frac{2k^2}{(4+k)^3}\,{}_4F_3\left({{\frac{4}{3},\frac{5}{3},1,1}\atop{2,2,2}};\frac{27k^2}{(4+k)^3}\right)\right. \\ & \left. -\frac{8k}{(k-2)^3}\,{}_4F_3\left({{\frac{4}{3},\frac{5}{3},1,1}\atop{2,2,2}};\frac{27k}{(k-2)^3}\right)\right). \end{aligned} \tag{2.1.36}$$

Equation (2.1.34) *is valid in* $\mathbb{C}\setminus\{0\}$*, while* (2.1.36) *holds in* $\mathbb{C}\setminus[-4,2]$*, and* (2.1.35) *is true for* $|k|$ *is sufficiently small.*

In certain cases we can reduce these hypergeometric functions further. Suppose that $k\in\mathbb{R}\setminus\{0\}$*, then*

$$Re\left(\log(k) - \frac{2}{k^2}\,{}_4F_3\left({{\frac{3}{2},\frac{3}{2},1,1}\atop{2,2,2}};\frac{16}{k^2}\right)\right) = Re\left(\frac{|k|}{4}\,{}_3F_2\left({{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\atop{1,\frac{3}{2}}};\frac{k^2}{16}\right)\right), \tag{2.1.37}$$

and

$$\begin{aligned} Re\left(\log(k) - \frac{2}{k^3}\,{}_4F_3\left({{\frac{4}{3},\frac{5}{3},1,1}\atop{2,2,2}};\frac{27}{k^3}\right)\right) =& s(k) Re\left(Ak\,{}_3F_2\left({{\frac{1}{3},\frac{1}{3},\frac{1}{3}}\atop{\frac{2}{3},\frac{4}{3}}};\frac{k^3}{27}\right)\right. \\ & \left. +Bk^2\,{}_3F_2\left({{\frac{2}{3},\frac{2}{3},\frac{2}{3}}\atop{\frac{4}{3},\frac{5}{3}}};\frac{k^3}{27}\right)\right), \end{aligned} \tag{2.1.38}$$

where $A = \frac{\sqrt[3]{2}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{8\sqrt{3}\pi^2}$*,* $B = \frac{\Gamma^3\left(\frac{2}{3}\right)}{16\pi^2}$*, and* $s(k) = \frac{1+3\operatorname{sgn}(k)}{4}$*.*

Equations (2.1.37) and (2.1.38) will often allow us to obtain convergent series expansions from divergent hypergeometric formulas. For example, applying the results of the last theorem to conjecture (2.1.30), we obtain

$$F(3,5) \stackrel{?}{=} \frac{16\pi^2}{15}\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{(1/16)^{2n+1}}{2n+1}. \tag{2.1.39}$$

It is hardly coincidental that (2.1.39) bears a striking resemblance to a famous formula that Ramanujan obtained for Catalan's constant [33]:

$$L\left(\chi_{-4},2\right) = \pi\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{(1/4)^{2n+1}}{2n+1}.$$

Ramanujan's formula follows easily from Boyd's evaluation of the degenerate Mahler measure $m(4)$.

The following list summarizes all of the known values of hypergeometric functions which reduce to special cases of $F(b,c)$. When possible, we have used equations (2.1.37) and (2.1.38) to obtain hypergeometric functions with convergent arguments. Since no hypergeometric expression is known for $r(-1)$, we have simply retained that Mahler measure in our list. Finally, because Mahler measures such as $g(2)$ and $n\left(3\sqrt[3]{2}\right)$ lead to identical hypergeometric expressions, this list contains fewer entries than we might otherwise expect. As in Theorem 2.1.4, define

$$A := \frac{\sqrt[3]{2}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{8\sqrt{3}\pi^2}, \qquad B := \frac{\Gamma^3\left(\frac{2}{3}\right)}{16\pi^2},$$

then the following results are numerically true:

$$\frac{9}{2\pi^2}F(1,1) = \frac{1}{9}\log(54) - \frac{1}{81}\,{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};\frac{1}{2}\right), \tag{2.1.40}$$

$$\frac{16}{\pi^2}F(1,2) = 2\log(2) + \frac{1}{8}\,{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};-\frac{1}{4}\right), \tag{2.1.41}$$

$$\frac{8}{\pi^2}F(1,2) = \frac{1}{\sqrt{2}}\,{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,\frac{3}{2}};\frac{1}{2}\right), \tag{2.1.42}$$

$$\frac{81}{4\pi^2}F(1,3) = \log(6) + \frac{1}{108}\,{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};-\frac{1}{8}\right), \tag{2.1.43}$$

$$\frac{25}{6\pi^2}F(1,5) \stackrel{?}{=} \sqrt[3]{2}A\,{}_3F_2\left({\frac{1}{3},\frac{1}{3},\frac{1}{3} \atop \frac{2}{3},\frac{4}{3}};\frac{2}{27}\right) + \sqrt[3]{4}B\,{}_3F_2\left({\frac{2}{3},\frac{2}{3},\frac{2}{3} \atop \frac{4}{3},\frac{5}{3}};\frac{2}{27}\right), \tag{2.1.44}$$

$$\frac{40}{3\pi^2}F(1,5) \stackrel{?}{=} \frac{5}{3}\log(2) - \frac{1}{16}\,{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};\frac{27}{32}\right), \tag{2.1.45}$$

$$\frac{77}{4\pi^2}F(1,11) = r(-1), \tag{2.1.46}$$

$$\frac{6}{\pi^2}F(2,3) \stackrel{?}{=} \frac{1}{2}\,{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,\frac{3}{2}};\frac{1}{4}\right), \tag{2.1.47}$$

$$\frac{24}{\pi^2}F(2,3) \stackrel{?}{=} 3\log(2) - \frac{1}{32}\,{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};\frac{1}{4}\right), \tag{2.1.48}$$

$$\frac{15}{\pi^2}F(2,3) \stackrel{?}{=} \frac{1}{2}\log(18) - \frac{1}{9}\,{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};\frac{8}{9}\right), \tag{2.1.49}$$

$$\frac{9}{\pi^2}F(2,3) \stackrel{?}{=} \frac{1}{2}\log(2) + {}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2},-8\right), \tag{2.1.50}$$

$$\frac{7}{\pi^2}F(2,7) \stackrel{?}{=} \frac{A}{2}\,{}_3F_2\left({\frac{1}{3},\frac{1}{3},\frac{1}{3} \atop \frac{2}{3},\frac{4}{3}};-\frac{1}{27}\right) - \frac{B}{2}\,{}_3F_2\left({\frac{2}{3},\frac{2}{3},\frac{2}{3} \atop \frac{4}{3},\frac{5}{3}};-\frac{1}{27}\right), \tag{2.1.51}$$

$$\frac{49}{2\pi^2}F(2,7) \stackrel{?}{=} \log(5) - \frac{2}{125}\,{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};\frac{27}{125}\right), \tag{2.1.52}$$

$$\frac{21}{\pi^2}F(2,7) \stackrel{?}{=} g(7), \tag{2.1.53}$$

$$\frac{15}{4\pi^2}F(3,5) \stackrel{?}{=} \frac{1}{4}{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\1,\frac{3}{2}\end{matrix};\frac{1}{16}\right), \tag{2.1.54}$$

$$\frac{45}{2\pi^2}F(3,5) \stackrel{?}{=} \log(5) - \frac{2}{25}{}_4F_3\left(\begin{matrix}\frac{3}{2},\frac{3}{2},1,1\\2,2,2\end{matrix};\frac{16}{25}\right), \tag{2.1.55}$$

$$\frac{165}{4\pi^2}F(3,5) \stackrel{?}{=} 4\log(2) - \frac{1}{128}{}_4F_3\left(\begin{matrix}\frac{3}{2},\frac{3}{2},1,1\\2,2,2\end{matrix};\frac{1}{16}\right), \tag{2.1.56}$$

$$\frac{75}{4\pi^2}F(3,5) \stackrel{?}{=} \log(3) + \frac{2}{9}{}_4F_3\left(\begin{matrix}\frac{3}{2},\frac{3}{2},1,1\\2,2,2\end{matrix};-\frac{16}{9}\right). \tag{2.1.57}$$

While most of these formulas remain unproven, a variety of partial results exist. For instance, identities (2.1.47) through (2.1.50) are equivalent to one another [48], formulas (2.1.40) through (2.1.43) follow from [51], and Brunault proved (2.1.46) in [42].

2.2 Reductions of $F(1,1)$, $F(1,2)$, $F(1,4)$, and $F(2,2)$ to integrals of hypergeometric functions

In the previous section we translated many of Boyd's conjectures into explicit identities between hypergeometric functions and lattice sums. This approach has two essential consequences. Not only does it eliminate any obvious connection with elliptic curves, but it also allows for the construction of proofs based upon series manipulation. We have used such an approach to reduce five cases of $F(b,c)$ to integrals of hypergeometric functions. In this section we will discuss the cases that occur when $(b,c) \in \{(1,1),(1,2),(1,4),(2,2)\}$. We will rely heavily on the q-series theorems contained in Ramanujan's notebooks (see [38] and [40]).

Definition 2.2.1. *Recall the following q-series notation:*

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \psi(q) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}},$$

$$f(-q) := \prod_{n=1}^{\infty}(1-q^n), \qquad (x;q)_\infty := \prod_{n=0}^{\infty}(1-xq^n).$$

Lemma 2.2.3 reduces the aforementioned cases of $F(b,c)$ to two-dimensional sums. Such identities exist because various eta-quotients can be written in terms theta functions. Euler's pentagonal number formula is probably the simplest such indentity:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} q^{\frac{n(3n+1)}{2}}.$$

Unfortunately, similar formulas are not known for $f^2(-q)$, $f(-q)f\left(-q^2\right)$, or $f(-q)f\left(-q^3\right)$ [45]. This fact represents the main obstruction to also proving Boyd's conjectures for $F(1,5)$, $F(2,3)$, $F(2,7)$, and $F(3,5)$.

Definition 2.2.2. *We will use the following notation:*

$$F_{(1,1)}(x) := 16 \sum_{\substack{n=-\infty \\ k=0}}^{\infty} \frac{(-1)^{n+k}(2k+1)}{\left[3(2k+1)^2 + x^2(6n+1)^2\right]^2}, \tag{2.2.1}$$

$$F_{(1,2)}(x) := \sum_{\substack{n=-\infty \\ k=0}}^{\infty} \frac{(-1)^{n+k}(2k+1)}{\left[(2k+1)^2 + x^2(2n)^2\right]^2}, \tag{2.2.2}$$

$$F_{(1,4)}(x) := 25 \sum_{n,k=-\infty}^{\infty} \frac{(-1)^{n}(3k+1)}{\left[4(3k+1)^2 + x^2(6n+1)^2\right]^2}, \tag{2.2.3}$$

$$F_{(2,2)}(x) := 9 \sum_{n,k=0}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}+k}(2k+1)}{\left[2(2k+1)^2 + x^2(2n+1)^2\right]^2}. \tag{2.2.4}$$

Lemma 2.2.3. *Suppose that* $(b,c) \in \{(1,1),(1,2),(1,4),(2,2)\}$*, then*

$$F_{(b,c)}(1) = F(b,c). \tag{2.2.5}$$

Proof. First notice that $F(b,c)$ has the following integral representation for all values of b and c:

$$\frac{24^2 F(b,c)}{(1+b)^2(1+c)^2} = \int_0^1 \int_0^{q_1} q^{\frac{(1+b)(1+c)}{24}} f(-q)\, f\left(-q^b\right) f\left(-q^c\right) f\left(-q^{bc}\right) \frac{\mathrm{d}q}{q}\frac{\mathrm{d}q_1}{q_1}.$$

We can apply Euler's pentagonal number theorem four times (once to each occurrence of $f(-q)$), to see the truth of this last formula. Taking note of the following identities:

$$q^{1/6} f^4(-q) = \left(q^{1/24} f(-q)\right)\left(q^{1/8} f^3(-q)\right),$$

$$q^{1/4} f^2(-q)\, f^2\left(-q^2\right) = \left(\frac{f^2(-q)}{f\left(-q^2\right)}\right)\left(q^{1/4} f^3\left(-q^2\right)\right),$$

$$q^{5/12}f^2\left(-q\right)f^2\left(-q^4\right)=\left(q^{1/12}f\left(-q^2\right)\right)\left(q^{1/3}\frac{f^2\left(-q\right)f^2\left(-q^4\right)}{f\left(-q^2\right)}\right),$$
$$q^{3/8}f\left(-q\right)f^2\left(-q^2\right)f\left(-q^4\right)=\left(q^{1/8}\frac{f\left(-q\right)f\left(-q^4\right)}{f\left(-q^2\right)}\right)\left(q^{1/4}f^3\left(-q^2\right)\right),$$

and then employing well known series expansions:

$$\frac{f^2\left(-q\right)}{f\left(-q^2\right)}=\varphi(-q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}, \tag{2.2.6}$$
$$q^{1/8}\frac{f\left(-q\right)f\left(-q^4\right)}{f\left(-q^2\right)}=q^{1/8}\psi\left(-q\right)=\sum_{n=0}^{\infty}(-1)^{\frac{n(n+1)}{2}}q^{\frac{(2n+1)^2}{8}}, \tag{2.2.7}$$
$$q^{1/24}f\left(-q\right)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(6n+1)^2}{24}}, \tag{2.2.8}$$
$$q^{1/8}f^3\left(-q\right)=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{(2n+1)^2}{8}}, \tag{2.2.9}$$
$$q^{1/3}\frac{f^2\left(-q\right)f^2\left(-q^4\right)}{f\left(-q^2\right)}=\sum_{n=-\infty}^{\infty}(3n+1)q^{\frac{(3n+1)^2}{3}}, \tag{2.2.10}$$

we recover equation (2.2.5) in every case.

□

We will use the next two propositions to reduce each of the two-dimensional sums to a q-series. Then, in Theorem 2.2.7, we will reduce $F_{(b,c)}(x)$ to integrals of hypergeometric functions for $x\in(0,\infty)$. For certain values of x, those formulas also translate into identities involving generalized hypergeometric functions and Mahler measures.

Proposition 2.2.4. *Assume that $\delta>0$ is sufficiently small, then*

$$F_{(1,1)}(x)=-\frac{\pi^2\mathrm{i}}{9x}\int_{\mathrm{i}\delta-\infty}^{\mathrm{i}\delta+\infty}\frac{\sinh(t)}{1+4\sinh^2(t)}\frac{\sec(\sqrt{3}xt)\tan(\sqrt{3}xt)}{t}\mathrm{d}t, \tag{2.2.11}$$
$$F_{(1,2)}(x)=\frac{\pi^3}{32}-\frac{\pi^2\mathrm{i}}{32x}\int_{\mathrm{i}\delta-\infty}^{\mathrm{i}\delta+\infty}\left(\operatorname{csch}(t)-\frac{1}{t}\right)\frac{\sec\left(xt\right)\tan\left(xt\right)}{t}\mathrm{d}t \tag{2.2.12}$$
$$F_{(1,4)}(x)=-\frac{25\pi^2\mathrm{i}}{144\sqrt{3}x}\int_{\mathrm{i}\delta-\infty}^{\mathrm{i}\delta+\infty}\frac{\sinh(t)}{1+4\sinh^2(t)}\frac{\csc^2\left(\frac{\pi}{3}-xt\right)-\csc^2\left(\frac{\pi}{3}+xt\right)}{t}\mathrm{d}t, \tag{2.2.13}$$

$$F_{(2,2)}(x) = -\frac{9\pi^2 \mathrm{i}}{128x}\int_{\mathrm{i}\delta-\infty}^{\mathrm{i}\delta+\infty}\frac{\sinh(t)}{\cosh(2t)}\frac{\sec\left(\sqrt{2}xt\right)\tan\left(\sqrt{2}xt\right)}{t}\mathrm{d}t \tag{2.2.14}$$

Proof. The proofs are all substantially the same. The idea is to use contour integration to pick off the n-index of summation in the corresponding two-dimensional sum. We will illustrate the proof of (2.2.12) explicitly. First assume that $0 < \delta < 1$, and let C denote a closed contour which runs along the line $(\mathrm{i}\delta - \infty, \mathrm{i}\delta + \infty)$ and then encircles the upper half plane. Since $\operatorname{csch}(t)$ has poles at $t = \pi \mathrm{i} n$, by the residue theorem

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\left[(2k+1)^2 + x^2(2n)^2\right]^2} = \frac{1}{2\pi\mathrm{i}}\int_C\left(\operatorname{csch}(t) - \frac{1}{t}\right)\frac{1}{\left[(2k+1)^2 - x^2(2t/\pi)^2\right]^2}\mathrm{d}t.$$

If $t = Re^{\mathrm{i}\theta}$, then the integrand has order $O\left(R^{-5}\right)$, and therefore the circular portion of the contour integral vanishes as R tends to ∞. Next observe that the sum

$$\frac{\pi^3}{32}\frac{\sec(tx)\tan(tx)}{tx} = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{\left[(2k+1)^2 - x^2\left(2t/\pi\right)^2\right]^2}$$

converges uniformly when $\operatorname{Im}(t) > 0$, hence

$$\sum_{\substack{k=0\\n=1}}^{\infty}\frac{(-1)^{n+k}(2k+1)}{\left[(2k+1)^2 + x^2(2n)^2\right]^2} = -\frac{\pi^2\mathrm{i}}{64x}\int_{\mathrm{i}\delta-\infty}^{\mathrm{i}\delta+\infty}\left(\operatorname{csch}(t) - \frac{1}{t}\right)\frac{\sec(xt)\tan(xt)}{t}\mathrm{d}t.$$

Equation (2.2.12) follows easily from this last result. $\square$

Proposition 2.2.5. *Let $\chi_{-3}(k)$ and $\chi_{-4}(k)$ denote Legendre symbols modulo three and four, and assume that $x > 0$.*

If $q = e^{-\pi x/\sqrt{12}}$ and $\omega = e^{\pi \mathrm{i}/6}$, then

$$F_{(1,1)}(x) = \frac{2\pi^2}{9x}\sum_{k=1}^{\infty}k\chi_{-4}(k)\log\left|\frac{1+wq^k}{1-wq^k}\right|. \tag{2.2.15}$$

If $q = e^{-\pi x}$, then

$$F_{(1,2)}(x) = \frac{\pi^3}{32} - \frac{\pi^2}{8x}\sum_{k=1}^{\infty}k\chi_{-4}(k)\log\left(1+q^k\right). \tag{2.2.16}$$

If $q = e^{-\pi x/3}$ and $\omega = e^{\pi\mathrm{i}/6}$, then

$$F_{(1,4)}(x) = \frac{25\pi^2}{72x}\sum_{k=1}^{\infty}k\chi_{-3}(k)\log\left|\frac{1+wq^k}{1-wq^k}\right|. \tag{2.2.17}$$

If $q = e^{-\pi x/\sqrt{8}}$ *and* $\omega = e^{\pi \mathrm{i}/4}$, *then*

$$F_{(2,2)}(x) = \frac{9\pi^2}{32x} \sum_{k=1}^{\infty} k\chi_{-4}(k) \log \left| \frac{1+wq^k}{1-wq^k} \right|. \tag{2.2.18}$$

Proof. All of the proofs are very similar, so we will only prove (2.2.15) in detail. Since $\sec(\sqrt{3}xt) \times \tan(\sqrt{3}xt)$ is periodic, we can rearrange (2.2.11) to obtain

$$F_{(1,1)}(x) = -\frac{\pi^2 \mathrm{i}}{9x} \int_{\mathrm{i}\delta}^{\mathrm{i}\delta + \frac{2\pi}{\sqrt{3}x}} \left(\sum_{n=-\infty}^{\infty} \frac{1}{t + \frac{2\pi n}{\sqrt{3}x}} \frac{\sinh\left(t + \frac{2\pi n}{\sqrt{3}x}\right)}{1 + 4\sinh^2\left(t + \frac{2\pi n}{\sqrt{3}x}\right)} \right) \frac{\sin(\sqrt{3}xt)}{\cos^2(\sqrt{3}xt)} \mathrm{d}t,$$

where the interchange of summation and integration can be justified by the fact that the summand has order $O\left(e^{-2\pi|n|/\sqrt{3}x}\right)$ as $n \to \pm\infty$. Observe that if $q = e^{-\pi x/\sqrt{12}}$, $\omega = e^{\pi \mathrm{i}/6}$, and $\mathrm{Im}\,(t) = \delta < x$, then

$$\begin{aligned} &\frac{2\pi}{\sqrt{3}x} \sum_{n=-\infty}^{\infty} \frac{1}{t + \frac{2\pi n}{\sqrt{3}x}} \frac{\sinh\left(t + \frac{2\pi n}{\sqrt{3}x}\right)}{1 + 4\sinh^2\left(t + \frac{2\pi n}{\sqrt{3}x}\right)} \\ &\qquad = \log\left(2+\sqrt{3}\right) + 2\sum_{k=1}^{\infty} \log \left| \frac{1+wq^k}{1-wq^k} \right| \cos\left(\sqrt{3}xkt\right). \end{aligned} \tag{2.2.19}$$

This new restriction, $\delta < x$, guarantees uniform convergence of the Fourier series, and is consistent with the prior assumption that $0 < \delta \ll 1$. The proof of (2.2.19) is a straightforward exercise in contour integration which we will skip. Substituting (2.2.19) into our integral yields

$$F_{(1,1)}(x) = -\frac{\pi^2 \mathrm{i}}{9x} \left(\log(2+\sqrt{3}) I_0 + 2\sum_{k=1}^{\infty} \log \left| \frac{1+wq^k}{1-wq^k} \right| I_k \right),$$

where

$$\begin{aligned} I_k =& \frac{\sqrt{3}x}{2\pi} \int_{\mathrm{i}\delta}^{\mathrm{i}\delta + \frac{2\pi}{\sqrt{3}x}} \frac{\cos\left(\sqrt{3}xkt\right)\sin\left(\sqrt{3}xt\right)}{\cos^2\left(\sqrt{3}xt\right)} \mathrm{d}t \\ =& \int_{\mathrm{i}\delta'}^{\mathrm{i}\delta'+1} \frac{\cos\left(2\pi kt\right)\sin\left(2\pi t\right)}{\cos^2\left(2\pi t\right)} \mathrm{d}t \\ =& \mathrm{i}k \sin\left(\pi k/2\right). \end{aligned}$$

The final step in this calculation follows from considering a closed rectangular contour with vertices at $\{0, \mathrm{i}\delta', 1+\mathrm{i}\delta', 1\}$ which avoids the boundary points $t = 1/4$ and $t = 3/4$. With this formula for I_k in hand, the proof of (2.2.15) is complete. The proofs of the other formulas follow in a similar manner from slightly different Fourier series expansions.

$\square$

At this point, a hypergeometric formula for $F_{(1,2)}(x)$ can be recovered. By combining equation (2.2.16) with formulas (2-9) and (2-16) in [48], it is easy to recognize that if $q = e^{-\pi x}$, then

$$F_{(1,2)}(x) = \frac{\pi^2}{16x}\mathrm{m}\left(\frac{\mathrm{i}f^4(-q)}{\sqrt{q}f^4\left(-q^4\right)} + y + y^{-1} + z + z^{-1}\right). \tag{2.2.20}$$

In the next section we will use values of class invariants to deduce explicit examples from (2.2.20). Unfortunately, we will require another theorem in order to obtain useful results on the other three lattice sums.

Theorem 2.2.6. *In this theorem we will always assume that $x > 0$. If $q = e^{-\pi x/\sqrt{12}}$, then*

$$F_{(1,1)}(x) = \frac{2\pi^2}{3\sqrt{3}x} Im \left[\int_0^{\mathrm{i}q} \frac{f^9\left(-u^3\right)}{f^3\left(-u\right)}\mathrm{d}u\right]. \tag{2.2.21}$$

If $q = e^{-\pi x}$, then

$$F_{(1,2)}(x) = \frac{\pi^3}{32} - \frac{\pi^2}{16x}\int_0^q \frac{\varphi^2(-u)\varphi^4(u) - 1}{u}\mathrm{d}u. \tag{2.2.22}$$

If $q = e^{-\pi x/3}$, then

$$F_{(1,4)}(x) = \frac{25\pi^2}{36x} Im \left[\int_0^{e^{\frac{2\pi\mathrm{i}}{3}}q} \varphi^2(u)\psi^4\left(u^2\right)\mathrm{d}u\right]. \tag{2.2.23}$$

If $q = e^{-\pi x/\sqrt{8}}$, then

$$F_{(2,2)}(x) = \frac{9\pi^2}{32\sqrt{2}x}\int_0^q \varphi(-u^2)\varphi(u^4)\left(3\psi^4(-u^2) - \psi^4(u^2)\right)\mathrm{d}u. \tag{2.2.24}$$

Proof. Equations (2.2.22) and (2.2.23) have similar proofs, so we will only prove the latter identity. Notice that (2.2.17) can be rearranged to obtain

$$\frac{72x}{25\pi^2}F_{(1,4)}(x) = \mathrm{Re}\left(\int_0^q \sum_{k=1}^{\infty} k^2\chi_{-3}(k)\frac{2\omega u^k}{1-\omega^2u^{2k}}\frac{\mathrm{d}u}{u}\right)$$

$$
\begin{aligned}
&=\sqrt{3}\int_0^q\sum_{k=1}^\infty k^2\chi_{-3}(k)\left(\frac{u^k-u^{5k}}{1+u^{6k}}\right)\frac{\mathrm{d}u}{u}\\
&=\mathrm{Im}\left(2\int_0^{e^{2\pi\mathrm{i}/3}q}\sum_{k=1}^\infty k^2\left(\frac{u^k-u^{3k}+u^{5k}}{1+u^{6k}}\right)\frac{\mathrm{d}u}{u}\right)\\
&=\mathrm{Im}\left(2\int_0^{e^{2\pi\mathrm{i}/3}q}\sum_{k=1}^\infty \frac{k^2u^k}{1+u^{2k}}\frac{\mathrm{d}u}{u}\right).
\end{aligned}
$$

Combining entries 10.1, 11.3, and 17.2 in Chapter 17 of [38], we deduce that for $|u|<1$:

$$
\sum_{k=1}^\infty \frac{k^2u^k}{1+u^{2k}} = u\varphi^2(u)\psi^4\left(u^2\right),
$$

which completes the proof of (2.2.23).

The proofs of equations (2.2.21) and (2.2.24) will require the following formula:

$$
\mathrm{Im}\left(g\left(\mathrm{i}u,t\right)\right)=\frac{1}{t}\sum_{k=0}^\infty(-1)^k\frac{(2k+1)^2u^{2k+1}}{1+u^{2(2k+1)}}\left(t^{2k+1}-t^{-(2k+1)}\right), \qquad (2.2.25)
$$

where

$$
g(u,t)=\frac{(u;u)_\infty^6\left(t^{-2}u;u\right)_\infty\left(t^2;u\right)_\infty}{\left(t^{-1}u;u\right)_\infty^4\left(t;u\right)_\infty^4},
$$

and

$$
(x;q)_\infty=(1-x)(1-xq)(1-xq^2)\dots
$$

Equation (2.2.25) is a direct consequence of identity (14.2.9) in [35], which follows from product expansions for the Weierstrass $\wp$-function [43].

Rearranging equation (2.2.15) we have

$$
\begin{aligned}
\frac{9x}{2\pi^2}F_{(1,1)}(x) &=\mathrm{Re}\left(\int_0^q\sum_{k=0}^\infty(-1)^k(2k+1)^2\frac{2\omega u^{2k+1}}{1-\omega^2u^{2(2k+1)}}\frac{\mathrm{d}u}{u}\right)\\
&=\sqrt{3}\int_0^q\sum_{k=0}^\infty(-1)^k(2k+1)^2\frac{u^{2k+1}-u^{5(2k+1)}}{1+u^{6(2k+1)}}\frac{\mathrm{d}u}{u}.
\end{aligned}
$$

Applying (2.2.25) after letting $u\to u^3$ and $t\to u^{-2}$, transforms this last integral into

$$
\frac{3\sqrt{3}x}{2\pi^2}F_{(1,1)}(x)=\mathrm{Im}\left(\int_0^{\mathrm{i}q}\frac{f^5\left(-u^2\right)f^4\left(-u^3\right)f\left(-u^6\right)}{f^4(-u)}\mathrm{d}u\right).
$$

By the following eta function identity:

$$\frac{f^5\left(-u^2\right)f^4\left(-u^3\right)f\left(-u^6\right)}{f^4(-u)}=\frac{f^9\left(-u^3\right)}{f^3\left(-u\right)}+u\frac{f^9\left(-u^6\right)}{f^3\left(-u^2\right)},\tag{2.2.26}$$

this becomes

$$\begin{aligned}\frac{3\sqrt{3}x}{\pi^2}F_{(1,1)}(x)=&\mathrm{Im}\left(\int_0^{\mathrm{i}q}\frac{f^9\left(-u^3\right)}{f^3\left(-u\right)}\mathrm{d}u\right)-\mathrm{Im}\left(\int_0^{q}u\frac{f^9\left(u^6\right)}{f^3\left(u^2\right)}\mathrm{d}u\right)\\=&\mathrm{Im}\left(\int_0^{\mathrm{i}q}\frac{f^9\left(-u^3\right)}{f^3\left(-u\right)}\mathrm{d}u\right)-0,\end{aligned}$$

which completes the proof of (2.2.21). Although we will not elaborate on the proof of (2.2.26) here, it suffices to say that it follows from algebraic transformations for the hypergeometric function ${}_2F_1\left(\frac{1}{3},\frac{2}{3},1,u\right)$. We will also point out that the eta-quotient $f^9\left(-q\right)/f^3\left(-q^3\right)$ curiously appears on page 1734 of [34].

The proof of (2.2.24) follows the same lines, but requires a few extra steps. Proceeding as before, we find that

$$\begin{aligned}\frac{32x}{9\pi^2}F_{(2,2)}(x)=&\mathrm{Re}\left(\int_0^{q}\sum_{k=0}^{\infty}(-1)^k(2k+1)^2\frac{2\omega u^{2k+1}}{1-\omega^2u^{2(2k+1)}}\frac{\mathrm{d}u}{u}\right)\\=&\int_0^{q}\sum_{k=0}^{\infty}(-1)^k(2k+1)^2\frac{u^{2k+1}-u^{3(2k+1)}}{1+u^{4(2k+1)}}\frac{\mathrm{d}u}{u}.\end{aligned}$$

If we apply equation (2.2.25) after letting $u\to u^2$ and $t\to u^{-1}$, this becomes

$$\frac{32x}{9\pi^2}F_{(2,2)}(x)=\mathrm{Re}\left(2\int_0^{\omega q}\frac{f^6\left(-u^2\right)f\left(-u^8\right)}{f\left(-u^4\right)}\frac{1}{(\omega u,u^2)_\infty^4\,(\bar{\omega}u,u^2)_\infty^4}\mathrm{d}u\right),$$

where $\omega=e^{\pi\mathrm{i}/4}$. For brevity of notation let us define a new function

$$\begin{aligned}g(u):=&\left(\omega u,u^2\right)_\infty\left(\bar{\omega}u,u^2\right)_\infty\\=&\prod_{n=0}^{\infty}\left(1-\sqrt{2}u^{2n+1}+u^{2(2n+1)}\right).\end{aligned}\tag{2.2.27}$$

Since (2.2.18) is odd with respect to q, our integral can be transformed into

$$\frac{32x}{9\pi^2}F_{(2,2)}(x)=\mathrm{Re}\left(\int_0^{\omega q}\frac{f^6\left(-u^2\right)f\left(-u^8\right)}{f\left(-u^4\right)}\frac{g^4(u)+g^4(-u)}{g^4(u)g^4(-u)}\mathrm{d}u\right).\tag{2.2.28}$$

Next we will reduce $\left(g^4(u)+g^4(-u)\right)/\left(g(u)g(-u)\right)^4$ to theta functions. Observe by equation (2.2.27) that

$$g(u)g(-u)=\prod_{n=0}^{\infty}\left(1+u^{4(2n+1)}\right)=\frac{\varphi\left(-u^8\right)}{f\left(-u^4\right)}. \tag{2.2.29}$$

With two applications of the Jacobi triple product [38], we also have

$$\begin{aligned}
g(u)+g(-u)=&\frac{1}{f\left(-u^2\right)}\left(\sum_{n=-\infty}^{\infty}(-\omega)^n u^{n^2}+\sum_{n=-\infty}^{\infty}\omega^n u^{n^2}\right)\\
=&\frac{2}{f\left(-u^2\right)}\sum_{n=-\infty}^{\infty}(-1)^n u^{16n^2}\\
=&\frac{2\varphi\left(-u^{16}\right)}{f\left(-u^2\right)}.
\end{aligned} \tag{2.2.30}$$

So finally, combining (2.2.29) and (2.2.30), we find that

$$\frac{g^4(u)+g^4(-u)}{g^4(u)g^4(-u)}=2\frac{f^4\left(-u^4\right)}{f^4\left(-u^2\right)}\left[8\frac{\varphi^4\left(-u^{16}\right)}{\varphi^4\left(-u^8\right)}-8\frac{\varphi^2\left(-u^{16}\right)\varphi\left(-u^2\right)}{\varphi^3\left(-u^8\right)}+\frac{\varphi^2\left(-u^2\right)}{\varphi^2\left(-u^8\right)}\right].$$

Recalling that $\varphi^2\left(-q^{16}\right)=\varphi\left(-q^8\right)\varphi\left(q^8\right)$, this becomes

$$\begin{aligned}
\frac{g^4(u)+g^4(-u)}{g^4(u)g^4(-u)}=&2\frac{f^4\left(-u^4\right)}{f^4\left(-u^2\right)}\left[\frac{8\varphi^2\left(u^8\right)-8\varphi\left(u^8\right)\varphi\left(-u^2\right)+\varphi^2\left(-u^2\right)}{\varphi^2\left(-u^8\right)}\right],\\
=&2\frac{f^5\left(-u^4\right)}{f^6\left(-u^2\right)}\left[\frac{8\varphi^2\left(u^8\right)\varphi\left(-u^2\right)-8\varphi\left(u^8\right)\varphi^2\left(-u^2\right)+\varphi^3\left(-u^2\right)}{\varphi^2\left(-u^8\right)}\right],
\end{aligned}$$

and therefore (2.2.28) simplifies to

$$\begin{aligned}
\frac{32x}{9\pi^2}F_{(2,2)}(x)=&\mathrm{Re}\left(2\int_0^{\omega q}f^4\left(-u^4\right)f\left(-u^8\right)\right.\\
&\left.\times\frac{8\varphi^2\left(u^8\right)\varphi\left(-u^2\right)-8\varphi\left(u^8\right)\varphi^2\left(-u^2\right)+\varphi^3\left(-u^2\right)}{\varphi^2\left(-u^8\right)}\mathrm{d}u\right).
\end{aligned}$$

Next, make the change of variables sending $u\to\omega u$, to obtain

$$\begin{aligned}
=&2\int_0^q\frac{f^4\left(u^4\right)f\left(-u^8\right)}{\varphi^2\left(-u^8\right)}\\
&\times\mathrm{Re}\left(\omega\left(8\varphi^2\left(u^8\right)\varphi\left(-\mathrm{i}u^2\right)-8\varphi\left(u^8\right)\varphi^2\left(-\mathrm{i}u^2\right)+\varphi^3\left(-\mathrm{i}u^2\right)\right)\right)\mathrm{d}u.
\end{aligned}$$

If we recall that $\varphi\left(-\mathrm{i}u^2\right)=\varphi\left(u^8\right)-2\mathrm{i}u^2\psi\left(u^{16}\right)$, then we are left with

$$\begin{aligned}\frac{32x}{9\pi^2}F_{(2,2)}(x)=&\sqrt{2}\int_0^q\frac{f^4\left(u^4\right)f\left(-u^8\right)}{\varphi^2\left(-u^8\right)}\Big(\left(\varphi\left(u^8\right)-2u^2\psi\left(u^{16}\right)\right)^3\\&-4u^2\varphi\left(u^8\right)\psi\left(u^{16}\right)\left(\varphi\left(u^8\right)-2u^2\psi\left(u^{16}\right)\right)\Big)\,\mathrm{d}u.\end{aligned}$$

In order to simplify this last formula, we will freely apply theta function identities on pages 34 and 40 of [38]. Therefore, we find that

$$\begin{aligned}&=\sqrt{2}\int_0^q\frac{f^4\left(u^4\right)f\left(-u^8\right)}{\varphi^2\left(-u^8\right)}\left(\varphi^3\left(-u^2\right)-4u^2\varphi\left(u^8\right)\psi\left(u^{16}\right)\varphi\left(-u^2\right)\right)\mathrm{d}u\\&=\sqrt{2}\int_0^q\frac{f^4\left(u^4\right)f\left(-u^8\right)}{\varphi^2\left(-u^8\right)}\varphi\left(-u^2\right)\left(\varphi^2\left(-u^2\right)-4u^2\psi^2\left(u^8\right)\right)\mathrm{d}u\\&=\frac{1}{\sqrt{2}}\int_0^q\frac{f^4\left(u^4\right)f\left(-u^8\right)}{\varphi^2\left(-u^8\right)}\varphi\left(-u^2\right)\left(3\varphi^2\left(-u^2\right)-\varphi^2\left(u^2\right)\right)\mathrm{d}u\\&=\frac{1}{\sqrt{2}}\int_0^q\frac{f^4\left(u^4\right)f\left(-u^8\right)}{\varphi^2\left(-u^8\right)\psi^2\left(u^4\right)}\varphi\left(-u^2\right)\left(3\psi^4\left(-u^2\right)-\psi^4\left(u^2\right)\right)\mathrm{d}u\\&=\frac{1}{\sqrt{2}}\int_0^q\varphi\left(u^4\right)\varphi\left(-u^2\right)\left(3\psi^4\left(-u^2\right)-\psi^4\left(u^2\right)\right)\mathrm{d}u,\end{aligned}$$

which completes the proof of (2.2.24).

□

The next theorem requires the signature-three theta functions. Recall that if $\omega=e^{2\pi\mathrm{i}/3}$, then the signature-three theta functions are defined by:

$$\begin{aligned}a(q)&:=\sum_{n,m=-\infty}^{\infty}q^{m^2+mn+n^2},\\b(q)&:=\sum_{n,m=-\infty}^{\infty}\omega^{m-n}q^{m^2+mn+n^2},\\c(q)&:=\sum_{n,m=-\infty}^{\infty}q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.\end{aligned}$$

The signature-three theta functions satisfy many interesting formulas, including the following cubic relation:

$$a^3(q)=b^3(q)+c^3(q).$$

Various other properties of $a(q)$, $b(q)$, and $c(q)$ have been catalogued (see [40], page 93).

Theorem 2.2.7. *We can reduce the two-dimensional lattice sums to integrals of hypergeometric functions.*

Suppose that $q = e^{-\pi x/\sqrt{12}}$, *then*

$$\frac{81x}{2\pi^2}F_{(1,1)}(x) = \begin{cases} 3\tilde{n}\left(3\frac{a(\mathrm{i}q)}{b(\mathrm{i}q)}\right) + \frac{4}{\sqrt{3}}n_2\left(\frac{b^3(\mathrm{i}q)}{a^3(\mathrm{i}q)}\right) & \text{if } x \in \left(0, \frac{1}{\sqrt{5}}\right), \\ 3\tilde{n}\left(3\frac{a(\mathrm{i}q)}{b(\mathrm{i}q)}\right) + \frac{1}{\sqrt{3}}n_2\left(\frac{b^3(\mathrm{i}q)}{a^3(\mathrm{i}q)}\right) & \text{if } x \in \left(\frac{1}{\sqrt{5}}, \sqrt{5}\right), \\ \frac{1}{\sqrt{3}}n_2\left(\frac{b^3(\mathrm{i}q)}{a^3(\mathrm{i}q)}\right) & \text{if } x \in \left(\sqrt{5}, \infty\right), \end{cases} \tag{2.2.31}$$

where

$$\tilde{n}(k) = Re\left(\log(k) - \frac{2}{k^3}{}_4F_3\left({1,1,\frac{4}{3},\frac{5}{3} \atop 2,2,2}; \frac{27}{k^3}\right)\right),$$

and

$$n_2(k) = Im\left(\int_k^1 \frac{{}_2F_1\left({\frac{1}{3},\frac{2}{3} \atop 1}; 1-u\right)}{u}\mathrm{d}u\right).$$

Notice that $\tilde{n}(k) = n(k)$ *whenever* $|k|$ *is sufficiently small.*

Suppose that $q = e^{-\pi x}$ *and* $x > 0$, *then*

$$\frac{16x}{\pi^2}F_{(1,2)}(x) = m\left(\frac{\mathrm{i}f^4(-q)}{\sqrt{q}f^4\left(-q^4\right)}\right), \tag{2.2.32}$$

where $m(k)$ *is defined in* (2.1.6).

Suppose that $q = e^{-\pi x/3}$ *and* $\omega = e^{2\pi\mathrm{i}/3}$, *then*

$$\frac{144x}{25\pi^2}F_{(1,4)}(x) = \begin{cases} m\left(4\frac{\varphi^2(\omega q)}{\varphi^2(-\omega q)}\right) - \frac{3}{4}m_2\left(\frac{\varphi^4(-\omega q)}{\varphi^4(\omega q)}\right) & \text{if } x \in \left(0, \frac{1}{\sqrt{2}}\right), \\ m\left(4\frac{\varphi^2(\omega q)}{\varphi^2(-\omega q)}\right) + \frac{1}{4}m_2\left(\frac{\varphi^4(-\omega q)}{\varphi^4(\omega q)}\right) & \text{if } x \in \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), \\ \frac{1}{4}m_2\left(\frac{\varphi^4(-\omega q)}{\varphi^4(\omega q)}\right) & \text{if } x \in \left(\sqrt{2}, \infty\right), \end{cases} \tag{2.2.33}$$

where $m(k)$ *is defined in* (2.1.6), *and*

$$m_2(k) := Im\left(\int_k^1 \frac{{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1}; 1-u\right)}{u}\mathrm{d}u\right).$$

If $q = e^{-\pi x/\sqrt{8}}$, *then*

$$F_{(2,2)}(x) = \frac{9\pi^2}{512x}\int_{\frac{\varphi^4(-q^2)}{\varphi^4(q^2)}}^1 \frac{(3\sqrt{u}-1)}{u^{3/4}\sqrt{1-\sqrt{u}}}{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1}; 1-u\right)\mathrm{d}u. \tag{2.2.34}$$

Proof. The proof of this theorem follows from our ability to invert theta functions. First recall the classical inversion formula for the theta function:

$$\varphi^2(q) = {}_2F_1\left({\tfrac{1}{2},\tfrac{1}{2} \atop 1}; 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}\right), \tag{2.2.35}$$

which holds whenever $q \in (-1, 1)$ [38]. If we use the notation $\alpha = 1 - \varphi^4(-q)/\varphi^4(q)$ and $z = \varphi^2(q)$, then many different theta functions can be expressed in terms of these two parameters. The following identities are true whenever $|q| < 1$ (see pages 122, and 123 in [38]):

$$\begin{aligned}
\varphi(q) =& \sqrt{z}, \\
\varphi(-q) =& (1-\alpha)^{1/4}\sqrt{z}, \\
\varphi\left(q^2\right) =& \left(1+\sqrt{1-\alpha}\right)^{1/2}\sqrt{\frac{z}{2}}, \\
\psi(-q) =& q^{-1/8}\left\{\alpha(1-\alpha)\right\}^{1/8}\sqrt{\frac{z}{2}}, \\
\psi(q) =& q^{-1/8}\alpha^{1/8}\sqrt{\frac{z}{2}},
\end{aligned}$$

and it is also well known that

$$\frac{\mathrm{d}\alpha}{\mathrm{d}q} = \frac{\alpha(1-\alpha)z^2}{q}.$$

Both (2.2.32) and (2.2.34) follow from applying these parameterizations to equations (2.2.22) and (2.2.24) respectively.

Since equation (2.2.35) does not hold in the entire open unit disk, we will need to generalize that result. First notice that z satisfies the hypergeometric differential equation with respect to α:

$$\alpha(1-\alpha)\frac{\mathrm{d}^2 z}{\mathrm{d}\alpha^2} + (1-2\alpha)\frac{\mathrm{d}z}{\mathrm{d}\alpha} - \frac{z}{4} = 0. \tag{2.2.36}$$

We can use the relation $\frac{\mathrm{d}}{\mathrm{d}\alpha} = \frac{1}{\frac{\mathrm{d}\alpha}{\mathrm{d}q}} \times \frac{\mathrm{d}}{\mathrm{d}q}$, to show that (2.2.36) holds (excluding possible poles) for $|q| < 1$. The most general solution of this differential equation has the form

$$z = C{}_2F_1\left({\tfrac{1}{2},\tfrac{1}{2} \atop 1}; \alpha\right) + D{}_2F_1\left({\tfrac{1}{2},\tfrac{1}{2} \atop 1}; 1-\alpha\right),$$

where C and D are undetermined constants. When q lies in a neighborhood of zero, (2.2.35) shows that $(C, D) = (1, 0)$. We can analytically continue that solution to a larger connected q-domain, provided that α (and $1-\alpha$ if $D \neq 0$) does not intersect the line $[1,\infty)$. In particular, the function ${}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\alpha\right)$ has a branch cut running along $[1,\infty)$.

If we consider values of $q \in (0,\omega)$ with $\omega = e^{2\pi \mathrm{i}/3}$, then α crosses $[1,\infty)$ at the point $q = \omega e^{-\pi\sqrt{2}/3}$. Similarly, $1-\alpha$ intersects the branch cut at $q = \omega e^{-\pi/3\sqrt{2}}$. It follows that we will have to solve the hypergeometric differential equation separately on each of the three line segments. If $u = \omega e^{-\pi x/3}$, then

$$\varphi^2(u) = \begin{cases} -3\,{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-\frac{\varphi^4(-u)}{\varphi^4(u)}\right) + 2\mathrm{i}\,{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\frac{\varphi^4(-u)}{\varphi^4(u)}\right) & \text{if } x \in \left(0,\frac{1}{\sqrt{2}}\right), \\ {}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-\frac{\varphi^4(-u)}{\varphi^4(u)}\right) + 2\mathrm{i}\,{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\frac{\varphi^4(-u)}{\varphi^4(u)}\right) & \text{if } x \in \left(\frac{1}{\sqrt{2}},\sqrt{2}\right), \\ {}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-\frac{\varphi^4(-u)}{\varphi^4(u)}\right) & \text{if } x \in \left(\sqrt{2},\infty\right). \end{cases} \tag{2.2.37}$$

The coefficients in (2.2.37) can be verified from the fact that $\varphi^2(u)$ is analytic when $x \in (0,\infty)$. For example, we can check the continuity of the right-hand side of (2.2.37) by letting $u \to \omega e^{-\pi\sqrt{2}/3}$. In that case $\alpha = 1 - \frac{\varphi^4(-u)}{\varphi^4(u)} \approx 5.828\ldots$, and we have:

$$\begin{aligned} 0 =& \varphi^2\left(\omega e^{-\pi\frac{\sqrt{2}+0}{3}}\right) - \varphi^2\left(\omega e^{-\pi\frac{\sqrt{2}-0}{3}}\right) \\ =& {}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\alpha+\mathrm{i}0\right) - {}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\alpha-\mathrm{i}0\right) - 2\mathrm{i}\,{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-\alpha\right). \end{aligned}$$

This vanishing of this last expression follows from basic properties of the hypergeometric function (see problem 1 on page 276 of [47]), and therefore the right-hand side of (2.2.37) is indeed continuous at $x = \sqrt{2}$. In practice, we simply discovered (2.2.37) numerically.

We will use the theory of signature-three theta functions to prove equation (2.2.31). Recall that $c(q)$ can be expressed as an infinite product ([40], page 109):

$$\frac{c^3(q)}{27q} = \frac{f^9(-q^3)}{f^3(-q)},$$

and that the signature-three theta functions obey a differentiation formula (which can be derived from formula 4.4 on page 106 of [40]):

$$\frac{c^3(q)}{q} = \frac{a(q)}{1-\frac{c^3(q)}{a^3(q)}}\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{c^3(q)}{a^3(q)}\right).$$

It follows immediately that equation (2.2.21) reduces to

$$F_{(1,1)}(x) = \frac{2\pi^2}{81\sqrt{3}x}\mathrm{Im}\left[\int_0^{\mathrm{i}q} \frac{a(u)}{1-\frac{c^3(u)}{a^3(u)}}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{c^3(u)}{a^3(u)}\right)\mathrm{d}u\right]. \tag{2.2.38}$$

Next recall that for $|u|$ sufficiently small ([40], page 99):

$$a(u) = {}_2F_1\left({\frac{1}{3},\frac{2}{3} \atop 1};\frac{c^3(u)}{a^3(u)}\right). \tag{2.2.39}$$

In order to apply (2.2.39) to our integral, we will need to establish a generalized inversion formula which holds for $u \in (0,\mathrm{i})$. The reasoning closely follows the proof of (2.2.37), except that $c^3(u)/a^3(u) \in [1,\infty)$ when $u = \mathrm{i}e^{-\pi\sqrt{5/12}}$, and $1 - c^3(u)/a^3(u) \in [1,\infty)$ when $u = \mathrm{i}e^{-\pi/\sqrt{60}}$. Suppose that $u = \mathrm{i}e^{-\pi x/\sqrt{12}}$, then we obtain

$$a(u) = \begin{cases} 4{}_2F_1\left({\frac{1}{3},\frac{2}{3} \atop 1};\frac{c^3(u)}{a^3(u)}\right) + \sqrt{3}\mathrm{i}{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-\frac{c^3(u)}{a^3(u)}\right) & \text{if } x \in \left(0,\frac{1}{\sqrt{5}}\right), \\ 2{}_2F_1\left({\frac{1}{3},\frac{2}{3} \atop 1};\frac{c^3(u)}{a^3(u)}\right) + \sqrt{3}\mathrm{i}{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-\frac{c^3(u)}{a^3(u)}\right) & \text{if } x \in \left(\frac{1}{\sqrt{5}},\sqrt{5}\right), \\ {}_2F_1\left({\frac{1}{3},\frac{2}{3} \atop 1};\frac{c^3(u)}{a^3(u)}\right) & \text{if } x \in \left(\sqrt{5},\infty\right). \end{cases} \tag{2.2.40}$$

Finally, (2.2.31) follows from substituting (2.2.40) into (2.2.38) and simplifying. □

Finally, we will conclude this section by summarizing the formulas that follow from setting $x = 1$ in Theorem 2.2.7.

Corollary 2.2.8. *The following identities are true:*

$$\frac{27}{2\pi^2}F(1,1) = \mathrm{m}\left(y^3+z^3+1-3\sqrt[3]{2}yz\right), \tag{2.2.41}$$

$$\frac{16}{\pi^2}F(1,2) = \mathrm{m}\left(4\mathrm{i}+y+y^{-1}+z+z^{-1}\right), \tag{2.2.42}$$

$$\begin{aligned}\frac{144}{25\pi^2}F(1,4) = &\mathrm{m}\left(\frac{4}{\sqrt{\theta_1}}+y+y^{-1}+z+z^{-1}\right) \\ &+\frac{1}{4}Im\left(\int_{\theta_1}^1 \frac{{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-u\right)}{u}\mathrm{d}u\right),\end{aligned} \tag{2.2.43}$$

$$\frac{256}{9\pi^2}F(2,2) = \int_{\theta_2}^1 \frac{3u-1}{\sqrt{u(1-u)}}{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-u^2\right)\mathrm{d}u, \tag{2.2.44}$$

where $\theta_1 = \frac{1}{2}+\mathrm{i}\left(2+\sqrt{3}\right)\sqrt[4]{12}$, *and* $\theta_2 = \sqrt{2}-1$.

2.2.1 More explicit examples

In this section we will use values of class invariants to deduce some explicit formulas for Mahler measures. Recall that if $q = e^{-\pi\sqrt{m}}$, then the class invariants are defined by

$$g_m := 2^{-1/4}q^{-1/24}\left(q;q^2\right)_\infty, \qquad G_m := 2^{-1/4}q^{-1/24}\left(-q;q^2\right)_\infty.$$

It is a classical fact that G_m and g_m are algebraic numbers whenever $m \in \mathbb{Q}$, and that they satisfy the following algebraic relation:

$$\left(g_m G_m\right)^8\left(G_m^8 - g_m^8\right) = \frac{1}{4}. \tag{2.2.45}$$

Since most tables only contain values of g_m when m is even, and G_m when m is odd, our calculations will require (2.2.45). The simplest examples that we will consider follow from equation (2.2.32), while equations (2.2.31), (2.2.33), and (2.2.34) lead to slightly more complicated results.

Theorem 2.2.9. *Suppose that $m \in \mathbb{N}$, then*

$$\mathrm{m}\left(8ig_m^8G_m^4 + y + y^{-1} + z + z^{-1}\right) = \frac{16\sqrt{m}}{\pi^2}\sum_{n=1}^{\infty}\frac{b_n}{n^2}, \tag{2.2.46}$$

where

$$\sum_{n=1}^{\infty} b_n q^n = q\prod_{n=1}^{\infty}\frac{\left(1-q^{8n}\right)^3\left(1-q^{4mn}\right)^2}{\left(1-q^{8mn}\right)}.$$

The following table gives evaluations of $8g_m^8G_m^4$, closed forms for $\sum_{n=1}^{\infty} b_n q^n$, and states whether or not b_n is multiplicative:

m	$8g_m^8G_m^4$	$\sum_{n=1}^{\infty} b_n q^n$	*Multiplicative?*
1	4	$qf^2\left(-q^4\right)f^2\left(-q^8\right)$	*Yes*
2	$4\sqrt{2+2\sqrt{2}}$	$q\frac{f^5\left(-q^8\right)}{f\left(-q^{16}\right)}$	*No*
3	$4\left(2+\sqrt{3}\right)$	$q\frac{f^3\left(-q^8\right)f^2\left(-q^{12}\right)}{f\left(-q^{24}\right)}$	*No*
7	$4\left(8+3\sqrt{7}\right)$	$q\frac{f^3\left(-q^8\right)f^2\left(-q^{28}\right)}{f\left(-q^{56}\right)}$	*No*
9	$4\left(7+4\sqrt[4]{12}+2\sqrt[4]{12^2}+\sqrt[4]{12^3}\right)$	$q\frac{f^3\left(-q^8\right)f^2\left(-q^{36}\right)}{f\left(-q^{72}\right)}$	*No*
15	$4\left(28+16\sqrt{3}+12\sqrt{5}+7\sqrt{15}\right)$	$q\frac{f^3\left(-q^8\right)f^2\left(-q^{60}\right)}{f\left(-q^{120}\right)}$	*No*

Proof. Setting $q = e^{-\pi\sqrt{m}}$ reduces equation (2.2.20) to

$$F_{(1,2)}\left(\sqrt{m}\right) = \frac{\pi^2}{16\sqrt{m}}\mathrm{m}\left(8\mathrm{i}g_m^8 G_m^4 + y + y^{-1} + z + z^{-1}\right). \tag{2.2.47}$$

Therefore, we can obtain Mahler measure formulas by appealing to tables of class invariants [40]. If $m \in \mathbb{N}$, we can also use the definition of $F_{(1,2)}(x)$ to show that

$$F_{(1,2)}(\sqrt{m}) = \sum_{n=1}^{\infty} \frac{b_n}{n^2},$$

where b_n has the stated generating function.

The only remaining task is to check the values of $8g_m^8 G_m^4$. In particular, we can solve (2.2.45) to show that

$$\begin{aligned} 8g_m^8 G_m^4 &= 4\left(G_m^{12} + \sqrt{G_m^{24} - 1}\right) \\ &= 4\sqrt{2}g_m^6\sqrt{g_m^{12} + \sqrt{g_m^{24} + 1}}. \end{aligned}$$

For example, since $G_1 = 1$, it follows that $8g_1^8 G_1^4 = 4$. While this type of argument naturally leads to equations involving nested radicals, many of those formulas simplify with sufficient effort.

□

If we consider examples involving $F_{(1,1)}(x)$, then we can obtain two distinct types of formulas. The first class of identities occurs when $\mathrm{Im}\left(a^3(\mathrm{i}q)/b^3(\mathrm{i}q)\right) = 0$. In two of those cases the n_2 term in (2.2.31) vanishes, yielding formulas that reduce to generalized hypergeometric functions.

Theorem 2.2.10. *Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio, then*

$$F_{(1,1)}(1) = \frac{2\pi^2}{27} n\left(3\sqrt[3]{2}\right), \tag{2.2.48}$$

$$F_{(1,1)}\left(\frac{1}{\sqrt{5}}\right) = \frac{2\sqrt{5}\pi^2}{27}\tilde{n}\left(\frac{3}{\sqrt[3]{\phi}}\right). \tag{2.2.49}$$

Recall that $n(k)$ is defined in equation (2.1.7).

In the case when $\mathrm{Im}\left(a^3(\mathrm{i}q)/b^3(\mathrm{i}q)\right) \neq 0$, we can establish many interesting formulas by setting $q = e^{-\pi\sqrt{\frac{a}{b}}}$. The next theorem provides two examples where $b = 1$ and a is a product of small primes.

Theorem 2.2.11. *Let $\omega = e^{\pi i/3}$ and recall that $n_2(k)$ is defined in Theorem 2.2.7. We have:*

$$L(g,2) = r_1 \frac{2\pi^2}{81\sqrt{3}} n_2(\theta) = r_2 F_{(1,1)}\left(\sqrt{m}\right), \tag{2.2.50}$$

for the following values of m, $g(q)$, θ, r_1, and r_2:

m	$g(q)$	θ	r_1	r_2
9	$qf^3\left(-q^2\right)f\left(-q^{18}\right)$	$\frac{9}{250}\left(7-19\sqrt[3]{2}w-2\sqrt[3]{4}w^2\right)$	3	9
25	$q^7f^3\left(-q^6\right)f\left(-q^{150}\right)$	$\frac{1}{1+z^6}$, *where* $\frac{\left(z^2+3z+1\right)^3}{z^6+1}=2$	$\frac{1}{5}$	1

We will conclude this section by pointing out that $F_{(1,1)}(x)$ can be expressed as a four-dimensional lattice sum for all values of x:

$$F_{(1,1)}(x) = 16 \sum_{\substack{n_i=-\infty \\ i\in\{1,2,3,4\}}}^{\infty} \frac{(-1)^{n_1+n_2+n_3+n_4}}{\left[(6n_1+1)^2+(6n_2+1)^2+(6n_3+1)^2+x^2(6n_4+1)^2\right]^2}.$$

While this formula for $F_{(1,1)}(x)$ resembles the definition of $F(b,c)$, we have shown that $F_{(1,1)}(x)$ is much easier to understand. By equation (2.2.31) we can obtain hypergeometric formulas for $F_{(1,1)}(x)$ whenever $x \in \mathbb{Q}$. For instance, applying equation (2.1.38) to formula (2.2.49), yields an interesting lattice sum identity:

$$\frac{3456}{\sqrt{15}} \sum_{\substack{n_i=-\infty \\ i\in\{1,2,3,4\}}}^{\infty} \frac{(-1)^{n_1+n_2+n_3+n_4}}{\left[(6n_1+1)^2+(6n_2+1)^2+(6n_3+1)^2+\frac{1}{5}(6n_4+1)^2\right]^2}$$
$$= \frac{C_1}{\sqrt[3]{\phi}}\,{}_3F_2\left({\frac{1}{3},\frac{1}{3},\frac{1}{3} \atop \frac{2}{3},\frac{4}{3}};\frac{1}{\phi}\right) + \frac{C_2}{\sqrt[3]{\phi^2}}\,{}_3F_2\left({\frac{2}{3},\frac{2}{3},\frac{2}{3} \atop \frac{4}{3},\frac{5}{3}};\frac{1}{\phi}\right), \tag{2.2.51}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, $C_1 = 2\sqrt[3]{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{6}\right)$, and $C_2 = 3\sqrt{3}\Gamma^3\left(\frac{2}{3}\right)$. Equation (2.2.51) also resembles formulas that Forrester and Glasser established for three-dimensional sums associated with NaCl lattices [46].

2.2.2 Remarks on $F(1,3)$ and higher values of $F(b,c)$

Finally, we will speculate on how one might reduce higher values of $F(b,c)$ to hypergeometric integrals. Our proof of the $F(1,3)$ formula will be instructive. Recall that Rodriguez-Villegas demonstrated that

$$\frac{4\pi^2}{81}n(-6) = \operatorname{Re}\left(\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{\chi_{-3}(n)}{\left(3\left(\frac{1+i\sqrt{3}}{2}\right)m+n\right)^2\left(3\left(\frac{1-i\sqrt{3}}{2}\right)m+n\right)}\right), \tag{2.2.52}$$

and then used Deuring's theorem to equate this Eisenstein series to the L series of a CM elliptic curve of conductor 27. A different proof could have been constructed from numerically observing that

$$\begin{aligned} &q\prod_{n=1}^{\infty}\left(1-q^{3n}\right)^2\left(1-q^{9n}\right)^2 \\ &\qquad = \frac{1}{4}\sum_{j=1}^{2}\chi_{-3}(j)\sum_{n,m=-\infty}^{\infty}\left[(6m+j)+3(6n+j)\right]q^{\frac{(6m+j)^2+3(6n+j)^2}{4}}. \end{aligned} \tag{2.2.53}$$

The modularity theorem implies that $qf^2\left(-q^3\right)f^2\left(-q^9\right)$ is associated to the correct elliptic curve, hence the Mellin transform of the left-hand side of equation (2.2.53) will equal $L(E,s)$. Since the Mellin transform (at $s=2$) of the right-hand side trivially equals the right-hand side of equation (2.2.52), it just remains to prove (2.2.53). By applying limiting cases of the triple and quintuple product identities, we can show that equation (2.2.53) is equivalent to an identity between eta functions:

$$\begin{aligned} &4qf^2\left(-q^3\right)f^2\left(-q^9\right) \\ &\quad = q\frac{f^5\left(-q^6\right)f\left(-q^{36}\right)f^2\left(-q^{54}\right)}{f^2\left(-q^{12}\right)f\left(-q^{18}\right)f\left(-q^{108}\right)} + 3q\frac{f\left(-q^{12}\right)f^7\left(-q^{18}\right)}{f\left(-q^6\right)f^3\left(-q^{36}\right)} \\ &\qquad - 2q^4\frac{f^2\left(-q^3\right)f^2\left(-q^{12}\right)f^2\left(-q^{18}\right)f\left(-q^{27}\right)f\left(-q^{108}\right)}{f\left(-q^6\right)f\left(-q^9\right)f\left(-q^{36}\right)f\left(-q^{54}\right)} \\ &\qquad - 6q^4\frac{f^2\left(-q^6\right)f^3\left(-q^9\right)f^3\left(-q^{36}\right)}{f\left(-q^3\right)f\left(-q^{12}\right)f^2\left(-q^{18}\right)}. \end{aligned} \tag{2.2.54}$$

Identities between modular forms, such as equation (2.2.54), are usually established by checking that both sides of the proposed formula have the same McLauren series expansion for sufficiently many terms. If we use the inversion formula for the eta function, then (2.2.54) can be viewed as an example of a *mixed modular equation*, i.e. an algebraic relation between the moduli of isogenous elliptic curves. From this perspective, it seems likely that the rest of Boyd's conjectures will follow from discovering appropriate mixed modular equations.

While it is probably unreasonable to expect to find useful series expansions for $qf(-q^A)f(-q^{Ab})f(-q^{Ac})f(-q^{Abc})$, when $(b,c)\in\{(1,5),(1,11),(2,3),(2,7),(3,5)\}$ and $A=24/\left((1+b)(1+c)\right)$, this does not seem to rule out the possibility of finding such results for linear combinations of related eta products. It might be interesting to examine whether or not a series expansion such as equation (2.2.53) exists for the following function:

$$\sum_{j=1}^{N} c_j q^j f(-q^{Aj})f(-q^{Abj})f(-q^{Acj})f(-q^{Abcj}), \tag{2.2.55}$$

for appropriate values of $c_j\in\mathbb{Q}$. Boyd's conjectures would most likely follow from such a theorem, as the Mellin transform (at $s=2$) of (2.2.55) equals a rational multiple of $L(E,2)$.

2.3 Connections with the elliptic dilogarithm

In this section we will point out that the method from Section 2.2 can be used to establish relations between values of $F(b,c)$ and the elliptic dilogarithm.

Definition 2.3.1. *Recall that the elliptic dilogarithm is defined by*

$$\mathscr{L}\left(x,q\right)=\sum_{n=-\infty}^{\infty} D\left(xq^n\right),$$

where $D(z) = \mathit{Im}\left(\mathrm{Li}_2(z)+\log|z|\log(1-z)\right)$ *is the Bloch-Wigner dilogarithm.*

In the previous section we integrated cusp forms to obtain identities between L-values and hypergeometric functions. For example, we used the following two-dimensional series:

$$qf^2\left(-q^4\right)f^2\left(-q^8\right)=\sum_{\substack{n=-\infty\\k=0}}^{\infty}(-1)^{n+k}(2k+1)q^{(2n)^2+(2k+1)^2},$$

to recover formula (2.1.13) for $F(1,2)$. In general, comparing different series expansions for the same cusp form will often lead to relations between hypergeometric functions and the elliptic dilogarithm. For instance, applying our method to the following identity

$$qf^2\left(-q^4\right)f^2\left(-q^8\right)=\sum_{n,k=0}^{\infty}(-1)^k(2k+1)q^{\frac{(2k+1)^2+(2n+1)^2}{2}},$$

yields

$$F(1,2)=\frac{\pi}{2}\mathscr{L}\left(\mathrm{i},-e^{\pi}\right), \tag{2.3.1}$$

which is comparable in difficulty to equation (2.1.13).

Theorem 2.3.2. *Suppose that $x\in\mathbb{R}$ is sufficiently large and $\omega=e^{2\pi\mathrm{i}/3}$, then the following identities are true:*

$$4\sum_{n,k=0}^{\infty}\frac{(-1)^k(2k+1)}{\left[(2n+1)^2+x^2(2k+1)^2\right]^2}=\frac{\pi}{2x^3}\mathscr{L}\left(\mathrm{i},-e^{\pi x}\right), \tag{2.3.2}$$

$$\sum_{n,k=-\infty}^{\infty}\frac{(3k+1)}{\left[(2n+1)^2+x^2(3k+1)^2\right]^2}=\frac{\pi}{2\sqrt{3}x^3}\mathscr{L}\left(\omega,-e^{\pi x}\right). \tag{2.3.3}$$

We will conclude this section by briefly pointing out a pair of identities involving the L-functions of modular forms. If we define $f_1(q)$ and $f_2(q)$ by

$$f_1(q):=q\prod_{n=1}^{\infty}\frac{\left(1-q^{8n}\right)^5}{\left(1-q^{16n}\right)},\qquad f_2(q):=q^3\prod_{n=1}^{\infty}\frac{\left(1-q^{16n}\right)^5}{\left(1-q^{8n}\right)},$$

then we have shown that

$$L\left(f_1,2\right)=\frac{\pi^2}{16\sqrt{2}}\mathrm{m}\left(4\mathrm{i}\sqrt{2+2\sqrt{2}}+y+y^{-1}+z+z^{-1}\right),$$

and

$$L\left(f_2,2\right)=\frac{\pi}{16\sqrt{2}}\mathscr{L}\left(\mathrm{i},-e^{\pi\sqrt{2}}\right).$$

Despite the fact that $f_1(q)$ and $f_2(q)$ are closely related, their L-values reduce to special cases of different functions. We will hypothesize that only L-functions of modular forms with nice arithmetic properties should be expressible in terms of both elliptic dilogarithms and Mahler measures.

2.4 Higher lattice sums and conclusion

Perhaps the moral of this paper is that lattice sums are difficult to deal with. While many hypergeometric formulas have been experimentally discovered for $F(b,c)$, only the simplest cases have been rigorously proved. Notice that our formula for $F(1,4)$, equation (2.2.44), involves Meijer's G-function disguised as a hypergeometric integral:

$$\mathrm{Im}\left(\int_k^1 \frac{{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};1-u\right)}{u}\mathrm{d}u\right) = \frac{1}{\pi^2}\mathrm{Im}\left(G_{3,3}^{3,2}\left(k\Big|{\frac{1}{2},\frac{1}{2},1 \atop 0,0,0}\right)\right).$$

This identity almost certainly rules out the possibility of expressing $F(1,4)$ as a Mahler measure, and it also indicates that any explicit formula for $F(b,c)$ should reduce to Meijer G functions in certain instances.

It seems difficult to conjecture what types of formulas should exist for higher dimensional lattice sums. Let us consider the k-dimensional lattice sum

$$F_k^{(j)}(x_1,\ldots,x_k) := \sum_{n_i=-\infty}^{\infty}\frac{(-1)^{n_1+\cdots+n_k}}{(x_1(6n_1+1)^2+\cdots+x_k(6n_k+1)^2)^j},$$

which arises from integrating eta products. Many non-trivial linear dependencies exist between different values of $F_k^{(j)}(x_1,\ldots,x_k)$ when $k\gg 4$. For example, clearing denominators and then integrating formula (2.2.26) leads to a linear dependence between 13-dimensional lattice sums:

$$\begin{aligned}F_{13}^{(j)}\left(\underbrace{2,\ldots,2}_{8},\underbrace{3,\ldots,3}_{4},6\right) =&F_{13}^{(j)}\left(1,\underbrace{2,\ldots,2}_{4},\underbrace{3,\ldots,3}_{8}\right)\\ &+F_{13}^{(j)}\left(\underbrace{1,\ldots,1}_{4},\underbrace{6,\ldots,6}_{9}\right).\end{aligned}$$

Although we have only proved hypergeometric formulas for $F_k^{(j)}$ when $k\le 6$, and $j\in\{2,3\}$, it might be interesting to search for higher dimensional examples numerically. Lattice sums with Euler products are the best candidates ($F(b,c)$ belongs to this class of functions when $(1+b)(1+c)$ divides 24), since they seem to be the only lattice sums which ever reduce to hypergeometric functions with rational arguments. To illustrate this principle, we can apply the methods of Section 2.2 to show that

$$F_6^{(2)}(1,1,1,m,m,m) = \frac{\pi^2}{64\sqrt{m}G_m^{12}}\,{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};\frac{1}{G_m^{24}}\right), \tag{2.4.1}$$

where G_m is the usual class invariant. For $m \in \{1, 3, 7\}$, we have the following table:

m	$g(q)$	$L(g,2)$
1	$q\prod_{n=1}^{\infty}\left(1-q^{4n}\right)^6$	$\frac{\pi^2}{16}{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};1\right)$
3	$q\prod_{n=1}^{\infty}\left(1-q^{2n}\right)^3\left(1-q^{6n}\right)^3$	$\frac{\pi^2}{8\sqrt{3}}{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};\frac{1}{4}\right)$
7	$q\prod_{n=1}^{\infty}\left(1-q^{n}\right)^3\left(1-q^{7n}\right)^3$	$\frac{\pi^2}{8\sqrt{7}}{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};\frac{1}{64}\right)$

In each of these three examples, $g(q)$ is a multiplicative weight 3 cusp form. On the other hand, if we consider a similar looking, but non-multiplicative eta product:

$$g_1(q) = q\prod_{n=1}^{\infty}\left(1-q^{3n}\right)^3\left(1-q^{5n}\right)^3,$$

then equation (2.4.1) gives a formula for $L(g_1, 2)$ involving $G_{5/3}^{24}$. Since $G_{5/3}$ is a root of the following polynomial equation:

$$0 = x^{12} - 8\sqrt[4]{8}\left(\frac{1+\sqrt{5}}{2}\right)^3 x^9 + 4\sqrt[4]{2}\left(\frac{1+\sqrt{5}}{2}\right)x^3 + 8\left(\frac{1+\sqrt{5}}{2}\right)^4,$$

it is not hard to see that $G_{5/3}^{24}$ is an irrational algebraic number.

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Chapter 3

Functional equations for Mahler measures

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3.1 History and introduction

The goal of this paper is to establish identities between the logarithmic Mahler measures of polynomials with zero varieties corresponding to genus-one curves. Recall that the logarithmic Mahler measure (which we shall henceforth simply refer to as the Mahler measure) of an n-variable Laurent polynomial $P(x_1, x_2, \ldots, x_n)$ is defined by

$$\mathrm{m}\left(P(x_1, \ldots, x_n)\right) = \int_0^1 \cdots \int_0^1 \log \left|P\left(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}\right)\right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n.$$

Many difficult questions surround the special functions defined by Mahler measures of elliptic curves.

The first example of the Mahler measure of a genus-one curve was studied by Boyd [63] and Deninger [65]. Boyd found that

$$\mathrm{m}\left(1 + x + \frac{1}{x} + y + \frac{1}{y}\right) \stackrel{?}{=} L'(E, 0), \tag{3.1.1}$$

where E denotes the elliptic curve of conductor 15 that is the projective closure of $1 + x + \frac{1}{x} + y + \frac{1}{y} = 0$. As usual, $L(E, s)$ is its L-function, and the question mark above the equals sign indicates numerical equality verified up to 60 decimal places.

Deninger [65] gave an interesting interpretation of this formula. He obtained the Mahler measure by evaluating the Bloch regulator of an element $\{x, y\}$ from a certain K-group. In other words, the Mahler measure is given

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by a value of an Eisenstein-Kronecker series. Therefore Bloch's and Beilinson's conjectures predict that

$$\mathrm{m}\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right)=cL'(E,0),$$

where c is some rational number. Let us add that, even if Beilinson's conjectures were known to be true, this would not suffice to prove equality (3.1.1), since we still would not know the height of the rational number c.

This picture applies to other situations as well. Boyd [63] performed extensive numerical computations within the family of polynomials $k+x+\frac{1}{x}+y+\frac{1}{y}$, as well as within some other genus-one families. Boyd's numerical searches led him to conjecture identities such as

$$\mathrm{m}\left(5+x+\frac{1}{x}+y+\frac{1}{y}\right)\stackrel{?}{=}6\mathrm{m}\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right),$$

$$\mathrm{m}\left(8+x+\frac{1}{x}+y+\frac{1}{y}\right)\stackrel{?}{=}4\mathrm{m}\left(2+x+\frac{1}{x}+y+\frac{1}{y}\right).$$

Boyd conjectured conditions predicting when formulas like Eq. (3.1.1) should exist for the Mahler measures of polynomials with integral coefficients. This was further studied by Rodriguez-Villegas [71] who interpreted these conditions in the context of Bloch's and Beilinson's conjectures. Furthermore, Rodriguez-Villegas used modular forms to express the Mahler measures as Kronecker-Eisenstein series in more general cases. In turn, this allowed him to prove some equalities such as

$$\mathrm{m}\left(4\sqrt{2}+x+\frac{1}{x}+y+\frac{1}{y}\right)=L'\left(E_{4\sqrt{2}},0\right), \tag{3.1.2}$$

$$\mathrm{m}\left(3\sqrt{2}+x+\frac{1}{x}+y+\frac{1}{y}\right)=\frac{5}{2}L'\left(E_{3\sqrt{2}},0\right), \tag{3.1.3}$$

where E_k denotes the projective closure of $k+x+\frac{1}{x}+y+\frac{1}{y}=0$. The first equality can be proved using the fact that the corresponding elliptic curve has complex multiplication, and therefore the conjectures are known for this case due to Bloch [62]. The second equality depends on the fact that one has the modular curve $X_0(24)$, and the conjectures then follow from a result of Beilinson.

Rodriguez-Villegas [72] subsequently used the relationship between Mahler measures and regulators to prove a conjecture of Boyd [63]:

$$\mathrm{m}(y^2+2xy+y-x^3-2x^2-x)=\frac{5}{7}\mathrm{m}(y^2+4xy+y-x^3+x^2).$$

It is important to point out that he proved this identity without actually expressing the Mahler measures in terms of L-series. Bertin [61] has also proved similar identities using these ideas.

Although the conjecture in Eq. (3.1.1) remains open, we will in fact prove two of Boyd's other conjectures in this paper.

Theorem 3.1.1. *The following identities are true:*

$$m\left(2+x+\frac{1}{x}+y+\frac{1}{y}\right)=L'\left(E_{3\sqrt{2}},0\right), \tag{3.1.4}$$

$$m\left(8+x+\frac{1}{x}+y+\frac{1}{y}\right)=4L'\left(E_{3\sqrt{2}},0\right). \tag{3.1.5}$$

Our proof of Theorem 3.1.1 follows from combining two interesting "functional equations" for the function

$$m(k):=\mathrm{m}\left(k+x+\frac{1}{x}+y+\frac{1}{y}\right).$$

Kurokawa and Ochiai [67] recently proved the first functional equation. They showed that if $k\in\mathbb{R}\backslash\{0\}$:

$$m\left(4k^2\right)+m\left(\frac{4}{k^2}\right)=2m\left(2\left(k+\frac{1}{k}\right)\right). \tag{3.1.6}$$

In Section 3.3 we use regulators to give a new proof of Eq. (3.1.6). We will also prove a second functional equation in Section 3.2.1 using q-series. In particular, if k is nonzero and $|k|<1$:

$$m\left(2\left(k+\frac{1}{k}\right)\right)+m\left(2\left(\mathrm{i}k+\frac{1}{\mathrm{i}k}\right)\right)=m\left(\frac{4}{k^2}\right). \tag{3.1.7}$$

Theorem 3.1.1 follows from setting $k=1/\sqrt{2}$ in both identities, and then showing that $5m\left(\mathrm{i}\sqrt{2}\right)=3m\left(3\sqrt{2}\right)$. We have proved this final equality in Section 3.3.6.

This paper is divided into two sections of roughly equal length. In Section 3.2 we will prove more identities like Eq. (3.1.7), which arise from expanding Mahler measures in q-series. In particular, we will look at identities for four special functions defined by the Mahler measures of genus-one curves (see equations (3.2.1) through (3.2.4) for notation). Equation (3.2.14) undoubtedly constitutes the most important result in this part of the paper, since it implies that infinitely many identities like Eq. (3.1.7) exist.

Subsections 3.2.1 and 3.2.2 are mostly devoted to transforming special cases of Eq. (3.2.14) into interesting identities between the Mahler measures of rational polynomials. While the theorems in those subsections rely heavily on Ramanujan's theory of modular equations to alternative bases, we have attempted to maximize readability by eliminating q-series manipulation wherever possible. Finally, we have devoted Subsection 3.2.3 to proving some useful computational formulas. As a corollary we establish several new transformations for hypergeometric functions, including:

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{k(1-k)^2}{(1+k)^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}&\\ =\frac{(1+k)^2}{\sqrt{(1+k^2)\left((1-k-k^2)^2-5k^2\right)}}&\\ \times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k^5\left(1+k-k^2\right)}{(1+k^2)^2\left((1-k-k^2)^2-5k^2\right)^2}\right).&\end{aligned} \tag{3.1.8}$$

We have devoted Section 3.3 to further studying the relationship between Mahler measures and regulators. We show how to recover the Mahler measure q-series expansions and the Kronecker-Einsenstein series directly from Bloch's formula for the regulator. This in turn shows that the Mahler measure identities can be viewed as consequences of functional identities for the elliptic dilogarithm.

Many of the identities in this paper can be interpreted from both a regulator perspective, and from a q-series perspective. The advantage of the q-series approach is that it simplifies the process of finding new identities. The fundamental result in Section 3.2, Eq. (3.2.14), follows easily from the Mahler measure q-series expansions. Unfortunately the q-series approach does not provide an easy way to explain identities like Eq. (3.1.6). Unlike most of the other formulas in Section 3.2, Kurokawa's and Ochiai's result *does not* follow from Eq. (3.2.14). An advantage of the regulator approach, is that it enables us to construct proofs of both Eq. (3.1.6) and Eq. (3.1.7) from a unified perspective. Additionally, the regulator approach seems to provide the only way to prove the final step in Theorem 3.1.1, namely to show that $5m\left(\mathrm{i}\sqrt{2}\right) = 3m\left(3\sqrt{2}\right)$. Thus, a complete view of this subject matter should incorporate both regulator and q-series perspectives.

3.2 Mahler measures and q-series

In this paper we will consider four important functions defined by Mahler measures:

$$\mu(t) = \mathrm{m}\left(\frac{4}{\sqrt{t}} + x + \frac{1}{x} + y + \frac{1}{y}\right), \tag{3.2.1}$$

$$n(t) = \mathrm{m}\left(x^3 + y^3 + 1 - \frac{3}{t^{1/3}}xy\right), \tag{3.2.2}$$

$$g(t) = \mathrm{m}\left((x+y)(x+1)(y+1) - \frac{1}{t}xy\right), \tag{3.2.3}$$

$$r(t) = \mathrm{m}\left((x+y+1)(x+1)(y+1) - \frac{1}{t}xy\right). \tag{3.2.4}$$

Throughout Section 3.2 will use the notation $\mu(t) = m\left(\frac{4}{\sqrt{t}}\right)$ for convenience. Recall from [71] and [76], that each of these functions has a simple q-series expansion when t is parameterized correctly. To summarize, if we let $(x;q)_\infty = (1-x)(1-xq)\left(1-xq^2\right)\ldots$, and

$$M(q) = 16q\frac{(q;q)_\infty^8 \left(q^4;q^4\right)_\infty^{16}}{\left(q^2;q^2\right)_\infty^{24}}, \tag{3.2.5}$$

$$N(q) = \frac{27q\left(q^3;q^3\right)_\infty^{12}}{(q;q)_\infty^{12} + 27q\left(q^3;q^3\right)_\infty^{12}}, \tag{3.2.6}$$

$$G(q) = q^{1/3}\frac{\left(q;q^2\right)_\infty}{\left(q^3;q^6\right)_\infty^3}, \tag{3.2.7}$$

$$R(q) = q^{1/5}\frac{\left(q;q^5\right)_\infty\left(q^4;q^5\right)_\infty}{\left(q^2;q^5\right)_\infty\left(q^3;q^5\right)_\infty}, \tag{3.2.8}$$

then for $|q|$ sufficiently small

$$\mu\left(M(q)\right) = -\operatorname{Re}\left[\frac{1}{2}\log(q) + 2\sum_{j=1}^{\infty} j\chi_{-4}(j)\log\left(1-q^j\right)\right], \tag{3.2.9}$$

$$n\left(N(q)\right) = -\operatorname{Re}\left[\frac{1}{3}\log(q) + 3\sum_{j=1}^{\infty} j\chi_{-3}(j)\log(1-q^j)\right], \tag{3.2.10}$$

$$g\left(G^3(q)\right) = -\operatorname{Re}\left[\log(q) + \sum_{j=1}^{\infty}(-1)^{j-1} j\chi_{-3}(j)\log(1-q^j)\right], \tag{3.2.11}$$

$$r\left(R^5(q)\right) = -\operatorname{Re}\left[\log(q) + \sum_{j=1}^{\infty} j\operatorname{Re}\left[(2-\mathrm{i})\chi_r(j)\right]\log\left(1-q^j\right)\right]. \quad (3.2.12)$$

In particular, $\chi_{-3}(j)$ and $\chi_{-4}(j)$ are the usual Dirichlet characters, and $\chi_r(j)$ is the character of conductor five with $\chi_r(2) = \mathrm{i}$. We have used the notation $G(q)$ and $R(q)$, as opposed to something like $\tilde{G}(q) = G^3(q)$, in order to preserve Ramanujan's notation. As usual, $G(q)$ corresponds to Ramanujan's cubic continued fraction, and $R(q)$ corresponds to the Rogers-Ramanujan continued fraction [53].

The first important application of the q-series expansions is that they can be used to calculate the Mahler measures numerically. For example, we can calculate $\mu(1/10)$ with Eq. (3.2.9), provided that we can first determine a value of q for which $M(q) = 1/10$. Fortunately, the theory of elliptic functions shows that if $\alpha = M(q)$, then

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2}\atop 1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2}\atop 1;\alpha\right)}\right). \quad (3.2.13)$$

Using Eq. (3.2.13) we easily compute $q = .01975\ldots$, and it follows that $\mu(1/10) = 2.524718\ldots$ The function defined in Eq. (3.2.13) is called the *elliptic nome*, and is sometimes denoted by $q_2(\alpha)$. Theorem 3.2.6 provides similarly explicit inversion formulas for Eqs. (3.2.5) through (3.2.8).

The second, and perhaps more significant fact that follows from these q-series, is that linear dependencies exist between the Mahler measures. In particular, if

$$f(q) \in \left\{\mu\left(M(q)\right), n\left(N(q)\right), g\left(G^3(q)\right), r\left(R^5(q)\right)\right\},$$

then for an appropriate prime p

$$\sum_{j=0}^{p-1} f\left(\mathrm{e}^{2\pi \mathrm{i} j/p} q\right) = (1+p^2\chi(p)) f\left(q^p\right) - p\chi(p) f\left(q^{p^2}\right), \quad (3.2.14)$$

where $\chi(j)$ is the character from the relevant q-series. The prime p satisfies the restriction that $p \neq 2$ when $f(q) = g\left(G^3(q)\right)$, and $p \not\equiv 2,3 \pmod 5$ when $f(q) = r\left(R^5(q)\right)$. The astute reader will immediately recognize that Eq. (3.2.14) is essentially a Hecke eigenvalue equation. A careful analysis of the exceptional case that occurs when $p = 2$ and $f(q) = g\left(G^3(q)\right)$ leads to the important and surprising inverse relation:

$$\begin{aligned} 3n\left(N(q)\right) &= g\left(G^3\left(q\right)\right) - 8g\left(G^3\left(-q\right)\right) + 4g\left(G^3\left(q^2\right)\right), \\ 3g\left(G^3(q)\right) &= n\left(N(q)\right) + 4n\left(N\left(q^2\right)\right). \end{aligned} \quad (3.2.15)$$

In the next two subsections we will discuss methods for transforming Eq. (3.2.14) and Eq. (3.2.15) into so-called functional equations.

3.2.1 Functional equations from modular equations

Since the primary goal of this paper is to find relations between the Mahler measures of *rational* (or at least algebraic) polynomials, we will require modular equations to simplify our results. For example, consider Eq. (3.2.14) when $f(q) = \mu\left(M(q)\right)$ and $p = 2$:

$$\mu\left(M(q)\right) + \mu\left(M(-q)\right) = \mu\left(M\left(q^2\right)\right). \tag{3.2.16}$$

For our purposes, Eq. (3.2.16) is only interesting if $M(q)$, $M(-q)$, and $M\left(q^2\right)$ are all simultaneously algebraic. Fortunately, it turns out that $M(q)$ and $M\left(q^2\right)$ (hence also $M(-q)$ and $M\left(q^2\right)$) satisfy a well known polynomial relation.

Definition 3.2.1. *Suppose that $F(q) \in \{M(q), N(q), G(q), R(q)\}$. An n'th degree modular equation is an algebraic relation between $F(q)$ and $F(q^n)$.*

We will not need to derive any new modular equations in this paper. Berndt proved virtually all of the necessary modular equations while editing Ramanujan's notebooks, see [53], [56], [57], and [58]. Ramanujan seems to have arrived at most of his modular equations through complicated q-series manipulations (of course this is speculation since he did not write down any proofs!). Modular equations involving $M(q)$ correspond to the classical modular equations [57], relations for $N(q)$ correspond to Ramanujan's signature three modular equations [58], and most of the known modular equations for $G(q)$ and $R(q)$ appear in [53].

Now we can finish simplifying Eq. (3.2.16). Since the classical second-degree modular equation shows that whenever $|q| < 1$

$$\frac{4M\left(q^2\right)}{\left(1+M\left(q^2\right)\right)^2} = \left(\frac{M(q)}{M(q)-2}\right)^2,$$

we easily obtain the parameterizations: $M(q) = \frac{4k^2}{(1+k^2)^2}$, $M(-q) = \frac{-4k^2}{(1-k^2)^2}$, and $M\left(q^2\right) = k^4$. Substituting these parametric formulas into Eq. (3.2.16) yields:

Theorem 3.2.2. *The following identity holds whenever* $|k|<1$*:*

$$\begin{aligned}\mathrm{m}\left(\frac{4}{k^2}+x+\frac{1}{x}+y+\frac{1}{y}\right)=&\mathrm{m}\left(2\left(k+\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right)\\&+\mathrm{m}\left(2\mathrm{i}\left(k-\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right).\end{aligned}\tag{3.2.17}$$

We need to make a few remarks about working with modular equations before proving the main theorem in this section. Suppose that for some algebraic function $P(X,Y)$:

$$P\left(F(q),F\left(q^p\right)\right)=0,$$

where $F(q)\in\{M(q),N(q),G(q),R(q)\}$. Using the elementary change of variables, $q\to \mathrm{e}^{2\pi \mathrm{i}j/p}q$, it follows that $P\left(F\left(\mathrm{e}^{2\pi\mathrm{i}j/p}q\right),F\left(q^p\right)\right)=0$ for every $j\in\{0,1,\dots,p-1\}$. If $P(X,Y)$ is symmetric in X and Y, it also follows that $P\left(F\left(q^{p^2}\right),F\left(q^p\right)\right)=0$. Therefore, if $P(X,Y)$ is sufficiently simple (for example a symmetric genus-zero polynomial), we can find simultaneous parameterizations for $F\left(q^p\right)$, $F\left(q^{p^2}\right)$, and $F\left(\mathrm{e}^{2\pi\mathrm{i}j/p}q\right)$ for all j. In such an instance, Eq. (3.2.14) reduces to an interesting functional equation for one of the four Mahler measures $\{\mu(t),n(t),g(t),r(t)\}$. Five basic functional equations follow from applying these ideas to Eq. (3.2.14).

Theorem 3.2.3. *For* $|k|<1$ *and* $k\neq 0$*, we have*

$$\mu\left(\frac{4k^2}{\left(1+k^2\right)^2}\right)+\mu\left(\frac{-4k^2}{\left(1-k^2\right)^2}\right)=\mu\left(k^4\right).\tag{3.2.18}$$

The following identities hold for $|u|$ *sufficiently small but non-zero:*

$$\begin{aligned}n\left(\frac{27u(1+u)^4}{2\left(1+4u+u^2\right)^3}\right)+&n\left(-\frac{27u(1+u)}{2\left(1-2u-2u^2\right)^3}\right)\\&=2n\left(\frac{27u^4(1+u)}{2\left(2+2u-u^2\right)^3}\right)-3n\left(\frac{27u^2(1+u)^2}{4\left(1+u+u^2\right)^3}\right).\end{aligned}\tag{3.2.19}$$

If $\zeta_3=\mathrm{e}^{2\pi\mathrm{i}/3}$*, and* $Y(t)=1-\left(\frac{1-t}{1+2t}\right)^3$*, then*

$$n\left(u^3\right)=\sum_{j=0}^{2}n\left(Y\left(\zeta_3^j u\right)\right).\tag{3.2.20}$$

If $\zeta_3 = \mathrm{e}^{2\pi \mathrm{i}/3}$*, and* $Y(t) = t\left(\frac{1-t+t^2}{1+2t+4t^2}\right)$*, then*

$$g(u^3) = \sum_{j=0}^{2} g\left(Y\left(\zeta_3^j u\right)\right). \tag{3.2.21}$$

If $\zeta_5 = \mathrm{e}^{2\pi \mathrm{i}/5}$*, and* $Y(t) = t\left(\frac{1-2t+4t^2-3t^3+t^4}{1+3t+4t^2+2t^3+t^4}\right)$*, then*

$$r\left(u^5\right) = \sum_{j=0}^{4} r\left(Y\left(\zeta_5^j u\right)\right). \tag{3.2.22}$$

Proof. We have already sketched a proof of Eq. (3.2.18) in the discussion preceding Theorem 3.2.2.

The proof of Eq. (3.2.19) requires the second-degree modular equation from Ramanujan's theory of signature three. If $\beta = N\left(q^2\right)$, and $\alpha \in \left\{N(q), N(-q), N\left(q^4\right)\right\}$, then

$$27\alpha\beta(1-\alpha)(1-\beta) - (\alpha+\beta-2\alpha\beta)^3 = 0. \tag{3.2.23}$$

If we choose u so that $N\left(q^2\right) = \frac{27u^2(1+u)^2}{4(1+u+u^2)^3}$, then we can use Eq. (3.2.23) to easily verify that $N(q) = \frac{27u(1+u)^4}{2(1+4u+u^2)^3}$, $N(-q) = -\frac{27u(1+u)}{2(1-2u-2u^2)^3}$, and $N\left(q^4\right) = \frac{27u^4(1+u)}{2(2+2u-u^2)^3}$. The proof of Eq. (3.2.19) follows from applying these parameterizations to Eq. (3.2.14) when $f(q) = n\left(N(q)\right)$, and $p = 2$.

The proof of Eq. (3.2.20) requires Ramanujan's third-degree, signature three modular equation. In particular, if $\alpha = N(q)$ and $\beta = N(q^3)$, then

$$\alpha = 1 - \left(\frac{1-\beta^{1/3}}{1+2\beta^{1/3}}\right)^3 = Y\left(\beta^{1/3}\right). \tag{3.2.24}$$

Since $N^{1/3}(q^3) = q \times \{\text{power series in } q^3\}$, a short computation shows that $N(\zeta_3^j q) = Y\left(\zeta_3^j N^{1/3}\left(q^3\right)\right)$ for all $j \in \{0,1,2\}$. Choosing u such that $N(q^3) = u^3$, we must have $N\left(\zeta_3^j q\right) = Y\left(\zeta_3^j u\right)$. Eq. (3.2.20) follows from applying these parametric formulas to Eq. (3.2.14) when $f(q) = n\left(N(q)\right)$, and $p = 3$.

Since the proofs of equations (3.2.21) and (3.2.22) rely on similar arguments to the proof of Eq. (3.2.20), we will simply state the prerequisite modular equations. In particular, Eq. (3.2.21) follows from Ramanujan's third-degree modular equation for the cubic continued fraction. If $\alpha = G(q)$ and $\beta = G\left(q^3\right)$, then

$$\alpha^3 = \beta\left(\frac{1-\beta+\beta^2}{1+2\beta+4\beta^2}\right). \tag{3.2.25}$$

Similarly, Eq. (3.2.22) follows from the fifth-degree modular equation for the Rogers-Ramanujan continued fraction. In particular, if $\alpha = R(q)$ and $\beta = R\left(q^5\right)$

$$\alpha^5 = \beta\left(\frac{1-2\beta+4\beta^2-3\beta^3+\beta^4}{1+3\beta+4\beta^2+2\beta^3+\beta^4}\right). \tag{3.2.26}$$

□

The functional equations in Theorem 3.2.3 only hold in restricted subsets of $\mathbb{C}$. To explain this phenomenon we will go back to Eq. (3.2.14). As a general rule, we have to restrict q to values for which *none* of the Mahler measure integrals in Eq. (3.2.14) vanish on the unit torus. In other words, we can only consider the set of q's for which each term in Eq. (3.2.14) can be calculated from the appropriate q-series. Next, we may need to further restrict the domain of q depending on where the relevant parametric formulas hold. For example, parameterizations such as $N(q) = \frac{27u(1+u)^4}{2(1+4u+u^2)^3}$ and $N(q^2) = \frac{27u^2(1+u)^2}{4(1+u+u^2)^3}$ hold for $|q|$ sufficiently small, but fail when q is close to 1. After determining the domain of q, we can calculate the domain of u by solving a parametric equation to express u in terms of a q-series.

Theorem 3.2.4. *Recall that $G(q)$ is defined in equation* (4.2.1). *For $|p|$ sufficiently small but non-zero*

$$3g(p) = n\left(\frac{27p}{(1+4p)^3}\right) + 4n\left(\frac{27p^2}{(1-2p)^3}\right). \tag{3.2.27}$$

Furthermore, for $|u|$ sufficiently small but non-zero

$$\begin{aligned} 3n\left(\frac{27u(1+u)^4}{2(1+4u+u^2)^3}\right) =& g\left(\frac{u}{2(1+u)^2}\right) - 8g\left(-\frac{u(1+u)}{2}\right) \\ &+ 4g\left(\frac{u^2}{4(1+u)}\right). \end{aligned} \tag{3.2.28}$$

Proof. We will prove Eq. (3.2.28) first. Recall that Eq. (3.2.15) shows that

$$3n\left(N(q)\right) = g\left(G^3(q)\right) - 8g\left(G^3(-q)\right) + 4g\left(G^3\left(q^2\right)\right).$$

Let us suppose that $q = q_2\left(\frac{u(2+u)^3}{(1+2u)^3}\right)$, where $q_2(\alpha)$ is the elliptic nome. Classical eta function inversion formulas (which we shall omit here) show that for $|u|$ sufficiently small: $G^3(q) = \frac{u}{2(1+u)^2}$, $G^3(-q) = -\frac{u(1+u)}{2}$, $G^3\left(q^2\right) = \frac{u^2}{4(1+u)}$, $N(q) = \frac{27u(1+u)^4}{2(1+4u+u^2)^3}$, and $N\left(q^2\right) = \frac{27u^2(1+u)^2}{4(1+u+u^2)^3}$.

To prove Eq. (3.2.27) first recall recall that

$$3g\left(G^3(q)\right) = n\left(N(q)\right) + 4n\left(N\left(q^2\right)\right).$$

If we let $p = \frac{u}{2(1+u)^2}$, then it follows that $G^3(q) = p$, $N(q) = \frac{27p}{(1+4p)^3}$, and $N\left(q^2\right) = \frac{27p^2}{(1+2p)^3}$. □

Theorem 3.2.4 shows that $g(t)$ and $n(t)$ are essentially interchangeable. In Section 3.2.3 we will use Eq. (3.2.27) to derive an extremely useful formula for calculating $g(t)$ numerically.

3.2.2 Identities arising from higher modular equations

The seven functional equations presented in Section 3.2.1 are certainly not the only interesting formulas that follow from Eq. (3.2.14). Rather those results represent the subset of functional equations in which every Mahler measure depends on a rational argument (possibly in a cyclotomic field). If we consider the higher modular equations, then we can establish formulas involving the Mahler measures of the modular polynomials themselves. Eq. (3.2.32) is the simplest formula in this class of results.

Consider Eq. (3.2.14) when $p = 3$ and $f(q) = \mu\left(M(q)\right)$:

$$\sum_{j=0}^{2}\mu\left(M\left(\zeta_3^j q\right)\right) = -8\mu\left(M\left(q^3\right)\right) + 3\mu\left(M\left(q^9\right)\right). \tag{3.2.29}$$

The third-degree modular equation shows that if
$\alpha \in \left\{M\left(q\right), M\left(\zeta_3 q\right), M\left(\zeta_3^2 q\right), M\left(q^9\right)\right\}$, and $\beta = M\left(q^3\right)$, then

$$G_3(\alpha,\beta) := \left(\alpha^2+\beta^2+6\alpha\beta\right)^2 - 16\alpha\beta\left(4(1+\alpha\beta) - 3(\alpha+\beta)\right)^2 = 0. \tag{3.2.30}$$

Since $G_3(\alpha,\beta) = 0$ defines a curve with genus greater than zero, it is impossible to find simultaneous rational parameterizations for all four zeros in α. For example, if we let $\beta = M(q^3) = p(2+p)^3/(1+2p)^3$, then we can obtain the rational expression $M\left(q^9\right) = p^3(2+p)/(1+2p)$, and three messy formulas involving radicals for the other zeros. Despite this difficulty, Eq. (3.2.29) still reduces to an interesting formula if we recall the factorization

$$G_3\left(\alpha, M\left(q^3\right)\right) = \left(\alpha - M\left(q^9\right)\right)\prod_{j=0}^{2}\left(\alpha - M\left(\zeta_3^j q\right)\right), \tag{3.2.31}$$

and then use the fact that Mahler measure satisfies $\mathrm{m}(P)+\mathrm{m}(Q) = \mathrm{m}(PQ)$.

Theorem 3.2.5. *If $G_3(\alpha,\beta)$ is defined in Eq.* (3.2.30)*, then for $|p|$ sufficiently small but non-zero*

$$\begin{aligned}\mathrm{m}&\left(G_3\left(\frac{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2}{16}, \frac{1}{p}\left(\frac{1+2p}{2+p}\right)^3\right)\right)\\ &= -16\log(2) - 16\mu\left(p\left(\frac{2+p}{1+2p}\right)^3\right) + 8\mu\left(p^3\left(\frac{2+p}{1+2p}\right)\right).\end{aligned} \tag{3.2.32}$$

Proof. First notice that from the elementary properties of Mahler's measure

$$\mu(t) = \frac{1}{2}\mathrm{m}\left(\frac{16}{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2} - t\right) - \frac{1}{2}\log|t|.$$

Applying this identity to Eq. (3.2.29), and then appealing to Eq. (3.2.31) yields

$$\begin{aligned}\mathrm{m}&\left(G_3\left(\frac{16}{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2}, M\left(q^3\right)\right)\right)\\ &= \log\left|M(q)\,M(\zeta_3 q)\,M\left(\zeta_3^2 q\right)M\left(q^9\right)\right| - 16\mu\left(M\left(q^3\right)\right) + 8\mu\left(M\left(q^9\right)\right).\end{aligned}$$

Elementary q-product manipulations show that
$M^4\left(q^3\right) = M(q)\,M(\zeta_3 q)\,M\left(\zeta_3^2 q\right)M\left(q^9\right)$, and since $\alpha^4\beta^4 G_3\left(\frac{1}{\alpha},\frac{1}{\beta}\right) = G_3(\alpha,\beta)$, we obtain

$$\begin{aligned}\mathrm{m}&\left(G_3\left(\frac{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2}{16}, \frac{1}{M\left(q^3\right)}\right)\right)\\ &= -16\log(2) - 16\mu\left(M\left(q^3\right)\right) + 8\mu\left(M\left(q^9\right)\right).\end{aligned}$$

Finally, if we choose p so that $M\left(q^3\right) = p\left(\frac{2+p}{1+2p}\right)^3$, then $M\left(q^9\right) = p^3\left(\frac{2+p}{1+2p}\right)$, and the theorem follows. □

Although we completely eliminated the q-series expressions from Eq. (3.2.32), this is not necessarily desirable (or even possible) in more complicated examples. Consider the identity involving resultants which follows from Eq. (3.2.14) (and some manipulation) when $p = 11$ and $f(q) = r\left(R^5(q)\right)$:

$$\begin{aligned}&\mathrm{m}\left(\operatorname*{Res}_z\left[z^5 - \frac{xy}{(x+1)(y+1)(x+y+1)}, P\left(z, R^5\left(q\right)\right)\right]\right)\\&\quad = -12\mathrm{m}\left(1+x+y\right) + 12\log\left|R^5\left(q\right)\right| + 122r\left(R^5\left(q\right)\right) - 11r\left(R^5\left(q^{11}\right)\right).\end{aligned} \tag{3.2.33}$$

In this formula $P(u, v)$ is the polynomial

$$P(u,v) = uv(1 - 11v^5 - v^{10})(1 - 11u^5 - u^{10}) - (u-v)^{12},$$

which also satisfies $P\left(R(q), R\left(q^{11}\right)\right) = 0$ [73]. Even if rational parameterizations existed for $R(q)$ and $R\left(q^{11}\right)$, substituting such formulas into Eq. (3.2.33) would probably just make the identity prohibitively complicated.

3.2.3 Computationally useful formulas, and a few related hypergeometric transformations

While many methods exist for numerically calculating each of the four Mahler measures $\{\mu(t), n(t), g(t), r(t)\}$, two simple and efficient methods are directly related to the material discussed so far.

The first computational method relies on the q-series expansions. For example, we can calculate $\mu(\alpha)$ with Eq. (3.2.9), provided that a value of q exists for which $M(q) = \alpha$. Amazingly, the elliptic nome function, defined in Eq. (3.2.13), furnishes a value of q whenever $|\alpha| < 1$. Similar inversion formulas exist for all of the q-products in equations (3.2.5) through (3.2.8). Suppose that for $j \in \{2, 3, 4, 6\}$

$$q_j(\alpha) = \exp\left(-\frac{\pi}{\sin\left(\frac{\pi}{j}\right)}\frac{{}_2F_1\left(\frac{1}{j}, 1-\frac{1}{j}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{j}, 1-\frac{1}{j}; 1; \alpha\right)}\right), \tag{3.2.34}$$

then we have the following theorem:

Theorem 3.2.6. *With α and q appropriately restricted, the following table gives inversion formulas for equations* (3.2.5) *through* (3.2.8)*:*

α	q
$M(q)$	$q_2(\alpha)$
$N(q)$	$q_3(\alpha)$
$G(q)$	$q_2\left(\frac{u(2+u)^3}{(1+2u)^3}\right)$, *where* $\alpha^3=\frac{u}{2(1+u)^2}$
$R(q)$	$q_4\left(\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right)$, *where* $\alpha^5=\frac{k(1-k)^2}{(1+k)^2}$

For example: If $|q|<1$ *and* $\alpha=M(q)$*, then* $q=q_2(\alpha)$*.*

Proof. The inversion formulas for $M(q)$ and $G(q)$ follow from classical eta function identities, and the inversion formula for $N(q)$ follows from eta function identities in Ramanujan's theory of signature three.

The inversion formula for $R(q)$ seems to be new, so we will prove it. Let us suppose that $\alpha=R(q)$ and $k=R(q)R^2\left(q^2\right)$, where q is fixed. A formula of Ramanujan [53] shows that $\alpha^5=\frac{k(1-k)^2}{(1+k)^2}$, which establishes the second part of the formula. Now suppose that $q=q_2(\alpha_2)$, where $\alpha_2=M(q)$. A classical identity shows that

$$q\left(-q;q\right)_\infty^{24}=\frac{\alpha_2}{16(1-\alpha_2)^2},$$

and comparing this to Ramanujan's identity

$$q\left(-q;q\right)_\infty^{24}=\left(\frac{k}{1-k^2}\right)\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5,$$

we deduce that

$$\frac{\alpha_2}{(1-\alpha_2)^2}=16\left(\frac{k}{1-k^2}\right)\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5. \tag{3.2.35}$$

Now recall that the theory of the signature 4 elliptic nome shows that

$$q=q_2(\alpha_2)=q_4\left(\frac{4\alpha_2}{(1+\alpha_2)^2}\right)=q_4\left(\frac{4\alpha_2/(1-\alpha_2)^2}{1+4\alpha_2/(1-\alpha_2)^2}\right).$$

Substituting Eq. (3.2.35) into this final result yields

$$q=q_4\left(\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right),$$

which completes the proof. □

The second method for calculating the four Mahler measures, $\{\mu(t), n(t), g(t), r(t)\}$, depends on reformulating them in terms of hypergeometric functions. For example, Rodriguez-Villegas proved [71] the formula

$$\mu(t) = -\frac{1}{2}\mathrm{Re}\left[\log(t/16) + \int_0^t \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;u\right)-1}{u}\mathrm{d}u\right].$$

Translated into the language of generalized hypergeometric functions, this becomes

$$\mu(t) = -\mathrm{Re}\left[\frac{t}{8}{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};t\right) + \frac{1}{2}\log(t/16)\right]. \tag{3.2.36}$$

He also proved a formula for $n(t)$ which is equivalent to

$$n(t) = -\mathrm{Re}\left[\frac{2t}{27}{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};t\right) + \frac{1}{3}\log(t/27)\right]. \tag{3.2.37}$$

Formulas like Eq. (3.2.36) and Eq. (3.2.37) hold some obvious appeal. From a computational perspective they are useful because most mathematics programs have routines for calculating generalized hypergeometric functions. For example, when $|t| < 1$ the Taylor series for the ${}_4F_3$ function easily gives better numerical accuracy than the Mahler measure integrals. Combining Eq. (3.2.37) with Eq. (3.2.27) also yields a useful formula for calculating $g(t)$ whenever $|t|$ is sufficiently small:

$$\begin{aligned} g(t) = -\,\mathrm{Re}\Bigg[&\frac{2t}{(1+4t)^3}{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};\frac{27t}{(1+4t)^3}\right) + \frac{8t^2}{(1-2t)^3}{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};\frac{27t^2}{(1-2t)^3}\right) \\ &+\log\left(\frac{t^3}{(1+4t)(1-2t)^4}\right)\Bigg]. \end{aligned} \tag{3.2.38}$$

So far we have been unable to to find a similar expression for $r(t)$.

Open Problem 2: Express $r(t)$ in terms of generalized hypergeometric functions.

Besides their computational importance, identities like Eq. (3.2.36) allow for a reformulation of Boyd's conjectures in the language of hypergeometric functions. For example, the conjecture

$$\mathrm{m}\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right) \stackrel{?}{=} L'(E,0),$$

where E is an elliptic curve with conductor 15, becomes

$$L'(E,0) \stackrel{?}{=} -2\mathrm{Re}\left[{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};16\right)\right].$$

A proof of this identity would certainly represent an important addition to the vast literature concerning transformations and evaluations of generalized hypergeometric functions.

In the remainder of this section we will apply our results to deduce a few interesting hypergeometric transformations. For example, differentiating Eq. (3.2.38) leads to an interesting corollary:

Corollary 3.2.7. *For $|t|$ sufficiently small*

$$\omega(t) := \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}^3 = \frac{1}{1-2t}{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{27t^2}{(1-2t)^3}\right), \tag{3.2.39}$$

furthermore

$$\omega\left(\frac{p}{2(1+p)^2}\right) = (1+p)\omega\left(\frac{p^2}{4(1+p)}\right), \tag{3.2.40}$$

whenever $|p|$ is sufficiently small.

Proof. We can prove Eq. (3.2.39) by differentiating each side of Eq. (3.2.38), and then by appealing to Stienstra's formulas [76]. A second possible proof follows from showing that both sides of Eq. (3.2.39) satisfy the same differential equation.

The shortest proof of Eq. (3.2.40) follows from a formula due to Zagier [76]:

$$\omega\left(G^3(q)\right) = \prod_{n=0}^{\infty} \frac{\left(1-q^{2n}\right)\left(1-q^{3n}\right)^6}{\left(1-q^n\right)^2\left(1-q^{6n}\right)^3}.$$

First use Zagier's identity to verify that $G^2(q)\omega\left(G^3(q)\right) = G\left(q^2\right)\omega\left(G^3\left(q^2\right)\right)$, and then apply the parameterizations for $G^3(q)$ and $G^3\left(q^2\right)$ from Theorem 3.2.4. □

We will also make a few remarks about the derivative of $r(t)$. Stienstra has shown that

$$r(t) = -\mathrm{Re}\left[\log(t) + \int_0^t \frac{\phi(u)-1}{u}\mathrm{d}u\right], \tag{3.2.41}$$

where $\phi(t)$ is defined by

$$\phi(t) = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}. \tag{3.2.42}$$

Even though we have not discovered a formula for $r(t)$ involving hypergeometric functions, we can still express $\phi(t)$ in terms of the hypergeometric function.

Theorem 3.2.8. *Let $\phi(t)$ be defined by Eq.* (3.2.42), *then for $|k|$ sufficiently small:*

$$\phi\left(k\left(\frac{1-k}{1+k}\right)^2\right) = \frac{(1+k)^2}{\sqrt{(1+k^2)\left((1-k-k^2)^2-5k^2\right)}} \times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k^5\left(1+k-k^2\right)}{(1+k^2)^2\left((1-k-k^2)^2-5k^2\right)^2}\right), \tag{3.2.43}$$

$$\phi\left(k^2\left(\frac{1+k}{1-k}\right)\right) = \frac{(1-k)}{\sqrt{(1+k^2)\left((1+11k-k^2)^2-125k^2\right)}} \times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right). \tag{3.2.44}$$

Furthermore, $\phi(t)$ satisfies the functional equation:

$$\phi\left(k^2\left(\frac{1+k}{1-k}\right)\right) = \frac{1-k}{(1+k)^2}\phi\left(k\left(\frac{1-k}{1+k}\right)^2\right). \tag{3.2.45}$$

Proof. We will prove Eq. (3.2.45) first. A result of Verrill [77] shows that

$$\phi^2\left(R^5(q)\right) = \frac{q}{R^5(q)}\frac{\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty}. \tag{3.2.46}$$

Combining Eq. (3.2.46) with the trivial formula $(q^2,q^2)_\infty = (q;q)_\infty(-q;q)_\infty$, we have

$$\frac{\phi^2\left(R^5(q)\right)}{\phi^2\left(R^5(q^2)\right)} = \frac{R^5\left(q^2\right)}{R^5(q)}\frac{\left\{q^{1/24}\left(-q;q\right)_\infty\right\}}{\left\{q^{5/24}\left(-q^5;q^5\right)_\infty\right\}^5}. \tag{3.2.47}$$

We will apply four of Ramanujan's formulas to finish the proof. If $k = R(q)R^2(q^2)$, then for $|q|$ sufficiently small [53]:

$$R^5(q) = k\left(\frac{1-k}{1+k}\right)^2, \tag{3.2.48}$$

$$R^5(q^2) = k^2\left(\frac{1+k}{1-k}\right), \tag{3.2.49}$$

$$q^{1/24}\left(-q;q\right)_\infty = \left(\frac{k}{1-k^2}\right)^{1/24}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^{5/24}, \tag{3.2.50}$$

$$q^{5/24}\left(-q^5;q^5\right)_\infty = \left(\frac{k}{1-k^2}\right)^{5/24}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^{1/24}. \tag{3.2.51}$$

Eq. (3.2.45) follows immediately from substituting these parametric formulas into Eq. (3.2.47).

Next we will prove Eq. (3.2.43). Combining Eq. (3.2.48) with Entry 3.2.15 in [53], we easily obtain

$$q^{5/24}\left(q^5;q^5\right)_\infty = \left\{\frac{k(1-k^2)^2}{\left(1+k-k^2\right)\left(1-4k-k^2\right)^2}\right\}^{1/6} q^{1/24}\left(q;q\right)_\infty. \tag{3.2.52}$$

Now we will evaluate the eta product $q^{1/24}\left(q;q\right)_\infty$. First recall that if $q = q_4(z)$, then (see [58], page 148)

$$q^{1/24}(q;q)_\infty = 2^{-1/4}z^{1/24}(1-z)^{1/12}\sqrt{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;z\right)}.$$

In Theorem 3.2.6 we showed that if $k = R(q)R^2\left(q^2\right)$ then $q = q_4\left(\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right)$, hence it follows that

$$q^{1/24}\left(q;q\right)_\infty = \left(\frac{k\left(1-k^2\right)^2\left(1+k-k^2\right)^5\left(1-4k-k^2\right)^{10}}{(1+k^2)^6\left((1+11k-k^2)^2-125k^2\right)^6}\right)^{1/24} \times\sqrt{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right)} \tag{3.2.53}$$

Substituting Eq. (3.2.53), Eq. (3.2.52), and Eq. (3.2.48) into Eq. (3.2.46) completes the proof of Eq. (3.2.43). The proof of Eq. (3.2.44) also follows from an extremely similar argument.

□

We will conclude this section by recording a few formulas which do not appear in [53], but which were probably known to Ramanujan. We will point out that Maier obtained several results along these lines in [69]. Notice that the functional equation for $\phi(t)$ (after substituting $z = k/(1-k^2)$) implies a new hypergeometric transformation:

$$\sqrt{\frac{(1+11z)^2-125z^2}{(1-z)^2-5z^2}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1,\frac{64z^5\,(1+z)}{(1+4z^2)\left((1-z)^2-5z^2\right)^2}\right)$$
$$= {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64z\,(1+z)^5}{(1+4z^2)\left((1+11z)^2-125z^2\right)^2}\right) \tag{3.2.54}$$

Perhaps not surprisingly, we can also use the arguments in this section to deduce that

$$q_4^5\left(\frac{64z\,(1+z)^5}{(1+4z^2)\left((1+11z)^2-125z^2\right)^2}\right) = q_4\left(\frac{64z^5\,(1+z)}{(1+4z^2)\left((1-z)^2-5z^2\right)^2}\right), \tag{3.2.55}$$

which implies a rational parametrization for the fifth-degree modular equation in Ramanujan's theory of signature 4.

3.3 A regulator explanation

Now we will reinterpret our identities in terms of the regulators of elliptic curves. The elliptic curves in question are defined by the zero varieties of the polynomials whose Mahler measure we studied. First we will explain the relationship between Mahler measures and regulators. Then we will use regulators to deduce formulas involving Kronecker-Eisenstein series, including equations (3.2.9), (3.2.10), (3.2.11), and (3.2.12).

We will follow some of the ideas of Rodriguez-Villegas [72].

3.3.1 The elliptic regulator

Let F be a field. By Matsumoto's Theorem, $K_2(F)$ is generated by the symbols $\{a,b\}$ for $a,b \in F$, which satisfy the bilinearity relations $\{a_1a_2,b\} =$

$\{a_1, b\}\{a_2, b\}$ and $\{a, b_1 b_2\} = \{a, b_1\}\{a, b_2\}$, and the Steinberg relation $\{a, 1-a\} = 1$ for all $a \neq 0$.

Recall that for a field F, with discrete valuation v, and maximal ideal $\mathcal{M}$, the tame symbol is given by

$$(x, y)_v \equiv (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \bmod \mathcal{M}$$

(see [71]). Note that this symbol is trivial if $v(x) = v(y) = 0$. In the case when $F = \mathbb{Q}(E)$ (from now on E denotes an elliptic curve), a valuation is determined by the order of the rational functions at each point $S \in E(\bar{\mathbb{Q}})$. We will denote the valuation determined by a point $S \in E(\bar{\mathbb{Q}})$ by v_S.

The tame symbol is then a map $K_2(\mathbb{Q}(E)) \to \mathbb{Q}(S)^*$.

We have

$$0 \to K_2(E) \otimes \mathbb{Q} \to K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \to \coprod_{S \in E(\bar{\mathbb{Q}})} \mathbb{Q}Q(S)^* \times \mathbb{Q},$$

where the last arrow corresponds to the coproduct of tame symbols.

Hence an element $\{x, y\} \in K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$ can be seen as an element in $K_2(E) \otimes \mathbb{Q}$ whenever $(x, y)_{v_S} = 1$ for all $S \in E(\bar{\mathbb{Q}})$. All of the families considered in this paper are tempered according to [71], and therefore they satisfy the triviality of tame symbols.

The regulator map (defined by Beilinson after the work of Bloch) may be defined by

$$\mathtt{r} : K_2(E) \to H^1(E, \mathbb{R})$$
$$\{x, y\} \to \left\{ \gamma \to \int_\gamma \eta(x, y) \right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$, and

$$\eta(x, y) := \log |x| \mathrm{d} \arg y - \log |y| \mathrm{d} \arg x.$$

Here we think of $H^1(E, \mathbb{R})$ as the dual of $H_1(E, \mathbb{Z})$. The regulator is well defined because $\eta(x, 1-x) = \mathrm{d}D(x)$, where

$$D(z) = \operatorname{Im}(\operatorname{Li}_2(z)) + \arg(1-z) \log |z|$$

is the Bloch-Wigner dilogarithm.

In terms of the general formulation of Beilinson's conjectures this definition is not completely correct. One needs to go a step further and consider $K_2(\mathcal{E})$, where $\mathcal{E}$ is a Néron model of E over $\mathbb{Z}$. In particular, $K_2(\mathcal{E})$ is a subgroup of $K_2(E)$. It seems [71] that a power of $\{x, y\}$ always lies in $K_2(\mathcal{E})$.

Assume that E is defined over $\mathbb{R}$. Because of the way that complex conjugation acts on η, the regulator map is trivial for the classes in $H_1(E, \mathbb{Z})^+$. In particular, these cycles remain invariant under complex conjugation. Therefore it suffices to consider the regulator as a function on $H_1(E, \mathbb{Z})^-$.

We write $E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, where τ is in the upper half-plane. Then $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$, where $z \bmod \Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$. Bloch [62] defines the regulator function in terms of a Kronecker-Eisenstein series

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right) = \frac{y_\tau^2}{\pi} \sum_{m,n\in\mathbb{Z}}{}' \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}, \tag{3.3.56}$$

where y_τ is the imaginary part of τ.

Let $J(z) = \log|z| \log|1-z|$, and let

$$D(x) = \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1-x)\log|x|$$

be the Bloch-Wigner dilogarithm.

Consider the following function on $E(\mathbb{C}) \cong \mathbb{C}^*/q^{\mathbb{Z}}$:

$$J_\tau(z) = \sum_{n=0}^{\infty} J(zq^n) - \sum_{n=1}^{\infty} J(z^{-1}q^n) + \frac{1}{3}\log^2|q| B_3\left(\frac{\log|z|}{\log|q|}\right), \tag{3.3.57}$$

where $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ is the third Bernoulli polynomial. If we recall that the elliptic dilogarithm is defined by

$$D_\tau(z) := \sum_{n\in\mathbb{Z}} D(zq^n), \tag{3.3.58}$$

then the regulator function (see [62]) is given by

$$R_\tau = D_\tau - \mathrm{i}J_\tau. \tag{3.3.59}$$

By linearity, R_τ extends to divisors with support in $E(\mathbb{C})$. Let x and y be non-constant functions on E with divisors

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

Following [62], and the notation in [71], we recall the diamond operation $\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

Here $\mathbb{Z}[E(\mathbb{C})]^-$ means that $[-P] \sim -[P]$.

Because R_τ is an odd function, we obtain a map

$$\mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{R}.$$

Theorem 3.3.1. *(Beilinson [55]) If $E/\mathbb{R}$ is an elliptic curve, x, y are non-constant functions in $\mathbb{C}(E)$, and $\omega \in \Omega^1$, then*

$$\int_{E(\mathbb{C})} \bar{\omega} \wedge \eta(x, y) = \Omega_0 R_\tau((x) \diamond (y)),$$

where Ω_0 is the real period.

Although a more general version of Beilinson's Theorem exists for elliptic curves defined over the complex numbers, the above version has a simpler formulation.

Corollary 3.3.2. *(after an idea of Deninger) If x and y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, then*

$$-\int_\gamma \eta(x, y) = Im\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right)$$

where $\Omega = \int_\gamma \omega$.

Proof. Notice that $\mathrm{i}\eta(x, y)$ is an element of the two-dimensional vector space $H^2_{\mathcal{D}}(E(\mathbb{C}), \mathbb{R}(2))$ generated by ω and $\bar{\omega}$. Then we may write

$$\mathrm{i}\eta(x, y) = \alpha[\omega] + \beta[\bar{\omega}],$$

from which we obtain

$$\int_\gamma \mathrm{i}\eta(x, y) = \alpha\Omega + \beta\overline{\Omega}.$$

On the other hand, we have

$$\int_{E(\mathbb{C})} \mathrm{i}\eta(x, y) \wedge \bar{\omega} = \alpha \int_{E(\mathbb{C})} \omega \wedge \bar{\omega} = \alpha \mathrm{i} 2\Omega_0^2 y_\tau,$$

and

$$\int_{E(\mathbb{C})} \mathrm{i}\eta(x, y) \wedge \omega = -\beta \mathrm{i} 2\Omega_0^2 y_\tau.$$

By Beilinson's Theorem

$$\int_\gamma \mathrm{i}\eta(x, y) = -\frac{R_\tau((x) \diamond (y))\Omega}{2\Omega_0 y_\tau} + \frac{\overline{R_\tau((x) \diamond (y))\Omega}}{2\Omega_0 y_\tau},$$

and the statement follows. □

3.3.2 Regulators and Mahler measure

From now on, we will set $k = \frac{4}{\sqrt{t}}$ in the first family (3.2.1).

Rodriguez-Villegas [71] proved that if $P_k(x,y) = k + x + \frac{1}{x} + y + \frac{1}{y}$ does not intersect the torus $\mathbb{T}^2$, then

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi}\, \mathtt{r}(\{x,y\})(\gamma). \tag{3.3.60}$$

Here the $\sim_{\mathbb{Z}}$ stands for "up to an integer number", and γ is a closed path that avoids the poles and zeros of x and y. In particular, γ generates the subgroup $H_1(E,\mathbb{Z})^-$ of $H_1(E,\mathbb{Z})$ where conjugation acts by -1.

We would like to use this property, however we need to exercise caution. In particular, $P_k(x,y)$ intersects the torus whenever $|k| \le 4$ and $k \in \mathbb{R}$. Let us recall the idea behind the proof of Eq. (3.3.60) for the special case of $P_k(x,y)$. Writing

$$yP_k(x,y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

we have

$$m(k) = \mathrm{m}(yP_k(x,y)) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} (\log^+ |y_{(1)}(x)| + \log^+ |y_{(2)}(x)|) \frac{\mathrm{d}x}{x}.$$

This last equality follows from applying Jensen's formula with respect to the variable y. When the polynomial does not intersect the torus, we may omit the "+" sign on the logarithm since each $y_{(i)}(x)$ is always inside or outside the unit circle. Indeed, there is always a branch inside the unit circle and a branch outside. It follows that

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x,y), \tag{3.3.61}$$

where $\mathbb{T}^1$ is interpreted as a cycle in the homology of the elliptic curve defined by $P_k(x,y) = 0$, namely $H_1(E,\mathbb{Z})$.

If $k \in [-4,4]$, then we may also assume that $k > 0$ since this particular Mahler measure does not depend on the sign of k. The equation

$$k + x + \frac{1}{x} + y + \frac{1}{y} = 0$$

certainly has solutions when $(x,y) \in \mathbb{T}^2$. However, for $|x| = 1$ and k real, the number $k + x + \frac{1}{x}$ is real, and therefore $y + \frac{1}{y}$ must be real. This forces two possibilities: either y is real or $|y| = 1$. Let $x = \mathrm{e}^{\mathrm{i}\theta}$, then for $-\pi \leq \theta \leq \pi$ we have

$$-k - 2\cos\theta = y + \frac{1}{y}. \tag{3.3.62}$$

The limiting case occurs when $|k + 2\cos\theta| = 2$. Since we have assumed that k is positive, this condition becomes $k + 2\cos\theta = 2$, which implies that $y = -1$. When $k + 2\cos\theta > 2$ one solution for y, say, $y_{(1)}$, becomes a negative number less than -1, thus $\left|y_{(1)}\right| > 1$ (the other solution $y_{(2)}$ is such that $\left|y_{(2)}\right| < 1$). When $k+2\cos\theta < 2$, y_i lies inside the unit circle and never reaches 1. What is important is that $\left|y_{(1)}\right| \geq 1$ and $\left|y_{(2)}\right| \leq 1$, so we can still write Eq. (3.3.61) even if there is a nontrivial intersection with the torus.

3.3.3 Functional identities for the regulator

First recall a result by Bloch [62] which studies the modularity of R_τ:

Proposition 3.3.3. *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$, *and let* $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$. *If we let*

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix},$$

then:

$$R_{\tau'}\left(\mathrm{e}^{2\pi\mathrm{i}(a'+b'\tau')}\right) = \frac{1}{\gamma\bar{\tau}+\delta} R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right).$$

We will need to use some functional equations for J_τ. First recall the following trivial property for $J(z)$:

$$J(z) = p \sum_{x^p = z} J(x). \tag{3.3.63}$$

Proposition 3.3.4. *Let p be an odd prime, let $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, and let $q_j = \mathrm{e}^{\frac{2\pi\mathrm{i}(\tau+j)}{p}}$ for $j \in \{0, 1, \ldots, p-1\}$. Suppose that $(N, k) = 1$, and $p \equiv \pm 1$ or $0 \pmod N$. Then*

$$(1 + \chi_{-N}(p)p^2) J_{N\tau}(q^k) = \sum_{j=0}^{p-1} p J_{\frac{N(\tau+j)}{p}}(q_j^k) + \chi_{-N}(p) J_{Np\tau}(q^{pk}), \tag{3.3.64}$$

and for any z we have

$$(\chi_{-N}(p) + p^2) J_{N\tau}(z) = \sum_{j=0}^{p-1} p J_{\frac{N(\tau+j)}{p}}(z) + \chi_{-N}(p) J_{Np\tau}(z). \tag{3.3.65}$$

Proof. First notice that

$$\sum_{j=0}^{p-1} J_{\frac{N(\tau+j)}{p}}(q_j^k) = \sum_{n=0}^{\infty}\sum_{j=0}^{p-1} J\left(q_j^{Nn+k}\right) - \sum_{n=1}^{\infty}\sum_{j=0}^{p-1} J\left(q_j^{Nn-k}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right).$$

By Eq. (3.3.63) this becomes

$$\begin{aligned}
&= \sum_{\substack{n=0\\ p\nmid Nn+k}}^{\infty} \frac{1}{p} J\left(q^{Nn+k}\right) - \sum_{\substack{n=1\\ p\nmid Nn-k}}^{\infty} \frac{1}{p} J\left(q^{Nn-k}\right)\\
&\quad + \sum_{\substack{n=0\\ p|Nn+k}}^{\infty} pJ\left(q^{\frac{Nn+k}{p}}\right) - \sum_{\substack{n=1\\ p|Nn-k}}^{\infty} pJ\left(q^{\frac{Nn-k}{p}}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right)\\
&= \sum_{n=0}^{\infty} \frac{1}{p} J\left(q^{Nn+k}\right) - \sum_{n=1}^{\infty} \frac{1}{p} J\left(q^{Nn-k}\right)\\
&\quad - \sum_{\substack{n=0\\ p|Nn+k}}^{\infty} \frac{1}{p} J\left(q^{Nn+k}\right) + \sum_{\substack{n=1\\ p|Nn-k}}^{\infty} \frac{1}{p} J\left(q^{Nn-k}\right)\\
&\quad + \sum_{\substack{n=0\\ p|Nn+k}}^{\infty} pJ\left(q^{\frac{Nn+k}{p}}\right) - \sum_{\substack{n=1\\ p|Nn-k}}^{\infty} pJ\left(q^{\frac{Nn-k}{p}}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right).
\end{aligned}$$

Rearranging, we find that

$$\begin{aligned}
&= \frac{1}{p} J_{N\tau}\left(q^k\right) - \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right)\\
&\quad - \sum_{\substack{n=0\\ p|Nn+k}}^{\infty} \frac{1}{p} J\left((q^p)^{\frac{Nn+k}{p}}\right) + \sum_{\substack{n=1\\ p|Nn-k}}^{\infty} \frac{1}{p} J\left((q^p)^{\frac{Nn-k}{p}}\right)\\
&\quad + \sum_{\substack{n=0\\ p|Nn+k}}^{\infty} pJ\left(q^{\frac{Nn+k}{p}}\right) - \sum_{\substack{n=1\\ p|Nn-k}}^{\infty} pJ\left(q^{\frac{Nn-k}{p}}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right)\\
&= \frac{1}{p} J_{N\tau}\left(q^k\right) - \frac{\chi_{-N}(p)}{p} J_{Np\tau}(q^{pk}) + \chi_{-N}(p) p J_{N\tau}(q^k),
\end{aligned}$$

which proves the assertion.

The second equality follows in a similar fashion. □

It is possible to prove analogous identities for D_τ and R_τ.

Proposition 3.3.5.

$$J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) = J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) - J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right) \tag{3.3.66}$$

Proof. Let $z = \mathrm{e}^{\pi \mathrm{i}\mu}$, then

$$\begin{aligned} J_{2\mu}(z) - J_{2\mu}(-z) =& J(z) - J(-z) \\ &+ \sum_{n=1}^{\infty}\left(J(zq^n) - J(-zq^n) - J(z^{-1}q^n) + J(-z^{-1}q^n)\right) \\ =& \sum_{n=0}^{\infty}\Big(J\left(\mathrm{e}^{\pi \mathrm{i}\mu(4n+1)}\right) - J\left(-\mathrm{e}^{\pi \mathrm{i}\mu(4n+1)}\right) \\ &\quad - J\left(\mathrm{e}^{\pi \mathrm{i}\mu(4n+3)}\right) + J\left(-\mathrm{e}^{\pi \mathrm{i}\mu(4n+3)}\right)\Big). \end{aligned}$$

On the other hand,

$$J_{\frac{2\mu+1}{2}}(z) = \sum_{n=0}^{\infty}\left(J\left((-1)^n \mathrm{e}^{\pi \mathrm{i}\mu(2n+1)}\right) - J\left((-1)^{n+1}\mathrm{e}^{\pi \mathrm{i}\mu(2n+1)}\right)\right),$$

which proves the equality. □

3.3.4 The first family

First we will write the equation

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

in Weierstrass form. Consider the rational transformation

$$X = \frac{k+x+y}{x+y} = -\frac{1}{xy}, \qquad Y = \frac{k(y-x)(k+x+y)}{2(x+y)^2} = \frac{(y-x)\left(1+\frac{1}{xy}\right)}{2xy},$$

which leads to

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

It is useful to state the inverse transformation:

$$x = \frac{kX - 2Y}{2X(X-1)}, \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

Notice that E_k contains a torsion point of order 4 over $\mathbb{Q}(k)$, namely $P = \left(1, \frac{k}{2}\right)$. Indeed, this family is the modular elliptic surface associated to $\Gamma_0(4)$.

We can show that $2P = (0,0)$, and $3P = \left(1, -\frac{k}{2}\right)$.

Now

$$(X) = 2(2P) - 2O,$$

and

$$\begin{aligned}(x) =& (2(P) + (2P) - 3O) - (2(2P) - 2O) - ((P) + (3P) - 2O)\\ =& (P) - (2P) - (3P) + O,\end{aligned}$$

$$\begin{aligned}(y) =& (2(3P) + (2P) - 3O) - (2(2P) - 2O) - ((P) + (3P) - 2O)\\ =& -(P) - (2P) + (3P) + O.\end{aligned}$$

Computing the diamond operation between the divisors of x and y yields

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

Now assume that $k \in \mathbb{R}$ and $k > 4$. We will choose an orientation for the curve and compute the real period. Because P is a point of order 4 and $\int_0^1 \omega$ is real, we may assume that P corresponds to $\frac{3\Omega_0}{4}$.

The next step is to understand the cycle $|x| = 1$ as an element of $H_1(E, \mathbb{Z})$. We would like to compute the value of $\Omega = \int_\gamma \omega$. First recall that

$$\omega = \frac{\mathrm{d}X}{2Y} = \frac{\mathrm{d}x}{x(y - y^{-1})}.$$

In the case when $k > 4$, consider conjugation of ω. This sends $x \to x^{-1}$, and $\frac{\mathrm{d}x}{x} \to -\frac{\mathrm{d}x}{x}$. There is no intersection with the torus, so y remains invariant. Therefore we conclude that Ω is the complex period, and $\frac{\Omega}{\Omega_0} = \tau$, where τ is purely imaginary.

Therefore for k real and $|k| > 4$

$$m(k) = \frac{4}{\pi} \mathrm{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right).$$

Now take $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$. By Proposition 3.3.3

$$R_\tau(-\mathrm{i}) = R_\tau\left(\mathrm{e}^{-\frac{2\pi \mathrm{i}}{4}}\right) = \bar{\tau} R_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4\tau}}\right),$$

therefore

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4\tau}}\right).$$

If we let $\mu = -\frac{1}{4\tau}$, then for $k \in \mathbb{R}$ we obtain

$$\begin{aligned} m(k) =& -\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right) = \mathrm{Im}\left(\frac{1}{\pi y_\mu} R_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right)\right) \\ =& \mathrm{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+4\mu n)^2 (m+4\bar{\mu}n)}\right), \end{aligned}$$

thus recovering a result of Rodriguez-Villegas. We can extend this result to all $k \in \mathbb{C}$, by arguing that both $m(k)$ and $-\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right)$ are the real parts of holomorphic functions that coincide at infinitely many points (see [70]).

Now we will show how to deduce equations (3.1.7) and (3.1.6). Applying Eq. (3.3.64) with $N = 4$, $k = 1$, and $p = 2$, we have

$$J_{4\mu}(q) = 2J_{2\mu}(q_0) + 2J_{2(\mu+1)}(q_1),$$

which translates into

$$\frac{1}{y_{4\mu}} J_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right) = \frac{1}{y_{2\mu}} J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) + \frac{1}{y_{2\mu}} J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right).$$

This is the content of Eq. (3.1.7). Setting $\tau = -\frac{1}{2\mu}$, we may also write

$$D_{\frac{\tau}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) + D_\tau(-\mathrm{i}\mathrm{e}^{\pi \mathrm{i}\tau}). \tag{3.3.67}$$

Next we will use Eq. (3.3.66):

$$J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) = J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) - J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right),$$

which translates into

$$\frac{1}{y_{\frac{2\mu+1}{2}}} J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) = \frac{2}{y_{2\mu}} J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) - \frac{2}{y_{2\mu}} J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right).$$

Setting $\tau = -\frac{1}{2\mu}$, and using $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ on the left-hand side, we have

$$D_{\frac{\tau-1}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) - D_\tau\left(-\mathrm{i}\mathrm{e}^{\pi \mathrm{i}\tau}\right). \tag{3.3.68}$$

Combining equations (3.3.67) and (3.3.68), we see that

$$2D_\tau(-\mathrm{i}) = D_{\frac{\tau}{2}}(-\mathrm{i}) + D_{\frac{\tau-1}{2}}(-\mathrm{i}).$$

This is the content of Eq. (3.1.6).

Similarly, we may deduce Eq. (3.2.14) from Eq. (3.3.64) when $k = 1$, $N = 4$, and p is an odd prime.

3.3.5 A direct approach

It is also possible to prove equations (3.1.6) and (3.1.7) directly, without considering the μ-parametrization or the explicit form of the regulator.

For those formulas, it is easy to explicitly write the isogenies at the level of the Weierstrass models. By using the well-known isogeny of degree 2 (see for example [64]):

$$\phi : \{E : y^2 = x(x^2 + ax + b)\} \to \{\widehat{E} : \hat{y}^2 = \hat{x}(\hat{x}^2 - 2a\hat{x} + (a^2 - 4b))\}$$

given by

$$(x, y) \to \left(\frac{y^2}{x^2}, \frac{y(b - x^2)}{x^2}\right)$$

(we require that $a^2 - 4b \neq 0$), we find

$$\phi_1 : E_{2\left(n+\frac{1}{n}\right)} \to E_{4n^2}, \qquad \phi_2 : E_{2\left(n+\frac{1}{n}\right)} \to E_{\frac{4}{n^2}},$$

$$\phi_1 : (X, Y) \to \left(\frac{X(n^2X + 1)}{X + n^2}, -\frac{n^3Y\left(X^2 + 2n^2X + 1\right)}{(X + n^2)^2}\right),$$

$$\phi_2 : (X, Y) \to \left(\frac{X(X + n^2)}{n^2X + 1}, -\frac{Y\left(n^2X^2 + 2X + n^2\right)}{n\left(n^2X + 1\right)^2}\right).$$

Let us write x_1, y_1, X_1, Y_1 for the rational functions and $\mathtt{r}_1$ for the regulator in E_{4n^2}, and x_2, y_2, X_2, Y_2, $\mathtt{r}_2$ for the corresponding objects in $E_{\frac{4}{n^2}}$.

It follows that

$$\begin{aligned}\pm m\left(4n^2\right) = &\mathtt{r}_1\left(\{x_1, y_1\}\right) = \frac{1}{2\pi}\int_{|X_1|=1} \eta(x_1, y_1)\\ =&\frac{1}{4\pi}\int_{|X|=1} \eta(x_1 \circ \phi_1, y_1 \circ \phi_1)\\ =&\frac{1}{2}\,\mathtt{r}\left(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}\right),\end{aligned}$$

where the factor of 2 follows from the degree of the isogeny. Similarly, we find that

$$\pm m\left(\frac{4}{n^2}\right) = \mathtt{r}_2\left(\{x_2, y_2\}\right) = \frac{1}{2}\,\mathtt{r}\left(\{x_2 \circ \phi_2, y_2 \circ \phi_2\}\right).$$

Now we need to compare the values of

$$\mathtt{r}\left(\{x_1\circ\phi_1, y_1\circ\phi_1\}\right), \quad \mathtt{r}\left(\{x_2\circ\phi_2, y_2\circ\phi_2\}\right), \quad \text{and} \quad \mathtt{r}\left(\{x,y\}\right).$$

Recall that $(x)\diamond(y) = 8(P)$, where $P = (1, \frac{k}{2})$. When $k = 2\left(n+\frac{1}{n}\right)$, we will also consider the point $Q = \left(-\frac{1}{n^2}, 0\right)$, which has order 2 (then $P+Q = \left(-1, n-\frac{1}{n}\right)$, $2P+Q = \left(-n^2, 0\right)$, etc).

Let P now denote the point in $E_{2\left(n+\frac{1}{n}\right)}$, and let P_1 denote the corresponding point in E_{4n^2}. We have the following table:

$$\phi_1: \qquad \begin{array}{rcll} 3P, & P+Q & \rightarrow & P_1 \\ 2P, & Q & \rightarrow & 2P_1 \\ P, & 3P+Q & \rightarrow & 3P_1 \\ O_0, & 2P+Q & \rightarrow & O_1 \end{array}.$$

Using this table, and the divisors (x_1) and (y_1) in E_{4n^2}, we can compute $(x_1\circ\phi_1)\diamond(y_1\circ\phi_1)$. We find that

$$(x_1\circ\phi_1)\diamond(y_1\circ\phi_1) = -16(P) + 16(P+Q),$$

and similarly

$$(x_2\circ\phi_2)\diamond(y_2\circ\phi_2) = -16(P) - 16(P+Q).$$

These computations show that

$$\frac{1}{2}\,\mathtt{r}_0\left(\{x_1\circ\phi_1, y_1\circ\phi_1\}\right) + \frac{1}{2}\,\mathtt{r}_0\left(\{x_2\circ\phi_2, y_2\circ\phi_2\}\right) = 2\,\mathtt{r}_0\left(\{x_0, y_0\}\right), \tag{3.3.69}$$

and therefore

$$\mathtt{r}_1\left(\{x_1, y_1\}\right) + \mathtt{r}_2\left(\{x_2, y_2\}\right) = 2\,\mathtt{r}_0\left(\{x_0, y_0\}\right). \tag{3.3.70}$$

We can conclude the proof of Eq. (3.1.6) by inspecting signs.

To prove Eq. (3.1.7), it is necessary to use the isomorphism ϕ from Eq. (3.3.71).

3.3.6 Relations among $m(2)$, $m(8)$, $m\left(3\sqrt{2}\right)$, and $m\left(\mathrm{i}\sqrt{2}\right)$

Setting $n = \frac{1}{\sqrt{2}}$ in Eq. (3.1.7), we obtain

$$m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right) = m(8).$$

Doing the same in Eq. (3.1.6), we find that

$$m(2) + m(8) = 2m\left(3\sqrt{2}\right).$$

In this section we will establish the identity

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}),$$

from which we can deduce expressions for $m(2)$ and $m(8)$.

Consider the functions f and $1-f$, where $f = \frac{\sqrt{2}Y-X}{2} \in \mathbb{C}(E_{3\sqrt{2}})$. Their divisors are

$$\begin{aligned}\left(\frac{\sqrt{2}Y-X}{2}\right) =& (2P) + 2(P+Q) - 3O,\\ \left(1-\frac{\sqrt{2}Y-X}{2}\right) =& (P) + (Q) + (3P+Q) - 3O.\end{aligned}$$

The diamond operation yields

$$(f)\diamond(1-f) = 6(P) - 10(P+Q).$$

But $(f)\diamond(1-f)$ is trivial in K-theory, hence

$$6(P) \sim 10(P+Q).$$

Now consider the isomorphism ϕ:

$$\phi : E_{2\left(n+\frac{1}{n}\right)} \to E_{2\left(\mathrm{i}n+\frac{1}{\mathrm{i}n}\right)}, \qquad (X,Y) \to (-X, \mathrm{i}Y) \tag{3.3.71}$$

This isomorphism implies that

$$\mathtt{r}_{\mathrm{i}\sqrt{2}}(\{x,y\}) = \mathtt{r}_{3\sqrt{2}}(\{x\circ\phi, y\circ\phi\}).$$

But we know that

$$(x\circ\phi)\diamond(y\circ\phi) = 8(P+Q).$$

This implies

$$6\,\mathtt{r}_{3\sqrt{2}}(\{x,y\}) = 10\,\mathtt{r}_{\mathrm{i}\sqrt{2}}(\{x,y\}),$$

and

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

Consequently, we may conclude that

$$m(8) = \frac{8}{5}m(3\sqrt{2}), \quad m(2) = \frac{2}{5}m(3\sqrt{2}),$$

and finally

$$m(8) = 4m(2).$$

3.3.7 The Hesse family

We will now sketch the case of the Hesse family:

$$x^3 + y^3 + 1 - \frac{3}{t^{\frac{1}{3}}} xy.$$

This family corresponds to $\Gamma_0(3)$. The diamond operation yields

$$(x) \diamond (y) = 9(P) + 9(A) + 9(B), \tag{3.3.72}$$

where P is a point of order 3, defined over $\mathbb{Q}(t^{1/3})$, and A, B are points of order 3 such that $A + B + P = O$.

For $0 < t < 1$, we have

$$n(t) = \frac{9}{2\pi} \mathrm{Im} \left(\frac{\tau}{y_\tau} \left(R_\tau \left(\mathrm{e}^{\frac{4\pi \mathrm{i}}{3}} \right) + R_\tau \left(\mathrm{e}^{\frac{4\pi \mathrm{i}(1+\tau)}{3}} \right) + R_\tau \left(\mathrm{e}^{\frac{2\pi \mathrm{i}(2+\tau)}{3}} \right) \right) \right).$$

If we let $\mu = -\frac{1}{\tau}$, we obtain, after several steps,

$$n(t) = \mathrm{Re} \left(\frac{27\sqrt{3} y_\mu}{4\pi^2} \sum_{k,n}{}' \frac{\chi_{-3}(n)}{(3\mu k + n)^2 (3\bar{\mu} k + n)} \right).$$

Following the previous example, this result may be extended to $\mathbb{C} \setminus \kappa$ by comparing holomorphic functions.

3.3.8 The $\Gamma_0^0(6)$ example

We will now sketch a treatment of Stienstra's example [76]:

$$(x+1)(y+1)(x+y) - \frac{1}{t} xy.$$

Applying the diamond operation, we have

$$(x) \diamond (y) = -6(P) - 6(2P),$$

where P is a point of order 6.

For t small, one can write

$$g(t) = \frac{3}{\pi} \mathrm{Im} \left(\frac{\tau}{y_\tau} R_\tau(\xi_6^{-1}) + R_\tau(\xi_3^{-1}) \right).$$

Eventually, one arrives to

$$\begin{aligned} g(t) = &\mathrm{Re}\left(\frac{36y_\mu}{\pi^2}\sum_{m,n}{}' \frac{\chi_{-3}(m)}{(m+6\mu n)^2(m+6\bar{\mu}n)}\right) \\ &+\mathrm{Re}\left(\frac{9y_\mu}{2\pi^2}\sum_{m,n}{}' \frac{\chi_{-3}(m)}{(m+3\mu n)^2(m+3\bar{\mu}n)}\right), \end{aligned}$$

thus recovering a result of [76].

3.3.9 The $\Gamma_0^0(5)$ example

Now we will consider our final example:

$$(x+y+1)(x+1)(y+1) - \frac{1}{t}xy.$$

Applying the diamond operation, we find that

$$(x) \diamond (y) = 10(P) + 5(2P),$$

where P is a torsion point of order 5.

For $t > 0$

$$r(t) = \frac{5}{2\pi}\mathrm{Im}\left(\frac{\tau}{y_\tau}\left(2R_\tau\left(\mathrm{e}^{\frac{8\pi \mathrm{i}}{5}}\right) + R_\tau\left(\mathrm{e}^{\frac{6\pi \mathrm{i}}{5}}\right)\right)\right).$$

Finally,

$$r(t) = -\mathrm{Re}\left(\frac{25\mathrm{i}y_\mu}{4\pi^2}\sum_{m,n}{}' \frac{2\left(\zeta_5^m - \zeta_5^{-m}\right) + \zeta_5^{2m} - \zeta_5^{-2m}}{(m+5\mu n)^2(m+5\bar{\mu}n)}\right).$$

In conclusion, we see that the modular structure comes from the form of the regulator function, and the functional identities are consequences of the functional identities of the elliptic dilogarithm.

3.4 Conclusion

We have used both regulator and q-series methods to prove a variety of identities between the Mahler measures of genus-one polynomials. We will conclude this paper with a final open problem.

Open Problem 3: How do you characterize all the functional equations of $\mu(t)$?

We have seen that there are identities like Eq. (3.1.6), stating that

$$2\mathrm{m}\left(2\left(k+\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right)=\mathrm{m}\left(4k^2+x+\frac{1}{x}+y+\frac{1}{y}\right)+\mathrm{m}\left(\frac{4}{k^2}+x+\frac{1}{x}+y+\frac{1}{y}\right).$$

While this formula does not follow from Eq. (3.2.14), it can be proved with regulators.

Indeed, the last section showed us that we can obtain functional identities for the Mahler measures by looking at functional equations for the elliptic dilogarithm. To have an idea of the dimensions of this problem, let us note that equation (3.3.64) corresponds to the integration of an identity for the Hecke operator T_p. This suggests that more identities will follow from looking at the general operator T_n. And this is just the beginning of the story. . .

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Chapter 4

New ${}_5F_4$ transformations and Mahler measures

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4.1 Introduction

In this paper we will study the consequences of some recent results of Bertin. Recall that Bertin proved q-series expansions for a pair of three-variable Mahler measures in [86]. As usual the Mahler measure of an n-variable polynomial, $P(z_1, \ldots, z_n)$, is defined by

$$\mathrm{m}\left(P(z_1, \ldots, z_n)\right) = \int_0^1 \cdots \int_0^1 \log \left| P\left(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}\right) \right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n.$$

We will define $g_1(u)$ and $g_2(u)$ in terms of the following three-variable Mahler measures

$$g_1(u) := \mathrm{m}\left(u + x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z}\right), \tag{4.1.1}$$

$$\begin{aligned} g_2(u) := \mathrm{m}\big(&-u + 4 + \left(x + x^{-1}\right)\left(y + y^{-1}\right) \\ &+ \left(x + x^{-1}\right)\left(z + z^{-1}\right) + \left(y + y^{-1}\right)\left(z + z^{-1}\right)\big). \end{aligned} \tag{4.1.2}$$

We can recover Bertin's original notation by observing that $g_1(u) = \mathrm{m}\left(P_u\right)$, and after substituting $(xz, y/z, z/x) \to (x, y, z)$ in Eq. (4.1.2) we see that $g_2(u + 4) = \mathrm{m}\left(Q_u\right)$ [86].

In Section 4.2 we will show how to establish a large number of interesting relations between $g_1(u)$, $g_2(u)$, and three more three-variable Mahler measures. For example, for $|u|$ sufficiently large Eq. (4.2.21) is equivalent

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to

$$\begin{aligned} g_1\left(3\left(u^2+u^{-2}\right)\right) =&\frac{1}{5}\mathrm{m}\left(x^4+y^4+z^4+1+\sqrt{3}\frac{\left(3+u^4\right)}{u^3}xyz\right)\\ &+\frac{3}{5}\mathrm{m}\left(x^4+y^4+z^4+1+\sqrt{3}\frac{\left(3+u^{-4}\right)}{u^{-3}}xyz\right). \end{aligned} \tag{4.1.3}$$

Rodriguez-Villegas briefly mentioned the Mahler measure $\mathrm{m}\left(x^4+y^4+z^4+1+uxyz\right)$ on the last page of [95].

We will also show that identities like Eq. (4.1.3) are equivalent to transformations for the ${}_5F_4$ hypergeometric function. Recall that the generalized hypergeometric function is defined by

$${}_pF_q\left({a_1,a_2,\ldots a_p \atop b_1,b_2,\ldots b_q};x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{x^n}{n!},$$

where $(y)_n=\Gamma(y+n)/\Gamma(y)$. We have restated Eq. (4.1.3) as a hypergeometric transformation in Eq. (4.2.24). As a special case of Eq. (4.1.3) we can also deduce that

$${}_5F_4\left({\frac{5}{4},\frac{3}{2},\frac{7}{4},1,1 \atop 2,2,2,2};1\right)=\frac{256}{3}\log(2)-\frac{5120\sqrt{2}}{3\pi^3}L(f,3),$$

where $f(q)=q\prod_{n=1}^{\infty}\left(1-q^n\right)^2\left(1-q^{2n}\right)\left(1-q^{4n}\right)\left(1-q^{8n}\right)^2$, and $L(f,s)$ is the usual L-series of $f(q)$. We will conclude Section 4.2 with a brief discussion of some related, but still unproven, evaluations of the ${}_4F_3$ and ${}_3F_2$ hypergeometric functions.

It turns out that $g_1(u)$ and $g_2(u)$ are also closely related to Watson's triple integrals. For appropriate values of u, Watson showed that $g_1'(u)$ and $g_2'(u)$ reduce to products of elliptical integrals (for relevant results see [96], [91], [92], and [90]). In Section 4.3 we will use some related transformations to prove new formulas for $1/\pi$. For example, we will show that

$$\frac{2}{\pi}=\sum_{n=0}^{\infty}(-1)^n\frac{(3n+1)}{32^n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2.$$

Notice that this formula for $1/\pi$ involves the Domb numbers. Chan, Chan and Liu obtained a similar formula for $1/\pi$ involving Domb numbers in [88], we have recovered their result in (4.3.10). Zudilin and Yang also discovered some related formulas for $1/\pi$ in [98]. All of the ${}_3F_2$ transformations that we will utilize in Section 4.3 follow from differentiating the ${}_5F_4$ identities established in Section 4.2.

4.2 Identities between Mahler measures and transformations for the ${}_5F_4$ function

Bertin proved that both $g_1(u)$ and $g_2(u)$ have convenient q-series expansions when u is parameterized correctly. Before stating her theorem, we will define some notation. As usual let

$$(x,q)_\infty = (1-x)(1-xq)\left(1-xq^2\right)\dots,$$

and define $G(q)$ by

$$G(q) = \operatorname{Re}\left[-\log(q) + 240\sum_{n=1}^{\infty} n^2 \log\left(1-q^n\right)\right]. \tag{4.2.1}$$

Notice that if $q \in (0,1)$ then $G'(q) = -M(q)/q$, where $M(q)$ is the Eisenstein series of weight 4 on the full modular group $\Gamma(1)$ [81].

Theorem 4.2.1. *(Bertin) For $|q|$ sufficiently small*

$$g_1\left(t_1(q)\right) = -\frac{1}{60}G(q) + \frac{1}{30}G(q^2) - \frac{1}{20}G(q^3) + \frac{1}{10}G(q^6), \tag{4.2.2}$$

$$g_2\left(t_2(q)\right) = \frac{1}{120}G(q) - \frac{1}{15}G\left(q^2\right) - \frac{1}{40}G\left(q^3\right) + \frac{1}{5}G\left(q^6\right), \tag{4.2.3}$$

where

$$t_1(q) = v_1 + \frac{1}{v_1}, \quad \text{and } v_1 = q^{1/2}\frac{\left(q;q^2\right)_\infty^6}{\left(q^3;q^6\right)_\infty^6},$$

$$t_2(q) = -\left(v_2 - \frac{1}{v_2}\right)^2, \quad \text{and } v_2 = q^{1/2}\frac{\left(q^2;q^2\right)_\infty^6\left(q^3;q^3\right)_\infty^2\left(q^{12};q^{12}\right)_\infty^4}{\left(q;q\right)_\infty^2\left(q^4;q^4\right)_\infty^4\left(q^6;q^6\right)_\infty^6}.$$

In this section we will show that both $g_1(u)$ and $g_2(u)$ reduce to linear combinations of ${}_5F_4$ hypergeometric functions. We will accomplish this goal by first expressing each of the functions

$$f_2(u) := 2\mathrm{m}\left(u^{1/2} + \left(x+x^{-1}\right)\left(y+y^{-1}\right)\left(z+z^{-1}\right)\right), \tag{4.2.4}$$

$$f_3(u) := \mathrm{m}\left(u - \left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2(1+z)^3 z^{-2}\right), \tag{4.2.5}$$

$$f_4(u) := 4\mathrm{m}\left(x^4+y^4+z^4+1+u^{1/4}xyz\right), \tag{4.2.6}$$

in terms of $G(q)$. We will then exploit those identities to establish linear relations between functions in the set $\{f_2(u), f_3(u), f_4(u), g_1(u), g_2(u)\}$. This is significant since $f_2(u)$, $f_3(u)$, and $f_4(u)$ all reduce to ${}_5F_4$ hypergeometric functions. In particular this implies the non-trivial fact that both $g_1(u)$ and $g_2(u)$ also reduce to linear combinations of ${}_5F_4$ functions.

Proposition 4.2.2. *The following identities hold for $|u|$ sufficiently large:*

$$f_2(u) = Re\left[\log(u) - \frac{8}{u}{}_5F_4\left({\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2,2}; \frac{64}{u}\right)\right], \tag{4.2.7}$$

$$f_3(u) = Re\left[\log(u) - \frac{12}{u}{}_5F_4\left({\frac{4}{3},\frac{3}{2},\frac{5}{3},1,1 \atop 2,2,2,2}; \frac{108}{u}\right)\right], \tag{4.2.8}$$

$$f_4(u) = Re\left[\log(u) - \frac{24}{u}{}_5F_4\left({\frac{5}{4},\frac{3}{2},\frac{7}{4},1,1 \atop 2,2,2,2}; \frac{256}{u}\right)\right]. \tag{4.2.9}$$

For $|u| > 6$

$$g_1(u) = Re\left[\log(u) - \sum_{n=1}^{\infty}\frac{(1/u)^{2n}}{2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2\right], \tag{4.2.10}$$

and if $|u| > 16$

$$g_2(u) = Re\left[\log(u) - \sum_{n=1}^{\infty}\frac{(1/u)^{n}}{n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2\right]. \tag{4.2.11}$$

Proof. We can prove each of these identities using a method due to Rodriguez-Villegas [95]. We will illustrate the proof of Eq. (4.2.7) explicitly. Rearranging the Mahler measure shows that

$$f_2(u) = \mathrm{Re}\left[\log(u) + \int_0^1\int_0^1\int_0^1 \log\left(1 - \frac{64}{u}\cos^2(2\pi t_1)\cos^2(2\pi t_2)\cos^2(2\pi t_3)\right)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\right].$$

If $|u| > 64$, then $\left|\frac{64}{u}\cos^2(2\pi t_1)\cos^2(2\pi t_2)\cos^2(2\pi t_3)\right| < 1$, hence by the Taylor series for the logarithm

$$\begin{aligned} f_2(u) &= \mathrm{Re}\left[\log(u) - \sum_{n=1}^{\infty}\frac{(64/u)^n}{n}\left(\int_0^1 \cos^{2n}(2\pi t)\mathrm{d}t\right)^3\right] \\ &= \mathrm{Re}\left[\log(u) - \sum_{n=1}^{\infty}\binom{2n}{n}^3\frac{(1/u)^n}{n}\right] \\ &= \mathrm{Re}\left[\log(u) - \frac{8}{u}{}_5F_4\left({\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2,2}; \frac{64}{u}\right)\right]. \end{aligned}$$

Notice that Eq. (4.2.7) holds whenever $u \notin [-64, 64]$, since $f_2(u)$ is harmonic in $\mathbb{C}\backslash[-64, 64]$. $\square$

While Proposition 4.2.2 shows that the results in this paper easily translate into the language of hypergeometric functions, the relationship to Mahler measure is more important than simple pedagogy. Bertin proved that for certain values of u the zero varieties of the (projectivized) polynomials from equations (4.1.1) and (4.1.2) define $K3$ hypersurfaces. She also proved formulas relating the L-functions of these $K3$ surfaces at $s = 3$ to rational multiples of the Mahler measures. Proposition 4.2.2 shows that these results imply explicit ${}_5F_4$ evaluations (see Corollary 4.2.6 for explicit examples). While it might also be interesting to interpret the polynomials from equations (4.2.4) through (4.2.6) in terms of $K3$ hypersurfaces, we will not pursue that direction here.

Theorem 4.2.3. *For $|q|$ sufficiently small*

$$f_2(s_2(q)) = -\frac{2}{15}G(q) - \frac{1}{15}G(-q) + \frac{3}{5}G\left(q^2\right), \tag{4.2.12}$$

$$f_3(s_3(q)) = -\frac{1}{8}G(q) + \frac{3}{8}G\left(q^3\right), \tag{4.2.13}$$

$$f_4(s_4(q)) = -\frac{1}{3}G(q) + \frac{2}{3}G\left(q^2\right), \tag{4.2.14}$$

where

$$\begin{aligned} s_2(q) =& q^{-1}\left(-q;q^2\right)_\infty^{24}, \\ s_3(q) =& \frac{1}{q}\left(27q\frac{\left(q^3;q^3\right)_\infty^6}{(q;q)_\infty^6} + \frac{(q;q)_\infty^6}{\left(q^3;q^3\right)_\infty^6}\right)^2, \\ s_4(q) =& \frac{1}{q}\frac{\left(q^2;q^2\right)_\infty^{24}}{(q;q)_\infty^{24}}\left(16q\frac{(q;q)_\infty^4\left(q^4;q^4\right)_\infty^8}{\left(q^2;q^2\right)_\infty^{12}} + \frac{\left(q^2;q^2\right)_\infty^{12}}{(q;q)_\infty^4\left(q^4;q^4\right)_\infty^8}\right)^4. \end{aligned}$$

The following inverse relations hold for $|q|$ sufficiently small:

$$\begin{aligned} G(q) =& -19f_2\left(s_2(q)\right) - 4f_2\left(s_2(-q)\right) + 24f_2\left(s_2\left(q^2\right)\right) \\ & -12f_2\left(s_2\left(-q^2\right)\right), \end{aligned} \tag{4.2.15}$$

$$\begin{aligned} G(q) =& -\frac{19}{2}f_3\left(s_3(q)\right) - \frac{3}{2}f_3\left(s_3\left(\mathrm{e}^{2\pi\mathrm{i}/3}q\right)\right) \\ & -\frac{3}{2}f_3\left(s_3\left(\mathrm{e}^{4\pi\mathrm{i}/3}q\right)\right) + \frac{9}{2}f_3\left(s_3\left(q^3\right)\right), \end{aligned} \tag{4.2.16}$$

$$G(q) = -5f_4\left(s_4(q)\right) - 2f_4\left(s_4(-q)\right) + 4f_4\left(s_4\left(q^2\right)\right). \tag{4.2.17}$$

Proof. We can use Ramanujan's theory of elliptic functions to verify the first half of this theorem. Recall that the *elliptic nome* is defined by

$$q_j(\alpha)=\exp\left(-\frac{\pi}{\sin(\pi/j)}\frac{{}_2F_1\left(\frac{1}{j},1-\frac{1}{j};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{j},1-\frac{1}{j};1;\alpha\right)}\right).$$

It is a well established fact that $s_j\left(q_j\left(\alpha\right)\right)$ is a rational function of α whenever $j\in\{2,3,4\}$. For example if $q=q_2(\alpha)$, then $s_2\left(q\right)=\frac{16}{\alpha(1-\alpha)}$. Therefore we can verify Eq. (4.2.12) by differentiating with respect to α, and by showing that the identity holds when $q\to 0$.

Observe that when $q\to 0$ both sides of Eq. (4.2.12) approach $-\log|q|+O(q)$. Differentiating with respect to α yields

$$\begin{aligned}&-\frac{(1-2\alpha)}{2\alpha(1-\alpha)}{}_3F_2\left({\textstyle\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};4\alpha(1-\alpha)\right)\\&\qquad=-\frac{1}{2q}\left(1-16\sum_{n=1}^{\infty}n^3\frac{q^n}{1-q^n}+256\sum_{n=1}^{\infty}n^3\frac{q^{4n}}{1-q^{4n}}\right)\frac{\mathrm{d}q}{\mathrm{d}\alpha}.\end{aligned}$$

This final identity follows from applying three well known formulas:

$$\begin{aligned}&\frac{\mathrm{d}q}{\mathrm{d}\alpha}=\frac{q}{\alpha(1-\alpha){}_2F_1{}^2\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)},\\&{}_3F_2\left({\textstyle\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};4\alpha(1-\alpha)\right)={}_2F_1{}^2\left(\frac{1}{2},\frac{1}{2};1;\alpha\right),\\&1-16\sum_{n=1}^{\infty}n^3\frac{q^n}{1-q^n}+256\sum_{n=1}^{\infty}n^3\frac{q^{4n}}{1-q^{4n}}=(1-2\alpha){}_2F_1{}^4\left(\frac{1}{2},\frac{1}{2};1;\alpha\right).\end{aligned}$$

We can verify equations (4.2.13) and (4.2.14) in a similar manner by using the fact that $s_3\left(q_3(\alpha)\right)=\frac{27}{\alpha(1-\alpha)}$, and $s_4\left(q_4(\alpha)\right)=\frac{64}{\alpha(1-\alpha)}$.

The crucial observation for proving equations (4.2.15) through (4.2.17) is the fact that $G(q)$ satisfies the following functional equation for any prime p:

$$\sum_{j=0}^{p-1}G\left(\mathrm{e}^{2\pi\mathrm{i}j/p}q\right)=\left(1+p^3\right)G\left(q^p\right)-p^2G\left(q^{p^2}\right).\tag{4.2.18}$$

We will only need the $p=2$ case to prove Eq. (4.2.15):

$$G(q)+G(-q)=9G\left(q^2\right)-4G\left(q^4\right).$$

Notice that this last formula always allows us to eliminate $G\left(q^4\right)$ from an equation. Applying the substitutions $q \to -q$, $q \to q^2$, and $q \to -q^2$ to Eq. (4.2.12) yields

$$\begin{pmatrix} -2/15 & -1/15 & 3/5 \\ -1/15 & -2/15 & 3/5 \\ -3/20 & -3/20 & 23/20 \end{pmatrix}\begin{pmatrix} G(q) \\ G(-q) \\ G\left(q^2\right) \end{pmatrix} = \begin{pmatrix} f_2\left(s_2(q)\right) \\ f_2\left(s_2(-q)\right) \\ 2f_2\left(s_2\left(q^2\right)\right) - f_2\left(s_2\left(-q^2\right)\right) \end{pmatrix}.$$

Since this system of equations is non-singular, we can invert the matrix to recover Eq. (4.2.15). We can prove equations (4.2.16) and (4.2.17) in a similar fashion. □

If we compare Theorem 4.2.3 with Bertin's results we can deduce some obvious relationships between the Mahler measures. For example, combining Eq. (4.2.14) with Eq. (4.2.3), and combining Eq. (4.2.13) with Eq. (4.2.2), we find that

$$g_1(t_1(q)) = \frac{1}{20} f_4\left(s_4(q)\right) + \frac{3}{20} f_4\left(s_4\left(q^3\right)\right), \tag{4.2.19}$$

$$g_2\left(t_2(q)\right) = -\frac{1}{15} f_3\left(s_3(q)\right) + \frac{8}{15} f_3\left(s_3\left(q^2\right)\right). \tag{4.2.20}$$

Notice that many more identities follow from substituting equations (4.2.15) through (4.2.17) into formulas (4.2.2), (4.2.3), (4.2.12), (4.2.13) and (4.2.14). However, for the remainder of this section we will restrict our attention to equations (4.2.19) and (4.2.20). In particular, we will appeal to the theory of elliptic functions to transform those results into identities which depend on rational arguments.

If we let $q = q_2(\alpha)$, then it is well known that $\frac{q^{j/24}\left(q^j;q^j\right)_\infty}{q^{1/24}(q;q)_\infty}$ is an algebraic function of α for $j \in \{1, 2, 3, \dots\}$ (for example see [82] or [83]). It follows immediately that $s_2(q)$, $s_3(q)$, $s_4(q)$, $t_1(q)$, and $t_2(q)$ are also algebraic functions of α. The following lemma lists several instances where those functions have rational parameterizations.

Lemma 4.2.4. *Suppose that $q = q_2(\alpha)$, where $\alpha = p(2+p)^3/(1+2p)^3$. The following identities hold for $|p|$ sufficiently small:*

$$s_2(q) = \frac{16(1+2p)^6}{p(1-p)^3(1+p)(2+p)^3}, \qquad s_2\left(q^3\right) = \frac{16(1+2p)^2}{p^3(1-p)(1+p)^3(2+p)},$$

$$s_2(-q) = -\frac{16(1-p)^6(1+p)^2}{p(2+p)^3(1+2p)^3}, \qquad s_2\left(-q^3\right) = -\frac{16(1-p)^2(1+p)^6}{p^3(2+p)(1+2p)},$$

$$s_2\left(-q^2\right)=\frac{16^2(1-p)^3(1+p)(1+2p)^3}{p^2(2+p)^6},\quad s_2\left(-q^6\right)=\frac{16^2(1-p)(1+p)^3(1+2p)}{p^6(2+p)^2},$$
$$s_3(q)=\frac{4\left(1+4p+p^2\right)^6}{p\left(1-p^2\right)^4(2+p)(1+2p)},\qquad s_3\left(q^2\right)=\frac{16\left(1+p+p^2\right)^6}{p^2\left(1-p^2\right)^2(2+p)^2(1+2p)^2},$$
$$s_3(-q)=-\frac{4\left(1-2p-2p^2\right)^6}{p(1-p^2)(2+p)(1+2p)^4},\qquad s_3\left(q^4\right)=\frac{4\left(2+2p-p^2\right)^6}{p^4\left(1-p^2\right)(2+p)^4(1+2p)},$$

$$\begin{aligned}
s_4(q)&=\frac{16\left(1+14p+24p^2+14p^3+p^4\right)^4}{p(1-p)^6(1+p)^2(2+p)^3(1+2p)^3},\\
s_4\left(q^3\right)&=\frac{16\left(1+2p+2p^3+p^4\right)^4}{p^3(1-p)^2(1+p)^6(2+p)(1+2p)},\\
s_4(-q)&=-\frac{16\left(1-10p-12p^2-4p^3-2p^4\right)^4}{p(1-p)^3(1+p)(1+2p)^6(2+p)^3},\\
s_4\left(-q^3\right)&=-\frac{16\left(1+2p-4p^3-2p^4\right)^4}{p^3(1-p)(1+p)^3(1+2p)^2(2+p)}.
\end{aligned}$$

Rational formulas also exist for certain values of $t_1^2(q)$ and $t_2(q)$:

$$\begin{aligned}
t_1^2(q)&=\frac{4\left(1+p+p^2\right)^2\left(1+4p+p^2\right)^2}{p(1-p^2)^2(2+p)(1+2p)},\\
t_1^2(-q)&=-\frac{4\left(1+p+p^2\right)^2\left(1-2p-2p^2\right)^2}{p(1-p^2)(2+p)(1+2p)^2},\\
t_2(q)&=-\frac{4\left(1-p^2\right)^2}{p(2+p)(1+2p)},\\
t_2(-q)&=-\frac{4(1+p+p^2)^2}{p(1-p^2)(2+p)}.
\end{aligned}$$

The main difficulty with Lemma 4.2.4 is the fact that very few values of $s_j(\pm q^n)$ reduce to rational functions of p. Consider the set $\left\{s_2(q), s_2(-q), s_2\left(-q^2\right), s_2\left(q^2\right)\right\}$ as an example. While Lemma 4.2.4 shows that $s_2(q)$, $s_2(-q)$, and $s_2\left(-q^2\right)$ are all rational with respect to p, the formula for $s_2\left(q^2\right)$ involves radicals. Recall that if $\alpha=p(2+p)^3/(1+2p)^3$, then

$$s_2\left(q^2\right)=\frac{4\left(1+\sqrt{1-\alpha}\right)^6}{\alpha^2\sqrt{1-\alpha}},$$

where $\sqrt{1-\alpha} = \frac{1-p}{(1+2p)^2}\sqrt{(1-p^2)(1+2p)}$. Since the curve $X^2 = (1-p^2)(1+2p)$ is elliptic with conductor 24, it follows immediately that rational substitutions for p will never reduce $s_2\left(q^2\right)$ to a rational function. For the sake of legibility, we will therefore avoid all identities which involve those four functions simultaneously. By avoiding pitfalls of this nature, we can derive several interesting results from Lemma 4.2.4.

Theorem 4.2.5. *For $|z|$ sufficiently large*

$$g_1\left(3\left(z+z^{-1}\right)\right) = \frac{1}{20}f_4\left(\frac{9\left(3+z^2\right)^4}{z^6}\right) + \frac{3}{20}f_4\left(\frac{9\left(3+z^{-2}\right)^4}{z^{-6}}\right), \tag{4.2.21}$$

$$g_2(z) = -\frac{1}{15}f_3\left(\frac{(16-z)^3}{z^2}\right) + \frac{8}{15}f_3\left(-\frac{(4-z)^3}{z}\right). \tag{4.2.22}$$

Proof. These identities follow from applying Lemma 4.2.4 to equations (4.2.19) and (4.2.20). If we consider Eq. (4.2.19), then Lemma 4.2.4 shows that $t_1^2(q)$, $s_4(q)$, and $s_4\left(q^3\right)$ are all rational functions of p. Forming a resultant with respect to p, we obtain

$$\begin{aligned} 0 = \operatorname*{Res}_p \Bigg[& \frac{4\left(1+p+p^2\right)^2\left(1+4p+p^2\right)^2}{p\left(1-p^2\right)^2(2+p)(1+2p)} - t_1^2(q), \\ & \frac{16\left(1+14p+24p^2+14p^3+p^4\right)^4}{p(1-p)^6(1+p)^2(2+p)^3(1+2p)^3} - s_4(q)\Bigg]. \end{aligned}$$

Simplifying with the aid of a computer, this becomes

$$0 = s_4^2(q) + \left(12+t_1^2(q)\right)^4 - s_4(q)\left(-288+352t_1^2(q)-42t_1^4(q)+t_1^6(q)\right).$$

If we choose z so that $t_1(q) = 3\left(z+z^{-1}\right)$, then $s_4(q) = 9\left(3+z^2\right)^4 z^{-6}$, and a formula for $s_4\left(q^3\right)$ follows in a similar fashion. $\square$

If we let $u = 1/z$ with $z \in \mathbb{R}$ and sufficiently large, then Eq. (4.2.22) reduces to the following infinite series identity:

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{u^n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2 \\ = & \frac{1}{5}\log\left(\frac{(1-16u)}{(1-4u)^8}\right) + \frac{4u}{5(1-16u)^3}\,{}_5F_4\left({\textstyle\frac{4}{3},\frac{3}{2},\frac{5}{3},1,1 \atop 2,2,2,2};-\frac{108u}{(1-16u)^3}\right) \\ & + \frac{32u^2}{5(1-4u)^3}\,{}_5F_4\left({\textstyle\frac{4}{3},\frac{3}{2},\frac{5}{3},1,1 \atop 2,2,2,2};\frac{108u^2}{(1-4u)^3}\right). \end{aligned} \tag{4.2.23}$$

Similarly, if we let $u = 1/z^2$ then Eq. (4.2.21) is equivalent to:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{u}{9(1+u)^2}\right)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2 \\ =&\frac{2}{5}\log\left(\frac{27(1+u)^5}{(3+u)^3(1+3u)}\right)+\frac{4u^3}{5(3+u)^4}{}_5F_4\left({\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4},1,1 \atop 2,2,2,2};\frac{256u^3}{9(3+u)^4}\right)\\ &+\frac{4u}{15(1+3u)^4}{}_5F_4\left({\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4},1,1 \atop 2,2,2,2};\frac{256u}{9(1+3u)^4}\right).\end{aligned} \tag{4.2.24}$$

In Section 4.3 we will differentiate equations (4.2.23) and (4.2.24) to obtain several new formulas for $1/\pi$. But first we will conclude this section by deducing some explicit ${}_5F_4$ evaluations.

Recall that for certain values of u, Bertin evaluated $g_1(u)$ and $g_2(u)$ in terms of the L-series of $K3$ surfaces. She also proved equivalent formulas involving twisted cusp forms. Amazingly, her formulas correspond to cases where the right-hand sides of equations (4.2.21) and (4.2.22) collapse to one hypergeometric term. We can combine her results with equations (4.2.23) and (4.2.24) to deduce several new ${}_5F_4$ evaluations.

Corollary 4.2.6. *If* $g(q) = q\left(q^2;q^2\right)_\infty^3\left(q^6;q^6\right)_\infty^3$, *then*

$${}_5F_4\left({\tfrac{4}{3},\tfrac{3}{2},\tfrac{5}{3},1,1 \atop 2,2,2,2};1\right) = 18\log(2)+27\log(3)-\frac{810\sqrt{3}}{\pi^3}L(g,3). \tag{4.2.25}$$

If $f(q) = q\left(q;q\right)_\infty^2\left(q^2;q^2\right)_\infty\left(q^4;q^4\right)_\infty\left(q^8;q^8\right)_\infty^2$, *then*

$${}_5F_4\left({\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4},1,1 \atop 2,2,2,2};1\right) = \frac{256}{3}\log(2)-\frac{5120\sqrt{2}}{3\pi^3}L(f,3). \tag{4.2.26}$$

While many famous ${}_5F_4$ identities, such as Dougall's formula [81], reduce special values of the ${}_5F_4$ function to gamma functions, equations (4.2.25) and (4.2.26) do not fit into this category. Rather these new formulas are higher dimensional analogues of Boyd's conjectures. In particular, Boyd has conjectured large numbers of identities relating two-variable Mahler measures (that mostly reduce to ${}_4F_3$ functions) to the L-series of elliptic curves [87]. The most famous outstanding conjecture of this type asserts that

$$\mathrm{m}\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right) = -2\mathrm{Re}\left[{}_4F_3\left({\tfrac{3}{2},\tfrac{3}{2},1,1 \atop 2,2,2};16\right)\right] \stackrel{?}{=} \frac{15}{4\pi^2}L(f,2),$$

where

$$f(q) = q\prod_{n=1}^{\infty}\left(1-q^{n}\right)\left(1-q^{3n}\right)\left(1-q^{5n}\right)\left(1-q^{15n}\right),$$

and "$\stackrel{?}{=}$" indicates numerical equality to at least 50 decimal places. Recently, Kurokawa and Ochiai proved a formula [93] which simplifies this last conjecture to

$${}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop \frac{3}{2},1};\frac{1}{16}\right) \stackrel{?}{=} \frac{15}{\pi^2}L(f,2).$$

Of course it would be highly desirable to rigorously prove Boyd's conjectures. Failing that, it might be interesting to search for more hypergeometric identities like equations (4.2.25) and (4.2.26). This line of thought suggests the following fundamental problem with which we shall conclude this section:
Open Problem: Determine every L-series that can be expressed in terms of generalized hypergeometric functions with algebraic parameters.

4.3 New formulas for $1/\pi$

In the previous section we produced several new transformations for the $_5F_4$ hypergeometric function. Now we will differentiate those formulas to obtain some new $_3F_2$ transformations, and several accompanying formulas for $1/\pi$. The following formula is a typical example of the identities in this section:

$$\frac{2}{\pi} = \sum_{n=0}^{\infty}(-1)^n\frac{(3n+1)}{32^n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2. \tag{4.3.1}$$

Ramanujan first proved identities like Eq. (4.3.1) in his famous paper "Modular equations and approximations to π" [94]. He showed that the following infinite series holds for certain constants A, B, and X:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}(An+B)\frac{(1/2)_n^3}{n!^3}X^n \tag{4.3.2}$$

Ramanujan determined many sets of algebraic values for A, B, and X by expressing them in terms of the classical singular moduli G_n and g_n. He also stated (but did not prove) several formulas for $1/\pi$ where $(1/2)_n^3$ is replaced by $(1/a)_n\,(1/2)_n\,(1-1/a)_n$ for $a \in \{3,4,6\}$ (for more details see [84] or [89]).

Ramanujan's formulas for $1/\pi$ have attracted a great deal of attention because of their intrinsic beauty, and because they converge extremely quickly. For example, *Mathematica* calculates π using a variant of a Ramanujan-type formula due to the Chudnovsky brothers [97]:

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty} \frac{(-1)^n(6n)!(13591409 + 54513013n)}{n!^3(3n)!\,(640320^3)^{n+1/2}}. \tag{4.3.3}$$

More recent mathematicians including Yang and Zudilin have derived formulas for $1/\pi$ which are not hypergeometric, but still similar to Eq. (4.3.3). For example, Yang showed that

$$\frac{18}{\pi\sqrt{15}} = \sum_{n=0}^{\infty} \frac{(4n+1)}{36^n} \sum_{k=0}^{n} \binom{n}{k}^4,$$

and Zudilin gave many infinite series for $1/\pi$ containing nested sums of binomial coefficients [98]. All of the formulas that we will prove, including Eq. (4.3.1), are essentially of this type.

Theorem 4.3.1. *For $|u|$ sufficiently small*

$${}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1}; \frac{108u^2}{(1-4u)^3}\right) = (1-4u)\sum_{n=0}^{\infty} u^n \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2. \tag{4.3.4}$$

If $|u|$ is sufficiently small

$${}_3F_2\left({\tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4} \atop 1,1}; \frac{256u}{9(1+3u)^4}\right) = \frac{(1+3u)}{(1+u)}\sum_{n=0}^{\infty} \left(\frac{u}{9(1+u)^2}\right)^n \binom{2n}{n} \sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2. \tag{4.3.5}$$

Proof. Applying the operator $u\frac{\mathrm{d}}{\mathrm{d}u}$ to Eq. (4.2.23), and then simplifying yields

$$\begin{aligned}\sum_{n=0}^{\infty} u^n \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2 = &-\frac{(1+32u)}{15(1-16u)}\,{}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1}; -\frac{108u}{(1-16u)^3}\right)\\ &+\frac{16(1+2u)}{15(1-4u)}\,{}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1}; \frac{108u^2}{(1-4u)^3}\right).\end{aligned}$$

Eq. (4.3.4) then follows from applying a standard ${}_3F_2$ transformation:

$${}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1}; -\frac{108u}{(1-16u)^3}\right) = \frac{(1-16u)}{(1-4u)}\,{}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1}; \frac{108u^2}{(1-4u)^3}\right). \tag{4.3.6}$$

We can prove Eq. (4.3.5) in a similar manner by differentiating Eq. (4.2.24) and then using

$$ {}_3F_2\left({\frac{1}{4},\frac{1}{2},\frac{3}{4} \atop 1,1};\frac{256u^3}{9(3+u)^4}\right)=\frac{(3+u)}{3(1+3u)}\,{}_3F_2\left({\frac{1}{4},\frac{1}{2},\frac{3}{4} \atop 1,1};\frac{256u}{9(1+3u)^4}\right). \tag{4.3.7} $$

□

While the infinite series in Theorem 4.3.1 are not hypergeometric since they involve nested binomial sums, they are still interesting. In particular, those formulas easily translate into unexpected integrals involving powers of modified Bessel functions. For $|x|$ sufficiently small, Eq. (4.3.4) is equivalent to

$$ \int_0^\infty \mathrm{e}^{-3\left(x+x^{-1}\right)u} I_0^3\left(2u\right)\mathrm{d}u = \frac{x}{3\left(1+3x^2\right)}\,{}_3F_2\left({\frac{1}{4},\frac{1}{2},\frac{3}{4} \atop 1,1}\frac{256x^2}{9(1+3x^2)^4}\right), \tag{4.3.8} $$

where $I_0(u)$ is the modified Bessel function of the first kind. Recall the series expansions for $I_0(2u)$ and $I_0^2(2u)$:

$$ I_0(2u)=\sum_{n=0}^\infty \frac{u^{2n}}{n!^2}, \qquad I_0^2(2u)=\sum_{n=0}^\infty \binom{2n}{n}\frac{u^{2n}}{n!^2}. $$

While it is not obvious that the Laplace transform of $I_0^3(2u)$ should equal a hypergeometric function, M. Lawrence Glasser has kindly pointed out that equation (4.3.8) is essentially a well known result. A variety of similar integrals have also been studied by Joyce [92], Glasser and Montaldi [90], and others.

Finally, we will list a few formulas for $1/\pi$. Notice that equation (4.3.10) first appeared in the work of Chan, Chan and Liu [88].

Corollary 4.3.2. *Let* $a_n=\sum_{k=0}^n\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2$, *then the following formulas are true:*

$$ \frac{2}{\pi}=\sum_{n=0}^\infty(-1)^n\frac{(3n+1)}{32^n}a_n, \tag{4.3.9} $$

$$ \frac{8\sqrt{3}}{3\pi}=\sum_{n=0}^\infty\frac{(5n+1)}{64^n}a_n, \tag{4.3.10} $$

$$ \frac{9+5\sqrt{3}}{\pi}=\sum_{n=0}^\infty(6n+3-\sqrt{3})\left(\frac{3\sqrt{3}-5}{4}\right)^n a_n. \tag{4.3.11} $$

Let $b_n = \binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2$, then the following identity holds:

$$\frac{2\left(64+29\sqrt{3}\right)}{\pi} = \sum_{n=0}^{\infty}\left(520n+159-48\sqrt{3}\right)\left(\frac{80\sqrt{3}-139}{484}\right)^n b_n. \quad (4.3.12)$$

Proof. We can use Eq. (4.3.4) to easily deduce that if

$$\sum_{n=0}^{\infty}(an+b)\frac{(1/3)_n(1/2)_n(2/3)_n}{n!^3}\left(\frac{108u^2}{(1-4u)^3}\right)^n = \sum_{n=0}^{\infty}(An+B)u^n a_n,$$

then $A = a(1-4u)/(2+4u)$, and $B = a(-4u)(1-4u)/(2+4u)+b(1-4u)$. Since the left-hand side of this last formula equals $1/\pi$ when $\left(a, b, \frac{108u^2}{(1-4u)^3}\right) \in \{\left(\frac{60}{27}, \frac{8}{27}, \frac{2}{27}\right), \left(\frac{2}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{1}{2}\right), \left(\frac{45}{11}-\frac{5}{33}\sqrt{3}, \frac{6}{11}-\frac{13}{99}\sqrt{3}, -\frac{194}{1331}+\frac{225}{2662}\sqrt{3}\right)\}$, it is easy to verify equations (4.3.9) through (4.3.11) [89].

We can verify Eq. (4.3.12) in a similar manner by combining Eq. (4.3.5) with Ramanujan's formula

$$\frac{8}{\pi} = \sum_{n=0}^{\infty}(20n+3)\frac{(1/4)_n(1/2)_n(3/4)_n}{n!^3}\left(\frac{-1}{4}\right)^n.$$

□

4.4 Conclusion

We will conclude the paper by suggesting two future projects. Firstly, it would be desirable to determine whether or not a rational series involving b_n exists for $1/\pi$. Secondly, it might be interesting to consider the Mahler measure

$$f_6(u) = \mathrm{m}\left(u-\left(z+z^{-1}\right)^6\left(y+y^{-1}\right)^2(1+x)^3x^{-2}\right),$$

since $f_6(u)$ arises from Ramanujan's theory of signature 6.

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Chapter 5

Trigonometric integrals and Mahler measures

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5.1 Introduction

In this paper we will undertake a systematic study of each of the inverse trigonometric integrals

$$\begin{aligned}
\mathrm{T}(v,w) &= \int_0^1 \frac{\tan^{-1}(vx)\tan^{-1}(wx)}{x}\mathrm{d}x,\\
\mathrm{S}(v,w) &= \int_0^1 \frac{\sin^{-1}(vx)\sin^{-1}(wx)}{x}\mathrm{d}x,\\
\mathrm{TS}(v,w) &= \int_0^1 \frac{\tan^{-1}(vx)\sin^{-1}(wx)}{x}\mathrm{d}x.
\end{aligned}$$

This class of integrals arises when trying to find closed form expressions for the Mahler measures of certain three-variable polynomials.

Recall that the Mahler measure of an n-dimensional polynomial, $P(x_1,\ldots,x_n)$, can be defined by

$$\mathrm{m}\left(P(x_1,\ldots,x_n)\right) = \int_0^1 \cdots \int_0^1 \log\left|P\left(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_n}\right)\right|\mathrm{d}\theta_1\ldots\mathrm{d}\theta_n.$$

In the last few years, numerous papers have established explicit formulas relating multi-variable Mahler measures to special constants. Smyth proved the first result [102] with

$$\mathrm{m}\left(1+x+y+z\right) = \frac{7}{2\pi^2}\zeta(3),$$

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where the Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

In this paper, we will prove a number of new formulas relating three-variable Mahler measures to the aforementioned trigonometric integrals. Many of our identities generalize previously known results. We will list a few of our main results in this introductory section.

For our first example, we can use various properties of $\mathrm{T}(v, w)$ to show that

$$\begin{aligned} &\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2 + \left(y + v^2\left(\frac{1-x}{1+x}\right)\right)^2 z\right) \\ &\qquad = \frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{8}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right) + \frac{1}{2}\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2\right). \end{aligned} \tag{5.1.1}$$

This reduces to one of Lalin's formulas [107] when $v = 1$:

$$\mathrm{m}\left((1+y)(1+z) + (1-z)(x-y)\right) = \frac{7}{2\pi^2}\zeta(3) + \frac{\log(2)}{2}. \tag{5.1.2}$$

We can use the double arcsine integral, $\mathrm{S}(v, w)$, to prove that if $v \in [0, 1]$:

$$\begin{aligned} \mathrm{m}\left(v(1+x) + y + z\right) &= \frac{2}{\pi}\int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u - \frac{4}{\pi^2}\mathrm{S}(v, 1) \\ &= \frac{4}{\pi^2}\left(\frac{\mathrm{Li}_3(v) - \mathrm{Li}_3(-v)}{2}\right). \end{aligned} \tag{5.1.3}$$

The second equality has been proved by Vandervelde [115]. Slightly more complicated arguments lead to expressions that include

$$\mathrm{m}\left(1 - x^{1/6} + y + z\right) = \frac{2}{\pi}\int_0^{\frac{1}{2}} \frac{\sin^{-1}(u)}{u}\mathrm{d}u - \frac{12}{\pi^2}\mathrm{S}\left(\frac{1}{2}, \frac{1}{2}\right) \tag{5.1.4}$$

This fractional Mahler measure is defined by

$$\mathrm{m}\left(1 - x^{1/6} + y + z\right) = \int_0^1 \mathrm{m}\left(1 - e^{2\pi i u/6} + y + z\right)\mathrm{d}u,$$

notice that $\mathrm{m}\left(1 - x^{1/6} + y + z\right) \neq \mathrm{m}(1 - x + y + z)$. We can simplify the right-hand side of Eq. (5.1.4) by either expressing $\mathrm{S}\left(\frac{1}{2}, \frac{1}{2}\right)$ as a linear combination of L-functions, or in terms of a famous binomial sum:

$$\mathrm{S}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}}.$$

Condon [105] proved an identity that Boyd and Rodriguez Villegas conjectured:

$$\mathrm{m}\left(1+x+(1-x)(y+z)\right)=\frac{28}{5\pi^2}\zeta(3). \tag{5.1.5}$$

His proof also showed (in a slightly disguised form) that

$$\mathrm{TS}(2,1)=\frac{\pi}{2}\int_0^2\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{7}{5}\zeta(3). \tag{5.1.6}$$

We have generalized Condon's identity to show that

$$\mathrm{m}\left(1+x+\frac{v}{2}(1-x)(y+z)\right)=\frac{2}{\pi}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{4}{\pi^2}\mathrm{TS}(v,1), \tag{5.1.7}$$

where Eq. (5.4.18) expresses $\mathrm{TS}(v,1)$ in terms of polylogarithms. We can use this result to prove a number of new formulas, including:

$$\begin{aligned}\mathrm{m}&\left(x+\frac{v^2}{4}(1+x)^2+\left(y+\frac{v}{2}(1+x)\right)^2z\right)\\&=\frac{2}{\pi}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{4}{\pi^2}\mathrm{TS}(v,1)+\frac{1}{2}\mathrm{m}\left(x+\frac{v^2}{4}(1+x)^2\right).\end{aligned} \tag{5.1.8}$$

When $v=2$ this reduces to an interesting identity for $\zeta(3)$ and the golden ratio:

$$\mathrm{m}\left(x+(1+x)^2+(1+x+y)^2z\right)=\frac{28}{5\pi^2}\zeta(3)+\log\left(\frac{1+\sqrt{5}}{2}\right). \tag{5.1.9}$$

We will show that all of the integrals $\mathrm{TS}(v,w)$, $\mathrm{T}(v,w)$, and $\mathrm{S}(v,w)$ have closed form expressions in terms of polylogarithms. The special case of $\mathrm{TS}(v,1)$ will warrant extra attention, as it is related to an interesting family of binomial sums. Our closed forms are all derived through elementary methods.

5.2 Preliminaries: A description of the method, and some two dimensional Mahler measures

Although there are many conjectured formulas for multi-variable Mahler measures, most are extremely difficult, if not impossible, to prove. Rather than attempting to prove any of these conjectures, we will take an easier approach. By investigating promising functions, and rewriting them as Mahler measures, we can recover a number of useful formulas.

Our first step was to determine a class of functions that we could relate to Mahler's measure. We chose the three integrals $\mathrm{TS}(v,w)$, $\mathrm{S}(v,w)$, and $\mathrm{T}(v,w)$, based on Condon's evaluation of $\mathrm{TS}(2,1)$, Eq. (5.1.6). Condon's formula naturally suggested the existence of a generalized Mahler measure formula involving $\mathrm{TS}(v,1)$. From there, it was a small step to consider the similar functions $\mathrm{TS}(v,w)$, $\mathrm{T}(v,w)$, and $\mathrm{S}(v,w)$.

We will use the following method to express $\mathrm{TS}(v,1)$, $\mathrm{S}(v,1)$, and $\mathrm{T}(v,1/v)$ as three-variable Mahler measures. First, a simple integration by parts changes each function into a two-dimensional integral, containing either a nested arcsine or arctangent integral. Recall that the following integrals define the arctangent and arcsine integrals respectively:

$$\int_0^w \frac{\tan^{-1}(u)}{u}\mathrm{d}u, \qquad \int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u.$$

A typical formula for $\mathrm{TS}(v,1)$, Eq. (5.3.8), can be proved with little trouble:

$$\mathrm{TS}(v,1) = \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \int_0^{\pi/2}\int_0^{v\sin(\theta)} \frac{\tan^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta.$$

Next, substituting a two-dimensional Mahler measure for the nested arctangent or arcsine integral will allow us to obtain a three-dimensional Mahler measure evaluation. Theorem 5.3.2, Proposition 5.5.2, and Theorem 5.7.3 contain our main results from using this method.

Expressing the arcsine and arctangent integrals in terms of Mahler's measure represents the main difficulty in this approach. In the remainder of this section we will establish four two-variable Mahler measures for the arctangent integral, and one two-variable Mahler measure for the arcsine integral.

Since many of our results involve polylogarithms, this will be a good place to define the polylogarithm.

Definition 5.2.1. *If $|z| < 1$, then the polylogarithm of order k is defined by*

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

We call $\mathrm{Li}_2(z)$ the dilogarithm, and we call $\mathrm{Li}_3(z)$ the trilogarithm.

Theorem 5.2.2 requires a formula of Cassaigne and Maillot [111]. In particular, Cassaigne and Mallot showed that

$$\pi\mathrm{m}(a+bx+cy) = \begin{cases} D\left(\frac{|a|}{|b|}e^{i\gamma}\right) + \alpha\log|a| + \beta\log|b| + \gamma\log|c|, & \text{if "}\triangle\text{"} \\ \pi\log\left(\max\{|a|,|b|,|c|\}\right), & \text{otherwise.} \end{cases}$$

The "△" condition states that $|a|$, $|b|$, and $|c|$ form the sides of a triangle. If "△" is true, then α, β, and γ denote the radian measures of the angles opposite to the sides of length $|a|$, $|b|$, and $|c|$ respectively. In this formula, $D(z)$ denotes the Bloch-Wigner dilogarithm. As usual,

$$D(z) = \mathrm{Im}\ (\mathrm{Li}_2(z)) + \log|z|\arg(1-z).$$

Now that we have stated Cassaigne and Maillot's formula, we will prove Theorem 5.2.2.

Theorem 5.2.2. *If* $0 \le v \le 1$ *and* $w \ge 0$*, then*

$$\int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u = \frac{\pi}{2}\mathrm{m}(2v+y+z) \tag{5.2.1}$$

$$\int_0^w \frac{\tan^{-1}(u)}{u}\mathrm{d}u = \frac{\pi}{2}\mathrm{m}\left(1+w^2+(y+w)^2z\right) - \frac{\pi}{4}\log\left(1+w^2\right) \tag{5.2.2}$$

Proof. To prove Eq. (5.2.1) first recall the usual formula for this arcsine integral,

$$\int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u = \frac{1}{2}\mathrm{Im}\left(\mathrm{Li}_2\left(e^{2i\sin^{-1}(v)}\right)\right) + \sin^{-1}(v)\log(2v), \tag{5.2.3}$$

which is valid whenever $0 \le v \le 1$.

Now apply Cassaigne and Maillot's formula to $\mathrm{m}(2v+y+z)$; we are in the "△" case since $0 \le v \le 1$. It follows from a little trigonometry that

$$\pi\mathrm{m}(2v+y+z) = D\left(e^{2i\sin^{-1}(v)}\right) + 2\sin^{-1}(v)\log(2v),$$

Since $\left|e^{2i\sin^{-1}(v)}\right| = 1$, $D\left(e^{2i\sin^{-1}(v)}\right) = \mathrm{Im}\left(\mathrm{Li}_2\left(e^{2i\sin^{-1}(v)}\right)\right)$, hence we obtain

$$\pi\mathrm{m}(2v+y+z) = \mathrm{Im}\left(\mathrm{Li}_2\left(e^{2i\sin^{-1}(v)}\right)\right) + 2\sin^{-1}(v)\log(2v).$$

Comparing this last formula to Eq. (5.2.3), we have

$$\frac{\pi}{2}\mathrm{m}(2v+y+z) = \int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u.$$

To prove Eq. (5.2.2) first recall that if $0 \leq w \leq 1$, then

$$\int_0^w \frac{\tan^{-1}(u)}{u} \mathrm{d}u = \mathrm{Im}\left(\mathrm{Li}_2(iw)\right).$$

Next observe that by Cassaigne and Maillot's formula

$$\begin{aligned}\pi \mathrm{m}\left(\sqrt{1+w^2}+wy+z\right) &= D\left(e^{\pi i/2}w\right)+\tan^{-1}(w)\log(w)+\frac{\pi}{2}\log\left(\sqrt{1+w^2}\right)\\ &= \mathrm{Im}\left(\mathrm{Li}_2(iw)\right)+\frac{\pi}{4}\log\left(1+w^2\right).\end{aligned}$$

After a change of variables in the Mahler measure, it is clear that

$$\begin{aligned}\mathrm{m}\left(\sqrt{1+w^2}+wy+z\right) &= \frac{1}{2}\left\{\mathrm{m}\left(\sqrt{1+w^2}+(1+wy)iz\right)+\mathrm{m}\left(\sqrt{1+w^2}-(1+wy)iz\right)\right\}\\ &= \frac{1}{2}\mathrm{m}\left(1+w^2+(1+wy)^2z^2\right)\\ &= \frac{1}{2}\mathrm{m}\left(1+w^2+(y+w)^2z\right).\end{aligned}$$

It follows that for $0 \leq w \leq 1$ we have

$$\int_0^w \frac{\tan^{-1}(u)}{u} \mathrm{d}u = \frac{\pi}{2}\mathrm{m}\left(1+w^2+(y+w)^2z\right)-\frac{\pi}{4}\log\left(1+w^2\right).$$

We can extend this formula to the entire positive real line. Suppose that $w = 1/w'$ where $w' \geq 1$, then

$$\begin{aligned}\int_0^{1/w'} \frac{\tan^{-1}(u)}{u} \mathrm{d}u &= \frac{\pi}{2}\mathrm{m}\left(1+\frac{1}{w'^2}+\left(y+\frac{1}{w'}\right)^2 z\right)-\frac{\pi}{4}\log\left(1+\frac{1}{w'^2}\right)\\ &= \frac{\pi}{2}\mathrm{m}\left(1+w'^2+\left(y+w'\right)^2 z\right)-\frac{\pi}{4}\log\left(1+w'^2\right)-\frac{\pi}{2}\log(w').\end{aligned}$$

Since the arctangent integral obeys the functional equation [112]

$$\int_0^{w'} \frac{\tan^{-1}(u)}{u} \mathrm{d}u = \frac{\pi}{2}\log(w')+\int_0^{1/w'} \frac{\tan^{-1}(u)}{u} \mathrm{d}u, \tag{5.2.4}$$

it follows that

$$\int_0^{w'} \frac{\tan^{-1}(u)}{u} \mathrm{d}u = \frac{\pi}{2}\mathrm{m}\left(1+w'^2+\left(y+w'\right)^2 z\right)-\frac{\pi}{4}\log\left(1+w'^2\right).$$

Therefore Eq. (5.2.2) holds for all $w \geq 0$. $\square$

The next theorem proves that Eq. (5.2.2) is not unique. Using results from Theorem 5.6.5, we can derive three more Mahler measures for the arctangent integral.

Theorem 5.2.3. *Suppose that $w \geq 0$, then*

$$\int_0^w \frac{\tan^{-1}(u)}{u}\mathrm{d}u = \frac{\pi}{4}\mathrm{m}\left((1+w^2)(1+y)+w(1-y)(z+z^{-1})\right), \tag{5.2.5}$$

$$\int_0^w \frac{\tan^{-1}(u)}{u}\mathrm{d}u = \frac{\pi}{2}\mathrm{m}\left((y-y^{-1})+w(z+z^{-1})\right), \tag{5.2.6}$$

$$\begin{aligned}\int_0^w \frac{\tan^{-1}(u)}{u}\mathrm{d}u =& \frac{\pi}{4}\mathrm{m}\begin{pmatrix}\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)(1+w^2)^2\\ +\left(z-z^{-1}\right)^2(1+y)^2(1-w^2)^2\end{pmatrix} \\ &-\frac{\pi}{4}\log(2)-\frac{\pi}{2}\log(1+w).\end{aligned} \tag{5.2.7}$$

Proof. Since all three of these formulas have similar proofs, we will only prove Eq. (5.2.5) and Eq. (5.2.7). It is necessary to remark, that while Eq. (5.2.5) follows from Eq. (5.6.25), and Eq. (5.2.7) follows from Eq. (5.6.19), we must start from Eq. (5.6.16) to prove Eq. (5.2.6).

Now we will proceed with the proof of Eq. (5.2.5). From Eq. (5.6.25) we have

$$\frac{\pi}{4k}\log\left(\frac{1+k}{1-k}\right)-\frac{2}{k}\mathrm{Im}\left(\mathrm{Li}_2(ir)\right)=\int_0^1\frac{\sin^{-1}(u)}{1-k^2u^2}\mathrm{d}u,$$

where $k=\frac{2r}{1+r^2}$, and $0<k<1$. After an integration by parts this becomes

$$\begin{aligned}\frac{\pi}{4k}\log\left(\frac{1+k}{1-k}\right)-&\frac{2}{k}\mathrm{Im}\left(\mathrm{Li}_2(ir)\right)\\ &=\frac{\pi}{4k}\log\left(\frac{1+k}{1-k}\right)-\frac{1}{2k}\int_0^1\log\left(\frac{1+ku}{1-ku}\right)\frac{\mathrm{d}u}{\sqrt{1-u^2}}.\end{aligned}$$

It follows immediately that

$$\begin{aligned}\mathrm{Im}\left(\mathrm{Li}_2(ir)\right)&=\frac{1}{4}\int_0^{\pi/2}\log\left(\frac{1+k\sin(t)}{1-k\sin(t)}\right)\mathrm{d}t\\ &=\frac{1}{8}\int_0^{2\pi}\log^+\left|\frac{1+k\sin(t)}{1-k\sin(t)}\right|\mathrm{d}t.\end{aligned}$$

Changing the "$\log^+|\cdot|$" term into a Mahler measure, which we can do by Jensen's formula, yields

$$\mathrm{Im}\left(\mathrm{Li}_2(ir)\right) = \frac{\pi}{4}\mathrm{m}\left(y + \frac{1+k\frac{z+z^{-1}}{2}}{1-k\frac{z+z^{-1}}{2}}\right).$$

Since $k = \frac{2}{r+r^{-1}}$, we have

$$\begin{aligned}
\mathrm{Im}\left(\mathrm{Li}_2(ir)\right) =& \frac{\pi}{4}\mathrm{m}\left(y + \frac{r+r^{-1}+(z+z^{-1})}{r+r^{-1}-(z+z^{-1})}\right) \\
=& \frac{\pi}{4}\mathrm{m}\left((1+y)(r+r^{-1})+(1-y)(z+z^{-1})\right) \\
&- \frac{\pi}{4}\mathrm{m}\left(r+r^{-1}-(z+z^{-1})\right) \\
=& \frac{\pi}{4}\mathrm{m}\left((1+y)(r+r^{-1})+(1-y)(z+z^{-1})\right) \\
&- \frac{\pi}{4}\left(\log^+(r)+\log^+\left(\frac{1}{r}\right)\right)
\end{aligned}$$

In order to substitute the arctangent integral for $\mathrm{Im}\left(\mathrm{Li}_2(ir)\right)$, we will assume that $0 < r < 1$. With this restriction, the formula becomes

$$\begin{aligned}
\int_0^r \frac{\tan^{-1}(u)}{u}\mathrm{d}u =& \frac{\pi}{4}\mathrm{m}\left((1+y)(r+r^{-1})+(1-y)(z+z^{-1})\right) \\
&- \frac{\pi}{4}\log\left(\frac{1}{r}\right) \\
=& \frac{\pi}{4}\mathrm{m}\left((1+y)(1+r^2)+r(1-y)(z+z^{-1})\right) \qquad (5.2.8)
\end{aligned}$$

We can manually verify that Eq. (5.2.8) holds when $r = 0$ and $r = 1$, and using Eq. (5.2.4) we can extend Eq. (5.2.8) to all $r > 1$. Therefore, Eq. (5.2.5) follows immediately.

Next we will prove Eq. (5.2.7). Using Eq. (5.6.16), we can show that

$$2\mathrm{Im}\left(\mathrm{Li}_2(ip)\right) = \frac{\pi}{2}\log(p) + \int_0^1 \frac{\sin^{-1}(u)}{u\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u,$$

where $k = \frac{1-p^2}{1+p^2}$, and $0 < k < 1$. To satisfy this restriction on k, we will assume that $0 < p < 1$. After several elementary simplifications, the right-hand side becomes

$$= \frac{\pi}{2}\log(p) + \frac{\pi}{2}\log\left(1+\frac{1}{\sqrt{1-k^2}}\right) + \int_0^1 \log\left(1+\sqrt{\frac{1-u^2}{1-k^2u^2}}\right)\frac{\mathrm{d}u}{\sqrt{(1-u^2)}}$$

$$=\frac{\pi}{2}\log(p)+\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2p}\right)+\int_0^{\pi/2}\log\left(1+\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right)\mathrm{d}\theta$$
$$=\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{1}{2}\int_0^{2\pi}\log^+\left|1+\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right|\mathrm{d}\theta$$

Since $\cos(\pi-\theta)=-\cos(\theta)$, we have

$$\begin{aligned}2\mathrm{Im}\,(\mathrm{Li}_2(ip))=&\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{1}{4}\int_0^{2\pi}\log^+\left|1+\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right|\mathrm{d}\theta\\&+\frac{1}{4}\int_0^{2\pi}\log^+\left|1-\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right|\mathrm{d}\theta.\end{aligned}$$

Applying Jensen's formula yields

$$\begin{aligned}2\mathrm{Im}\,(\mathrm{Li}_2(ip))=&\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{1}{4}\int_0^{2\pi}\mathrm{m}\left((1+y)^2-\frac{\cos^2(\theta)}{1-k^2\sin^2(\theta)}\right)\mathrm{d}\theta\\=&\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{\pi}{2}\mathrm{m}\left((1+y)^2-\frac{\left(z+z^{-1}\right)^2}{4+k^2\left(z-z^{-1}\right)^2}\right)\\=&\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)-\frac{\pi}{2}\mathrm{m}\left(4+k^2\left(z-z^{-1}\right)^2\right)\\&+\frac{\pi}{2}\mathrm{m}\left(\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)+k^2(1+y)^2\left(z-z^{-1}\right)^2\right)\end{aligned}$$

We can simplify the one-dimensional Mahler measure as follows:

$$\begin{aligned}\mathrm{m}\left(4+k^2\left(z-z^{-1}\right)^2\right)&=2\mathrm{m}\left(2+ik\left(z-z^{-1}\right)\right)\\&=2\log\left(1+\sqrt{1-k^2}\right)\\&=2\log\left(\frac{(1+p)^2}{1+p^2}\right).\end{aligned}$$

Eliminating k yields

$$
\begin{aligned}
2\mathrm{Im}\left(\mathrm{Li}_2(ip)\right) =& \frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right) - \pi\log\left(\frac{(1+p)^2}{1+p^2}\right) \\
&+ \frac{\pi}{2}\mathrm{m}\left(\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)+\left(\frac{1-p^2}{1+p^2}\right)^2(1+y)^2\left(z-z^{-1}\right)^2\right) \\
=& -\frac{\pi}{2}\log(2) - \pi\log(1+p) \\
&+ \frac{\pi}{2}\mathrm{m}\left(\begin{matrix}\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)(1+p^2)^2 \\ +(1+y)^2\left(z-z^{-1}\right)^2(1-p^2)^2\end{matrix}\right).
\end{aligned}
$$

Since $0<p<1$, it follows that

$$
\begin{aligned}
2\int_0^p \frac{\tan^{-1}(u)}{u}\mathrm{d}u =& \frac{\pi}{2}\mathrm{m}\left(\begin{matrix}\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)(1+p^2)^2 \\ +(1+y)^2\left(z-z^{-1}\right)^2(1-p^2)^2\end{matrix}\right) \\
&- \frac{\pi}{2}\log(2) - \pi\log(1+p).
\end{aligned} \tag{5.2.9}
$$

It is relatively easy to verify that Eq. (5.2.9) holds when $p=0$ and $p=1$. Using Eq. (5.2.4), we can also extend Eq. (5.2.9) to $p>1$, which completes the proof of Eq. (5.2.7).

□

5.3 Relations between $\mathrm{TS}(v,1)$ and Mahler's measure, and a reduction of $\mathrm{TS}(v,w)$ to multiple polylogarithms

The first goal of this section is to establish five identities relating $\mathrm{TS}(v,1)$ to three-variable Mahler measures. We will prove these formulas in Theorem 5.3.2, using the methods outlined in Section 5.2. Corollary 5.3.3 examines a few special cases of these results.

Theorem 5.3.5 accomplishes the second goal of this section, which is to express $\mathrm{TS}(v,w)$ in terms of multiple polylogarithms. This result, which appears to be new, is stated in Eq. (5.3.14). The importance of Eq. (5.3.14) lies in its easy proof, and more importantly in the fact that it immediately reduces $\mathrm{TS}(v,1)$ to multiple polylogarithms. Finally, Proposition 5.3.6 will demonstrate that the multiple polylogarithms in Eq. (5.3.14) always reduce to standard polylogarithms.

We will need the following simple lemma to prove Theorem 5.3.1.

Lemma 5.3.1. *Assume that v and w are real numbers with $v > 0$ and $w \in (0, 1]$, then*

$$\mathrm{TS}(v,w) = \tan^{-1}(v)\int_0^w \frac{\sin^{-1}(z)}{z}\mathrm{d}z - \int_0^{\tan^{-1}(v)}\int_0^{\frac{w}{v}\tan(\theta)} \frac{\sin^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta, \tag{5.3.1}$$

$$\mathrm{TS}(v,w) = \sin^{-1}(w)\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \int_0^{\sin^{-1}(w)}\int_0^{\frac{v}{w}\sin(\theta)} \frac{\tan^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta. \tag{5.3.2}$$

Proof. To prove Eq. (5.3.1) first integrate $\mathrm{TS}(v,w)$ by parts to obtain:

$$\begin{aligned}\mathrm{TS}(v,w) = &\tan^{-1}(v)\int_0^w \frac{\sin^{-1}(z)}{z}\mathrm{d}z \\ &- \int_0^1 \frac{\mathrm{d}}{\mathrm{d}u}\left(\tan^{-1}(vu)\right)\int_0^{wu}\frac{\sin^{-1}(z)}{z}\mathrm{d}z\mathrm{d}u.\end{aligned}$$

Making the u-substitution $\theta = \tan^{-1}(vu)$ we have:

$$\mathrm{TS}(v,w) = \tan^{-1}(v)\int_0^w \frac{\sin^{-1}(z)}{z}\mathrm{d}z - \int_0^{\tan^{-1}(v)}\int_0^{\frac{w}{v}\tan(\theta)} \frac{\sin^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta,$$

which completes the proof of the identity.

The proof of Eq. (5.3.2) follows in a similar manner. □

The fact that Lemma 5.3.1 expresses $\mathrm{TS}(v,w)$ as a double integral in two different ways, makes $\mathrm{TS}(v,w)$ more versatile than either $\mathrm{S}(v,w)$ or $\mathrm{T}(v,w)$. These two different expansions will allow us to combine $\mathrm{TS}(v,w)$ with Mahler measures for both arctangent and arcsine integrals.

Theorem 5.3.2. *The following Mahler measures hold whenever $v \geq 0$:*

$$\mathrm{m}\left(1 + x + \frac{v}{2}(1-x)(y+z)\right) = \frac{2}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{4}{\pi^2}\mathrm{TS}(v,1) \tag{5.3.3}$$

$$\begin{aligned}&\mathrm{m}\left(x + \frac{v^2}{4}(1+x)^2 + \left(y + \frac{v}{2}(1+x)\right)^2 z\right) \\ &\quad = \frac{2}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{4}{\pi^2}\mathrm{TS}(v,1) + \frac{1}{2}\mathrm{m}\left(x + \frac{v^2}{4}(1+x)^2\right)\end{aligned} \tag{5.3.4}$$

$$\begin{aligned}&\mathrm{m}\left((1+y)\left(1+\frac{v^2}{4}(x+x^{-1})^2\right)+\frac{v}{2}(1-y)\left(x+x^{-1}\right)\left(z+z^{-1}\right)\right)\\&\qquad=\frac{4}{\pi}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{8}{\pi^2}\mathrm{TS}(v,1)\end{aligned}\tag{5.3.5}$$

$$\begin{aligned}&\mathrm{m}\left((z-z^{-1})+\frac{v}{2}(x+x^{-1})(y+y^{-1})\right)\\&\qquad=\frac{2}{\pi}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{4}{\pi^2}\mathrm{TS}(v,1)\end{aligned}\tag{5.3.6}$$

$$\begin{aligned}&\mathrm{m}\left(\begin{array}{l}\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)\left(1+\frac{v^2}{4}\left(x+x^{-1}\right)^2\right)^2\\+\left(z-z^{-1}\right)^2(1+y)^2\left(1-\frac{v^2}{4}\left(x+x^{-1}\right)^2\right)^2\end{array}\right)\\&\qquad=\frac{4}{\pi}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{8}{\pi^2}\mathrm{TS}(v,1)+\frac{4}{\pi}\int_0^{\pi/2}\log\left(1+v\sin(\theta)\right)\mathrm{d}\theta\\&\qquad\quad+\log(2)\end{aligned}\tag{5.3.7}$$

Proof. We will prove Eq. (5.3.3) first, since it has the most difficult proof. Letting $w=1$ in Eq. (5.3.1) yields

$$\mathrm{TS}(v,1)=\frac{\pi}{2}\log(2)\tan^{-1}(v)-\int_0^{\tan^{-1}(v)}\int_0^{\tan(\theta)/v}\frac{\sin^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta.$$

Since $0\le\frac{\tan(\theta)}{v}\le 1$, we may substitute Eq. (5.2.1) for the nested arcsine integral to obtain

$$\begin{aligned}\mathrm{TS}(v,1)=&\frac{\pi}{2}\log(2)\tan^{-1}(v)-\frac{\pi}{2}\int_0^{\tan^{-1}(v)}\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)\mathrm{d}\theta\\=&\frac{\pi}{2}\log(2)\tan^{-1}(v)-\frac{\pi}{2}\int_0^{\pi/2}\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)\mathrm{d}\theta\\&+\frac{\pi}{2}\int_{\tan^{-1}(v)}^{\pi/2}\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)\mathrm{d}\theta.\end{aligned}$$

In the right-hand integral $\frac{\tan(\theta)}{v}\ge 1$, hence by Cassaigne and Maillot's formula

$$\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)=\log\left(\frac{2}{v}\tan(\theta)\right).$$

Substituting this result yields:

$$\begin{aligned}\mathrm{TS}(v,1) =&\frac{\pi}{2}\log(2)\tan^{-1}(v)+\frac{\pi}{2}\int_{\tan^{-1}(v)}^{\pi/2}\log\left(\frac{2}{v}\tan(\theta)\right)\mathrm{d}\theta\\ &-\frac{\pi}{2}\int_0^{\pi/2}\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)\mathrm{d}\theta\\ =&\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{\pi}{2}\int_0^{\pi/2}\mathrm{m}\left(\tan(\theta)+\frac{v}{2}(y+z)\right)\mathrm{d}\theta\\ =&\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{\pi^2}{4}\mathrm{m}\left(1+x+\frac{v}{2}(1-x)(y+z)\right).\end{aligned}$$

Eq. (5.3.3) follows immediately from rearranging this final identity.

The proofs of equations (5.3.4) through (5.3.7) are virtually identical, hence we will only prove Eq. (5.3.5). Letting $w=1$ in Eq. (5.3.2), we have

$$\mathrm{TS}(v,1)=\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\int_0^{\pi/2}\int_0^{v\sin(\theta)}\frac{\tan^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta. \tag{5.3.8}$$

Substituting Eq. (5.2.5) for the nested arctangent integral yields

$$\begin{aligned}\mathrm{TS}(v,1) =&\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u\\ &-\frac{\pi}{4}\int_0^{\pi/2}\mathrm{m}\left((1+y)\left(1+v^2\sin^2(\theta)\right)+v\sin(\theta)(1-y)(z+z^{-1})\right)\mathrm{d}\theta\\ =&\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u\\ &-\frac{\pi^2}{8}\mathrm{m}\left((1+y)\left(1-\frac{v^2}{4}(x-x^{-1})^2\right)+\frac{v}{2i}(1-y)(x-x^{-1})(z+z^{-1})\right).\end{aligned}$$

Letting $x\to ix$, we obtain

$$\begin{aligned}\mathrm{TS}(v,1) =&\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u\\ &-\frac{\pi^2}{8}\mathrm{m}\left((1+y)\left(1+\frac{v^2}{4}(x+x^{-1})^2\right)+\frac{v}{2}(1-y)(x+x^{-1})(z+z^{-1})\right).\end{aligned}$$

Eq. (5.3.5) follows immediately from rearranging this final equality.

Finally, we will remark that the while Eq. (5.3.5) follows from substituting Eq. (5.2.5) into Eq. (5.3.8), we must substitute Eq. (5.2.2) to prove Eq. (5.3.4), Eq. (5.3.6) follows from substituting Eq. (5.2.6), and Eq. (5.3.7) follows from substituting Eq. (5.2.7). □

Corollary 5.3.3. *The formulas in Theorem 5.3.2 reduce, in order, to the following identities when $v = 2$:*

$$\mathrm{m}\left((1+x)+(1-x)(y+z)\right) = \frac{28}{5\pi^2}\zeta(3), \tag{5.3.9}$$

$$\mathrm{m}\left(x+(1+x)^2+(1+x+y)^2z\right) = \frac{28}{5\pi^2}\zeta(3)+\log\left(\frac{1+\sqrt{5}}{2}\right), \tag{5.3.10}$$

$$\begin{aligned}&\mathrm{m}\left((1+x+z)\left(1+x^{-1}+z^{-1}\right)+y(1+x-z)\left(1+x^{-1}-z^{-1}\right)\right)\\ &\qquad = \frac{56}{5\pi^2}\zeta(3),\end{aligned} \tag{5.3.11}$$

$$\mathrm{m}\left(\left(z-z^{-1}\right)+\left(x+x^{-1}\right)\left(y+y^{-1}\right)\right) = \frac{28}{5\pi^2}\zeta(3), \tag{5.3.12}$$

$$\begin{aligned}\mathrm{m}\left(\begin{matrix}\left(4z(1+y)^2-(1+z)^2\right)\left(1+3x+x^2\right)^2\\ +(1-z)^2(1+y)^2\left(1+x+x^2\right)^2\end{matrix}\right) =& \frac{56}{5\pi^2}\zeta(3)+\frac{16}{3\pi}G\\ &+\log(2).\end{aligned} \tag{5.3.13}$$

In Eq. (5.3.13)*, and throughout the rest of the paper, G denotes Catalan's constant. In particular, $G = 1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots$*

Proof. As we have already stated, Condon proved Eq. (5.3.9) in [105]. His proof also showed that

$$\mathrm{TS}(2,1) = \frac{\pi}{2}\int_0^2 \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{7}{5}\zeta(3).$$

Using this formula, equations (5.3.10) through (5.3.13) follow immediately from Theorem 5.3.2. □

Theorem 5.3.2 shows that we can obtain closed forms for several three-variable Mahler measures by reducing $\mathrm{TS}(v,1)$ to polylogarithms. We have proved a convenient closed form for $\mathrm{TS}(v,1)$ in Eq. (5.4.18). Corollary 5.4.6 also shows that this closed form immediately implies Condon's evaluation of $\mathrm{TS}(2,1)$. We will postpone further discussion of Eq. (5.4.18) until Section 5.4.

We will devote the remainder of this section to deriving a closed form for $\mathrm{TS}(v,w)$ in terms of multiple polylogarithms. For convenience, we will use a slightly non-standard notation for our multiple polylogarithms.

Definition 5.3.4. *Define* $\mathrm{F}_j(x)$ *by*

$$\mathrm{F}_j(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)^j} = \frac{\mathrm{Li}_j(x) - \mathrm{Li}_j(-x)}{2},$$

and define $\mathrm{F}_{j,k}(x,y)$ *by*

$$\mathrm{F}_{j,k}(x,y) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)^j} \sum_{m=0}^{n} \frac{y^{2m+1}}{(2m+1)^k}.$$

We will employ this notation throughout the rest of the paper.

Theorem 5.3.5. *If* $\frac{v}{w} \notin (-i\infty, -i] \cup [i, i\infty)$ *and* $w \in [-1,1]$, *then we can express* $\mathrm{TS}(v,w)$ *in terms of multiple polylogarithms. Let* $R = \frac{\frac{v}{w}}{1+\sqrt{1+\left(\frac{v}{w}\right)^2}}$, *and let* $S = iw + \sqrt{1-w^2}$, *then*

$$\begin{aligned}\mathrm{TS}(v,w) =& 2\mathrm{F}_3(R) - \mathrm{F}_3(RS) - \mathrm{F}_3(R/S) - 4\mathrm{F}_{1,2}(R,1) \\ &+ 2\mathrm{F}_{1,2}(R,S) + 2\mathrm{F}_{1,2}(R,1/S) \\ &+ i\sin^{-1}(w)\left\{\mathrm{F}_2(RS) - \mathrm{F}_2(R/S)\right. \\ &\qquad \left. -2\mathrm{F}_{1,1}(R,S) + 2\mathrm{F}_{1,1}(R,1/S)\right\}.\end{aligned} \tag{5.3.14}$$

Proof. First note that by u-substitution

$$\mathrm{TS}(v,w) = \int_0^{\sin^{-1}(w)} \tan^{-1}\left(\frac{v}{w}\sin(\theta)\right)\cot(\theta)\theta \mathrm{d}\theta. \tag{5.3.15}$$

Since $w \in [-1,1]$, it follows that our path of integration is along the real axis. Next substitute the Fourier series

$$\tan^{-1}\left(\frac{v}{w}\sin(\theta)\right) = 2\sum_{n=0}^{\infty} \frac{R^{2n+1}}{2n+1}\sin\left((2n+1)\theta\right), \tag{5.3.16}$$

into Eq. (5.3.15). Swapping the order of summation and integration, we have

$$\mathrm{TS}(v,w) = 2\sum_{n=0}^{\infty} \frac{R^{2n+1}}{2n+1} \int_0^{\sin^{-1}(w)} \sin\left((2n+1)\theta\right)\cot(\theta)\theta \mathrm{d}\theta.$$

Uniform convergence justifies this interchange of summation and integration. In particular, Eq. (5.3.16) converges uniformly whenever $|R| < 1$ and $\theta \in \mathbb{R}$. It is easy to show that $|R| < 1$ except when $\frac{v}{w} \in (-i\infty, -i] \cup [i, i\infty)$, in which case $|R| = 1$. If $|R| = 1$, then Eq. (5.3.16) no longer converges uniformly, and hence the following arguments do not apply.

Evaluating the nested integral yields

$$\begin{aligned}\mathrm{TS}(v,w) = 4\sum_{n=0}^{\infty}\frac{R^{2n+1}}{2n+1}\Bigg\{&\sin^{-1}(w){\sum_{k=0}^{n}}'\frac{\sin\left((2k+1)\sin^{-1}(w)\right)}{2k+1}\\ &-{\sum_{k=0}^{n}}'\frac{1-\cos\left((2k+1)\sin^{-1}(w)\right)}{(2k+1)^2}\Bigg\},\end{aligned} \tag{5.3.17}$$

where ${\sum\limits_{k=0}^{n}}' a_k = a_0 + \cdots + a_{n-1} + \frac{a_n}{2}$. Simplifying Eq. (5.3.17) completes our proof. □

Eq. (5.3.14) deserves a few remarks, since it is a fairly general result. Firstly, observe that a closer analysis of Eq. (5.3.16) would probably allow us to relax the restriction that $w \in [-1,1]$. Secondly, Eq. (5.3.14) most likely has applications beyond the scope of this paper. For example, we can use Eq. (5.3.14) to reduce the right-hand side of the following equation

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\binom{2n}{n}\left(\frac{w}{2}\right)^{2n+1}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\\ &\quad= \mathrm{TS}(1,w) - \frac{\pi}{4}\int_0^w \frac{\sin^{-1}(t)}{t}\mathrm{d}t + \frac{\log(2)}{2}\int_0^w\frac{\sinh^{-1}(t)}{t}\mathrm{d}t,\end{aligned} \tag{5.3.18}$$

to multiple polylogarithms.

We can use the final result of this section, Proposition 5.3.6, to reduce $\mathrm{TS}(v,w)$ to regular polylogarithms. This proposition allows us to equate $\mathrm{TS}(v,w)$ with a formula involving around twenty trilogarithms. While a clever usage of trilogarithmic functional equations might simplify this result, it seems more convenient to simply leave Eq. (5.3.14) in its current form.

Proposition 5.3.6. *The functions* $\mathrm{F}_{1,1}(x,y)$ *and* $\mathrm{F}_{1,2}(x,y)$ *can be expressed in terms of polylogarithms, we have:*

$$\begin{aligned}4\mathrm{F}_{1,1}(x,y) =&\mathrm{Li}_2\left(\frac{x(1+y)}{1+x}\right) - \mathrm{Li}_2\left(\frac{x(1-y)}{1+x}\right)\\ &- \mathrm{Li}_2\left(\frac{-x(1+y)}{1-x}\right) + \mathrm{Li}_2\left(\frac{-x(1-y)}{1-x}\right).\end{aligned} \tag{5.3.19}$$

To reduce $\mathrm{F}_{1,2}(x,y)$ *to polylogarithms, apply Lewin's formula, Eq.* (5.7.5)*, four times to the following identity:*

$$\begin{aligned}\mathrm{F}_{1,2}(x,y) =&\mathrm{F}_3(xy) - \frac{1}{2}\log\left(1-x^2\right)\mathrm{F}_2(xy)\\ &+\frac{1}{4}\int_0^x \frac{\log\left(1-u^2\right)\log\left(\frac{1+yu}{1-yu}\right)}{u}\mathrm{d}u.\end{aligned} \tag{5.3.20}$$

Proof. To prove Eq. (5.3.20), first swap the order of summation to obtain

$$\mathrm{F}_{1,2}(x,y) = \mathrm{F}_3(xy) + \mathrm{F}_1(x)\mathrm{F}_2(y) - \sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)^2}\sum_{k=0}^{n}\frac{x^{2k+1}}{2k+1}.$$

Substituting an integral for the nested sum yields

$$\begin{aligned}\mathrm{F}_{1,2}(x,y) &= \mathrm{F}_3(xy) + \mathrm{F}_1(x)\mathrm{F}_2(y) - \sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)^2}\int_0^x\frac{1-u^{2n+2}}{1-u^2}\mathrm{d}u\\ &= \mathrm{F}_3(xy) + \int_0^x\frac{u}{1-u^2}\mathrm{F}_2(yu)\mathrm{d}u.\end{aligned}$$

Integrating by parts, the identity becomes

$$\begin{aligned}\mathrm{F}_{1,2}(x,y) =&\mathrm{F}_3(xy) - \frac{1}{2}\log(1-x^2)\mathrm{F}_2(xy)\\ &+\frac{1}{4}\int_0^x \frac{\log\left(1-u^2\right)\log\left(\frac{1+yu}{1-yu}\right)}{u}\mathrm{d}u,\end{aligned}$$

which completes the proof of Eq. (5.3.20).

We can verify Eq. (5.3.19) by differentiating each side of the equation with respect to y.

□

Finally, observe that we can obtain simple closed forms for $\mathrm{F}_{1,2}(x,1)$ and $\mathrm{F}_{2,1}(1,x)$ from Eq. (5.4.9).

5.4 An evaluation of $\mathrm{TS}(v,1)$ using infinite series

This evaluation of $\mathrm{TS}(v,1)$ generalizes a theorem due to Condon. Condon proved a formula that Boyd and Rodriguez Villegas conjectured:

$$m\big(1+x+(1-x)(y+z)\big) = \frac{28}{5\pi^2}\zeta(3).$$

Condon's result is equivalent to evaluating $\mathrm{TS}(2,1)$ in closed form. As Theorem 5.3.2 has shown, generalizing this Mahler measure depends on finding a closed form for $\mathrm{TS}(v,1)$. Eq. (5.4.18) accomplishes this goal by expressing $\mathrm{TS}(v,1)$ in terms of polylogarithms.

This calculation of $\mathrm{TS}(v,1)$ is based on several series transformations. The first step is to expand $\mathrm{TS}(v,1)$ in a Taylor series; observe that the following formula holds whenever $|v|<1$:

$$\mathrm{TS}(v,1)=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}v^{2k+1}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\frac{(2v)^{2k+1}}{\binom{2k}{k}}. \tag{5.4.1}$$

We can easily prove Eq. (5.4.1) by starting from Eq. (5.3.8). Formula (5.4.1) shows that $\mathrm{TS}(v,1)$ is analytic in the open unit disk. Unfortunately Eq. (5.4.1) does not converge when $v=2$, and hence it can not be used to calculate $\mathrm{TS}(2,1)$. It will be necessary to find an analytic continuation of $\mathrm{TS}(v,1)$ in order to carry out any useful computations.

The following family of functions will play a crucial role in our calculations.

Definition 5.4.1. *Define $h_n(v)$ by the infinite series,*

$$h_n(v)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}\frac{(2v)^{2k+1}}{\binom{2k}{k}}. \tag{5.4.2}$$

Using the definition of $h_3(v)$, combined with the identity

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}v^{2k+1}=\int_0^v\frac{\tan^{-1}(u)}{u}du,$$

it follows that Eq. (5.4.1) can be rewritten as

$$\mathrm{TS}(v,1)=\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}du-\frac{1}{2}h_3(v). \tag{5.4.3}$$

Finding a closed form for $\mathrm{TS}(v,1)$ we will entail finding a closed form for $h_3(v)$. Theorem 5.4.5 accomplishes this goal, however several auxiliary lemmas are needed first. The idea behind our proof is very simple: first find a closed form for $h_2(v)$, and then integrate it to find a closed form for $h_3(v)$.

Batir recently used this method in an interesting paper [99] to obtain a formula that is equivalent to Eq. (5.4.15). Unfortunately Batir seems to have missed Eq. (5.4.12), so we will provide a full derivation of this important result.

Lemma 5.4.2. *The function $h_2(v)$ is analytic if $v \notin (-i\infty, -i] \cup [i, i\infty)$. Furthermore, we can express $h_2(v)$ in terms of the dilogarithm,*

$$\begin{aligned} h_2(v) &= 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\left(\frac{v}{1+\sqrt{1+v^2}}\right)^{2k+1} \\ &= 2\mathrm{Li}_2\left(\frac{v}{1+\sqrt{1+v^2}}\right) - 2\mathrm{Li}_2\left(\frac{-v}{1+\sqrt{1+v^2}}\right). \end{aligned} \tag{5.4.4}$$

Proof. We use the following elementary identity to prove Eq. (5.4.4),

$$\frac{2^{4k}}{(2k+1)^2\binom{2k}{k}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \frac{(2k)!}{(k+j+1)!(k-j)!}. \tag{5.4.5}$$

Substituting Eq. (5.4.5) into the definition of $h_2(v)$, we have

$$\begin{aligned} h_2(v) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \frac{(2v)^{2k+1}}{\binom{2k}{k}} \\ &= 4\sum_{k=0}^{\infty} (-1)^k \left(\frac{v}{2}\right)^{2k+1} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \frac{(2k)!}{(k+j+1)!(k-j)!}. \end{aligned}$$

If we assume that $|v| < 1$, then the series converges uniformly, hence we may swap the order of summation to obtain

$$\begin{aligned} h_2(v) &= 4\sum_{j=0}^{\infty} \frac{1}{2j+1} \sum_{k=0}^{\infty} (-1)^{k+2j} \frac{(2k+2j)!}{(k+2j+1)!k!} \left(\frac{v}{2}\right)^{2k+2j+1} \\ &= 4\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \left(\frac{v}{2}\right)^{2j+1} \sum_{k=0}^{\infty} \frac{(j+\frac{1}{2})_k (j+1)_k}{(2j+2)_k} \frac{(-v^2)^k}{k!}, \end{aligned}$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. But then we have

$$h_2(v) = 4\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \left(\frac{v}{2}\right)^{2j+1} {}_2F_1\left[\begin{matrix} j+\frac{1}{2}, j+1 \\ 2j+2 \end{matrix}\middle| -v^2\right],$$

where ${}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix}\middle| x\right]$ is the usual hypergeometric function. A standard hypergeometric identity [106] shows that

$${}_2F_1\left[\begin{matrix} j+\frac{1}{2}, j+1 \\ 2j+2 \end{matrix}\middle| -v^2\right] = \frac{2^{2j+1}}{(1+\sqrt{1+v^2})^{2j+1}},$$

from which we obtain

$$h_2(v) = 4\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2}\left(\frac{v}{1+\sqrt{1+v^2}}\right)^{2j+1},$$

concluding the proof of the identity.

We can use Eq. (5.4.4) to analytically continue $h_2(v)$ to a larger domain. Recall that $\mathrm{Li}_2(r)-\mathrm{Li}_2(-r)$ is analytic whenever $r \notin (-\infty,-1]\cup[1,\infty)$, and $\frac{v}{1+\sqrt{1+v^2}}$ is analytic whenever $v \notin (-i\infty,-i]\cup[i,i\infty)$. Since we have already assumed that $v \notin (-i\infty,-i]\cup[i,i\infty)$, we simply have to show that the range of $r = \frac{v}{1+\sqrt{1+v^2}}$ does not intersect the set $\{(-\infty,-1]\cup[1,\infty)\}$.

Some elementary calculus shows that $|r| = \left|\frac{v}{1+\sqrt{1+v^2}}\right| \leq 1$ for all $v \in \mathbb{C}$, with equality occurring only when $v \in (-i\infty,-i]\cup[i,i\infty)$. It follows that $h_2(v)$ is analytic on $\mathbb{C}-\{(-i\infty,-i]\cup[i,i\infty)\}$. $\square$

Since we have now expressed $h_2(v)$ in terms of dilogarithms, we can find a closed form for $h_1(v)$ by differentiating Eq. (5.4.4):

$$h_1(v) = \frac{2}{\sqrt{1+v^2}}\log\left(v+\sqrt{1+v^2}\right). \tag{5.4.6}$$

In Theorem 5.4.5, we will integrate Eq. (5.4.4) to find a closed form for $h_3(v)$ involving trilogarithms. To prove this theorem, we first need to establish two lemmas. Lemma 5.4.3 evaluates a necessary integral, while Lemma 5.4.4 expresses $\mathrm{F}_{2,1}(1,x)$ in terms of polylogarithms.

Lemma 5.4.3. *If $j \geq 0$ is an integer, and $r = \frac{v}{1+\sqrt{1+v^2}}$, then we have the following identity:*

$$\int_0^v \frac{1}{u}\left(\frac{u}{1+\sqrt{1+u^2}}\right)^{2j+1}\mathrm{d}u = \log\left(\frac{1+r}{1-r}\right)+\frac{r^{2j+1}}{2j+1}-2\sum_{k=0}^{j}\frac{r^{2k+1}}{2k+1}. \tag{5.4.7}$$

Proof. To evaluate the integral

$$w_j(v) = \int_0^v \frac{1}{u}\left(\frac{u}{1+\sqrt{1+u^2}}\right)^{2j+1}\mathrm{d}u,$$

first make the substitution $z = \frac{u}{1+\sqrt{1+u^2}}$. In particular we can show that $u = \frac{2z}{1-z^2}$ and $\frac{\mathrm{d}u}{\mathrm{d}z} = 2\frac{(1+z^2)}{(1-z^2)^2}$. Therefore we have

$$\begin{aligned} w_j(v) &= \int_0^r z^{2j}\left(\frac{1+z^2}{1-z^2}\right)\mathrm{d}z \\ &= \int_0^r \frac{2}{1-z^2}\mathrm{d}z - \int_0^r \frac{1-z^{2j}}{1-z^2}\mathrm{d}z - \int_0^r \frac{1-z^{2j+2}}{1-z^2}\mathrm{d}z. \end{aligned}$$

Next substitute the geometric series $\frac{1-z^{2j}}{1-z^2} = \sum_{k=0}^{j-1} z^{2k}$ into each of the right-hand integrals, and swap the order of summation and integration to obtain

$$\begin{aligned} w_j(v) &= \int_0^r \frac{2}{1-z^2}\mathrm{d}z - \sum_{k=0}^{j-1} \frac{r^{2k+1}}{2k+1} - \sum_{k=0}^{j} \frac{r^{2k+1}}{2k+1} \\ &= \log\left(\frac{1+r}{1-r}\right) + \frac{r^{2j+1}}{2j+1} - 2\sum_{k=0}^{j} \frac{r^{2k+1}}{2k+1}. \end{aligned}$$

□

Lemma 5.4.4. *The following double polylogarithm*

$$\mathrm{F}_{2,1}(1,x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sum_{k=0}^{n} \frac{x^{2k+1}}{2k+1} \tag{5.4.8}$$

can be evaluated in closed form. If $|x| < 1$,

$$\begin{aligned} 8\mathrm{F}_{2,1}(1,x) =& 4\mathrm{Li}_3(x) - \mathrm{Li}_3(x^2) - 4\mathrm{Li}_3(1-x) - 4\mathrm{Li}_3\left(\frac{x}{1+x}\right) + 4\zeta(3) \\ &+ \log\left(\frac{1+x}{1-x}\right)\mathrm{Li}_2(x^2) + \frac{\pi^2}{2}\log(1+x) + \frac{\pi^2}{6}\log(1-x) \\ &+ \frac{2}{3}\log^3(1+x) - 2\log(x)\log^2(1-x) \end{aligned} \tag{5.4.9}$$

Proof. We will verify Eq. (5.4.9) by differentiating each side of the identity. First observe that the infinite series in Eq. (5.4.8) converges uniformly whenever $|x| \le 1$, hence term by term differentiation is justified at all points in the open unit disk. It follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}_{2,1}(1,x) =& \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\left(\frac{1-x^{2n+2}}{1-x^2}\right) \\ =& \frac{\pi^2}{8}\left(\frac{1}{1-x^2}\right) - \frac{x}{1-x^2}\left(\mathrm{Li}_2(x) - \frac{1}{4}\mathrm{Li}_2(x^2)\right), \end{aligned} \tag{5.4.10}$$

whenever $|x| < 1$.

Let $\varphi(x)$ denote the right-hand side of Eq. (5.4.9). Taking the derivative of $\varphi(x)$ we obtain:

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}x} =& \frac{4}{x}\mathrm{Li}_2(x) - \frac{2}{x}\mathrm{Li}_2(x^2) + \frac{4}{1-x}\mathrm{Li}_2(1-x) \\ & - 4\left(\frac{1}{x} - \frac{1}{1+x}\right)\mathrm{Li}_2\left(\frac{x}{1+x}\right) + \frac{2}{1-x^2}\mathrm{Li}_2(x^2) \\ & - \frac{2}{x}\left(\log^2(1+x) - \log^2(1-x)\right) + \frac{\pi^2}{2}\left(\frac{1}{1+x}\right) - \frac{\pi^2}{6}\left(\frac{1}{1-x}\right) \\ & + \frac{2}{1+x}\log^2(1+x) - \frac{2}{x}\log^2(1-x) + \frac{4}{1-x}\log(x)\log(1-x)\end{aligned} \tag{5.4.11}$$

We can simplify Eq. (5.4.11) by eliminating $\mathrm{Li}_2(1-x)$ and $\mathrm{Li}_2\left(\frac{x}{1+x}\right)$ with the functional equations:

$$\begin{aligned}\mathrm{Li}_2(1-x) &= \frac{\pi^2}{6} - \log(x)\log(1-x) - \mathrm{Li}_2(x), \\ \mathrm{Li}_2\left(\frac{x}{1+x}\right) &= -\frac{1}{2}\log^2(1+x) + \mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2(x^2).\end{aligned}$$

Substituting these identities into Eq. (5.4.11) and simplifying, we are left with

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}x} &= \pi^2\left(\frac{1}{1-x^2}\right) - \frac{8x}{1-x^2}\left(\mathrm{Li}_2(x) - \frac{1}{4}\mathrm{Li}_2(x^2)\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{8\mathrm{F}_{2,1}(1,x)\right\}.\end{aligned}$$

Eq. (5.4.10) justifies this final step. Since the derivatives of $8\mathrm{F}_{2,1}(1,x)$ and $\varphi(x)$ are equal on the open unit disk, and since both functions vanish at zero, we may conclude that $8\mathrm{F}_{2,1}(1,x) = \varphi(x)$. $\square$

The proof of Eq. (5.4.9) requires a remark. Despite the fact that the right-hand side of Eq. (5.4.9) is single valued and analytic whenever $|x| < 1$, the individual terms involving $\mathrm{Li}_3(1-x)$ and $\log(x)$ are multivalued for $x \in (-1, 0)$. To avoid all ambiguity, we can simply use $\mathrm{F}_{2,1}(1,x) = -\mathrm{F}_{2,1}(1,-x)$ to calculate the function at negative real arguments .

Theorem 5.4.5. *The function $h_3(v)$ is analytic on $\mathbb{C} - \{(-i\infty, -i] \cup [i, i\infty)\}$. If $v \notin (-i\infty, -i] \cup [i, i\infty)$, then $h_3(v)$ can be expressed in terms of polyloga-*

rithms. Let $r = \frac{v}{1+\sqrt{1+v^2}}$*, then*

$$\begin{aligned} h_3(v) =& \frac{1}{2}\mathrm{Li}_3(r^2) + 4\mathrm{Li}_3(1-r) + 4\mathrm{Li}_3\left(\frac{r}{1+r}\right) - 4\zeta(3) \\ & - \log\left(\frac{1+r}{1-r}\right)\mathrm{Li}_2(r^2) - \frac{2\pi^2}{3}\log(1-r) - \frac{2}{3}\log^3(1+r) \\ & + 2\log(r)\log^2(1-r). \end{aligned} \tag{5.4.12}$$

We can recover an equivalent form of Condon's identity by letting $v = 2$*:*

$$h_3(2) = \frac{14}{5}\zeta(3). \tag{5.4.13}$$

Proof. This proof is very simple since we have already completed all of the hard computations. Observe from Eq. (5.4.2) that if $|v| < 1$,

$$h_3(v) = \int_0^v \frac{h_2(u)}{u}\mathrm{d}u. \tag{5.4.14}$$

Lemma 5.4.2 shows that $h_2(v)$ is analytic provided that $v \notin (-i\infty, -i] \cup [i, i\infty)$. If we assume that the path of integration does not pass through either of these branch cuts, then it is easy to see that Eq. (5.4.14) provides an analytic continuation of $h_3(v)$ to $\mathbb{C} - \{(-i\infty, -i] \cup [i, i\infty)\}$.

Next we will prove Eq. (5.4.12). Substituting Eq. (5.4.4) into Eq. (5.4.14) yields an infinite series for $h_3(v)$ that is valid whenever $v \notin (-i\infty, -i] \cup [i, i\infty)$. We have

$$h_3(v) = 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^v \frac{1}{u}\left(\frac{u}{1+\sqrt{1+u^2}}\right)^{2n+1}\mathrm{d}u.$$

The nested integrals can be evaluated by Lemma 5.4.3. Letting $r = \frac{v}{1+\sqrt{1+v^2}}$ it is clear that

$$\begin{aligned} h_3(v) =& 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(\log\left(\frac{1+r}{1-r}\right) + \frac{r^{2n+1}}{2n+1} - 2\sum_{j=0}^{n}\frac{r^{2j+1}}{2j+1}\right) \\ =& \frac{\pi^2}{2}\log\left(\frac{1+r}{1-r}\right) + 4\mathrm{Li}_3(r) - \frac{1}{2}\mathrm{Li}_3(r^2) - 8\mathrm{F}_{2,1}(1, r), \end{aligned} \tag{5.4.15}$$

where $\mathrm{F}_{2,1}(1, r)$ has a closed form provided by Eq. (5.4.9). Since $|r| < 1$ whenever $v \notin (-i\infty, -i] \cup [i, i\infty)$, we may substitute Eq. (5.4.9) to finish the calculation.

Observe that when $v = 2$, we have $r = \frac{\sqrt{5}-1}{2}$. It is easy to verify that $\frac{3-\sqrt{5}}{2} = r^2 = 1 - r = \frac{r}{1+r}$. Using Eq. (5.4.12), it follows that

$$\begin{aligned} h_3(2) =& \frac{17}{2}\mathrm{Li}_3\left(\frac{3-\sqrt{5}}{2}\right) - 4\zeta(3) - 3\log\left(\frac{1+\sqrt{5}}{2}\right)\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) \\ &+ \frac{4\pi^2}{3}\log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{26}{3}\log^3\left(\frac{1+\sqrt{5}}{2}\right). \end{aligned} \tag{5.4.16}$$

Eq. (5.4.13) follows immediately from substituting the classical formulas for $\mathrm{Li}_3\left(\frac{3-\sqrt{5}}{2}\right)$ and $\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right)$ into Eq. (5.4.16) (see [100], pages 248 and 249). □

Notice that Eq. (5.4.13) is equivalent to a new evaluation of the ${}_4F_3$ hypergeometric function,

$$ {}_4F_3\left[\left.\begin{matrix}1,1,\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2},\frac{3}{2}\end{matrix}\right| -4\right] = \frac{7}{10}\zeta(3). \tag{5.4.17}$$

Corollary 5.4.6. *If $r = \frac{v}{1+\sqrt{1+v^2}}$ and $v \notin (-i\infty, -i] \cup [i, i\infty)$, then*

$$\begin{aligned} \mathrm{TS}(v,1) =& \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{1}{4}\mathrm{Li}_3(r^2) - 2\mathrm{Li}_3(1-r) - 2\mathrm{Li}_3\left(\frac{r}{1+r}\right) \\ &+ 2\zeta(3) + \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)\mathrm{Li}_2(r^2) + \frac{\pi^2}{3}\log(1-r) \\ &+ \frac{1}{3}\log^3(1+r) - \log(r)\log^2(1-r), \end{aligned} \tag{5.4.18}$$

$$\mathrm{TS}(2,1) = \frac{\pi}{2}\int_0^2 \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{7}{5}\zeta(3). \tag{5.4.19}$$

Proof. Eq. (5.4.18) follows immediately from substituting Eq. (5.4.12) into Eq. (5.4.3), while Eq. (5.4.19) follows from combining Eq. (5.4.13) with Eq. (5.4.3). □

The fact that we can reduce $h_1(v)$, $h_2(v)$ and $h_3(v)$ to standard polylogarithms is somewhat miraculous. Integrating Eq. (5.4.15) again, we can show that

$$h_4(v) = \frac{\pi^2}{4}\left(\log(1-r^2)\log\left(\frac{1-r}{1+r}\right) + 2\mathrm{Li}_2\left(\frac{1-r}{2}\right) - 2\mathrm{Li}_2\left(\frac{1+r}{2}\right)\right) + \pi^2\mathrm{F}_2(r) + 4\mathrm{F}_3(r) - 8\mathrm{F}_{3,1}(1,r) - 8\mathrm{F}_{2,2}(1,r) + 16\mathrm{F}_{2,1,1}(1,1,r). \tag{5.4.20}$$

Considering the complexity of these multiple polylogarithms, it seems unlikely that $h_n(v)$ will reduce to standard polylogarithms for $n \geq 4$.

5.5 Relations between $\mathrm{S}(v,1)$ and Mahler's measure, and a closed form for $\mathrm{S}(v,w)$.

In this section we will study the double arcsine integral, $\mathrm{S}(v,w)$. Recall that we defined $\mathrm{S}(v,w)$ with an integral:

$$\mathrm{S}(v,w) = \int_0^1 \frac{\sin^{-1}(vx)\sin^{-1}(wx)}{x}\mathrm{d}x.$$

First, we will show that both $\mathrm{S}(v,1)$ and $\mathrm{S}(v,v)$ reduce to standard polylogarithms. Next, we will discuss several interesting results relating $\mathrm{S}(v,1)$ and $\mathrm{S}(v,v)$ to Mahler's measure and binomial sums. Finally, Theorem 5.5.4 concludes this section by expressing $\mathrm{S}(v,w)$ in terms of polylogarithms.

Theorem 5.5.1. *If $0 \leq v \leq 1$, then $\mathrm{S}(v,v)$ and $\mathrm{S}(v,1)$ both have simple closed forms:*

$$\mathrm{S}(v,1) = \frac{\pi}{2}\int_0^v \frac{\sin^{-1}(x)}{x}\mathrm{d}x - \left(\frac{\mathrm{Li}_3(v) - \mathrm{Li}_3(-v)}{2}\right), \tag{5.5.1}$$

$$\mathrm{S}(v,v) = \left(\frac{\mathrm{Li}_3\left(e^{2i\sin^{-1}(v)}\right) + \mathrm{Li}_3\left(e^{-2i\sin^{-1}(v)}\right)}{4}\right) - \frac{\zeta(3)}{2} + \sin^{-1}(v)\left(\frac{\mathrm{Li}_2\left(e^{2i\sin^{-1}(v)}\right) - \mathrm{Li}_2\left(e^{-2i\sin^{-1}(v)}\right)}{2i}\right) + \left(\sin^{-1}(v)\right)^2\log(2v). \tag{5.5.2}$$

Proof. To prove Eq. (5.5.1), we will substitute the Taylor series for $\sin^{-1}(vx)$ into the integral $\mathrm{S}(v,1) = \int_0^1 \frac{\sin^{-1}(vx)\sin^{-1}(x)}{x}\mathrm{d}x$. After swapping the order of summation and integration, we have

$$\begin{aligned}
\mathrm{S}(v,1) &= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\binom{2n}{n}\left(\frac{v}{2}\right)^{2n+1}\int_0^1 \sin^{-1}(x)x^{2n}\mathrm{d}x \\
&= \pi\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\binom{2n}{n}\left(\frac{v}{2}\right)^{2n+1} - \sum_{n=0}^{\infty}\frac{v^{2n+1}}{(2n+1)^3} \\
&= \frac{\pi}{2}\int_0^v \frac{\sin^{-1}(x)}{x}\mathrm{d}x - \left(\frac{\mathrm{Li}_3(v)-\mathrm{Li}_3(-v)}{2}\right).
\end{aligned}$$

To prove (5.5.2) make the u-substitution $x = \frac{\sin(t)}{v}$, and then integrate by parts as follows:

$$\begin{aligned}
\mathrm{S}(v,v) &= \int_0^1 \frac{\left(\sin^{-1}(vx)\right)^2}{x}\mathrm{d}x = \int_0^{\sin^{-1}(v)} t^2\cot(t)\mathrm{d}t \\
&= \left(\sin^{-1}(v)\right)^2\log(v) - 2\int_0^{\sin^{-1}(v)} t\log(\sin(t))\mathrm{d}t.
\end{aligned}$$

Next substitute the Fourier series for $\log(\sin(t))$ into the previous equation. Recall that

$$\log(\sin(t)) = -\log(2) - \sum_{n=1}^{\infty}\frac{\cos(2nt)}{n}$$

is valid for $0 < t < \pi$. Integrating by parts a second time completes the proof. □

The function $\mathrm{S}(v,v)$ provides a connection to a second family of interesting binomial sums. If we recall the formula (see [106], page 61)

$$(\sin^{-1}(x))^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2\binom{2n}{n}},$$

then it is immediately obvious that if $|v|\leq 1$ we must have

$$\mathrm{S}(v,v) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(2v)^{2n}}{n^3\binom{2n}{n}}. \tag{5.5.3}$$

Comparing Eq. (5.5.3) with Eq. (5.5.2) yields a classical formula:

$$\mathrm{S}\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n}{3}\right)}{n^3} - \frac{\zeta(3)}{2} + \frac{\pi}{6}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n}{3}\right)}{n^2}. \tag{5.5.4}$$

Proposition 5.5.2. *If $v \in [0,1]$ and $w \in (0,1]$, we have*

$$\begin{aligned}\mathrm{S}(v,w) =& \sin^{-1}(w)\int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u \\ &- \frac{\pi}{2}\int_0^{\sin^{-1}(w)} \mathrm{m}\left(\frac{2v}{w}\sin(\theta)+y+z\right)\mathrm{d}\theta.\end{aligned} \tag{5.5.5}$$

Proof. This proof is similar to the proof of Proposition 5.3.1. After an integration by parts, and the u-substitution $u = \sin(\theta)/w$, we obtain

$$\mathrm{S}(v,w) = \sin^{-1}(w)\int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u - \frac{\pi}{2}\int_0^{\sin^{-1}(w)}\int_0^{\frac{v}{w}\sin(\theta)} \frac{\sin^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta.$$

Since $0 \le v \le 1$ and $0 < w \le 1$, it follows that $0 \le \frac{v}{w}\sin(\theta) \le 1$. Therefore we may complete the proof by substituting Eq. (5.2.1) for the nested arcsine integral. $\square$

Corollary 5.5.3. *We can recover Vandervelde's formula by letting $w = 1$ in Eq.* (5.5.5)*:*

$$\begin{aligned}\mathrm{m}\left(v(1+x)+y+z\right) &= \frac{2}{\pi}\int_0^v \frac{\sin^{-1}(u)}{u}\mathrm{d}u - \frac{4}{\pi^2}\mathrm{S}(v,1) \\ &= \frac{4}{\pi^2}\left(\frac{\mathrm{Li}_3(v)-\mathrm{Li}_3(-v)}{2}\right)\end{aligned} \tag{5.5.6}$$

Notice that if $v = w = \frac{1}{2}$ in Eq. (5.5.5), we have

$$\begin{aligned}\mathrm{S}\left(\frac{1}{2},\frac{1}{2}\right) &= \frac{\pi}{6}\int_0^{1/2} \frac{\sin^{-1}(u)}{u}\mathrm{d}u - \frac{\pi}{2}\int_0^{\pi/6} \mathrm{m}\left(2\sin(\theta)+y+z\right)\mathrm{d}\theta \\ &= \frac{\pi}{6}\int_0^{1/2} \frac{\sin^{-1}(u)}{u}\mathrm{d}u - \frac{\pi^2}{12}\mathrm{m}\left(1-x^{1/6}+y+z\right)\end{aligned} \tag{5.5.7}$$

Comparing Eq. (5.5.7) to Eq. (5.5.4) allows us to express a famous binomial sum as the Mahler measure of a three-variable algebraic function.

The final result of this section allows us to express $\mathrm{S}(v,w)$ in terms of standard polylogarithms.

Theorem 5.5.4. *Suppose that* $0 \le v < w \le 1$, *and let* $\theta = \sin^{-1}(w) - \sin^{-1}(v)$. *Then we have*

$$\begin{aligned} 2\mathrm{S}(v,w) =& \mathrm{S}(v,v) + \mathrm{S}(w,w) - \mathrm{S}\left(\sin(\theta),\sin(\theta)\right) \\ & - 2\mathrm{Li}_3\left(\frac{v}{w}\right) + \mathrm{Li}_3\left(\frac{v}{w}e^{i\theta}\right) + \mathrm{Li}_3\left(\frac{v}{w}e^{-i\theta}\right) \\ & - i\theta\mathrm{Li}_2\left(\frac{v}{w}e^{i\theta}\right) + i\theta\mathrm{Li}_2\left(\frac{v}{w}e^{-i\theta}\right) \\ & + \frac{\theta^2}{2}\log\left(1 + \frac{v^2}{w^2} - \frac{2v}{w}\cos(\theta)\right). \end{aligned} \tag{5.5.8}$$

Notice that Eq. (5.5.2) *reduces* $\mathrm{S}(v,v)$, $\mathrm{S}(w,w)$, *and* $\mathrm{S}\left(\sin(\theta),\sin(\theta)\right)$ *to standard polylogarithms.*

Proof. The details of this proof are not particularly difficult. First observe the following trivial formula:

$$\mathrm{S}(v,v) - 2\mathrm{S}(v,w) + \mathrm{S}(w,w) = \int_0^1 \frac{\left(\sin^{-1}(wu) - \sin^{-1}(vu)\right)^2}{u}\mathrm{d}u.$$

Rearranging, and then applying the arcsine addition formula yields

$$\begin{aligned} 2\mathrm{S}(v,w) =& \mathrm{S}(v,v) + \mathrm{S}(w,w) \\ & - \int_0^1 \frac{\left(\sin^{-1}\left(wu\sqrt{1-v^2u^2} - vu\sqrt{1-w^2u^2}\right)\right)^2}{u}\mathrm{d}u. \end{aligned} \tag{5.5.9}$$

This substitution is justified by the monotonicity of the arcsine function. In particular, $0 \le v < w \le 1$ implies that $0 \le \sin^{-1}(wu) - \sin^{-1}(vu) \le \frac{\pi}{2}$ for all $u \in [0,1]$.

Next we will make the u-substitution $z = wu\sqrt{1-v^2u^2} - vu\sqrt{1-w^2u^2}$. In particular, we can show that

$$u^2 = \frac{z^2}{w^2 + v^2 - 2vw\sqrt{1-z^2}},$$

and we can easily verify that

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{1}{z} - \frac{vwz}{(v^2+w^2-2vw\sqrt{1-z^2})\sqrt{1-z^2}}.$$

Observe that the new path of integration will run from $z=0$ to $z=\sin(\theta)=w\sqrt{1-v^2}-v\sqrt{1-w^2}$. Therefore, Eq. (5.5.9) becomes

$$\begin{aligned}2\mathrm{S}(v,w)=&\mathrm{S}(v,v)+\mathrm{S}(w,w)\\&-\int_0^{\sin(\theta)}\left(\sin^{-1}(z)\right)^2\left(\frac{1}{z}-\frac{vwz}{(v^2+w^2-2vw\sqrt{1-z^2})\sqrt{1-z^2}}\right)\mathrm{d}z\\=&\mathrm{S}(v,v)+\mathrm{S}(w,w)-\mathrm{S}\left(\sin(\theta),\sin(\theta)\right)\\&+\int_0^{\sin(\theta)}\left(\sin^{-1}(z)\right)^2\frac{vwz}{(v^2+w^2-2vw\sqrt{1-z^2})\sqrt{1-z^2}}\mathrm{d}z.\end{aligned}$$

If we let $t=\sin^{-1}(z)$, then this last integral becomes

$$\begin{aligned}2\mathrm{S}(v,w)=&\mathrm{S}(v,v)+\mathrm{S}(w,w)-\mathrm{S}\left(\sin(\theta),\sin(\theta)\right)\\&+\int_0^{\theta}t^2\frac{vw\sin(t)}{v^2+w^2-2vw\cos(t)}\mathrm{d}t.\end{aligned}\tag{5.5.10}$$

Since $0\le v<w\le 1$, a formula from [106], page 48, shows that

$$\frac{vw\sin(t)}{v^2+w^2-2vw\cos(t)}=\sum_{n=1}^{\infty}\left(\frac{v}{w}\right)^n\sin(nt).\tag{5.5.11}$$

The Fourier series in Eq. (5.5.11) converges uniformly since $v<w$. It follows that we may substitute Eq. (5.5.11) into Eq. (5.5.10), and then swap the order of summation and integration to obtain:

$$\begin{aligned}2\mathrm{S}(v,w)=&\mathrm{S}(v,v)+\mathrm{S}(w,w)-\mathrm{S}\left(\sin(\theta),\sin(\theta)\right)\\&+\sum_{n=1}^{\infty}\left(\frac{v}{w}\right)^n\int_0^{\theta}t^2\sin(nt)\mathrm{d}t.\end{aligned}\tag{5.5.12}$$

Simplifying Eq. (5.5.12) completes the proof of Eq. (5.5.8). □

5.6 q-series for the dilogarithm, and some associated trigonometric integrals

In this section we will prove several double q-series expansions for the dilogarithm. While these formulas are relatively simple, it appears that they are new. The first of these formulas, Eq. (5.6.8), follows from a few simple manipulations of Eq. (5.5.1). The remaining formulas follow from integrals that we have evaluated in Theorem 5.6.5. Recall that Theorem 5.6.5 figured prominently in the proof of Theorem 5.2.3.

In this section, the twelve Jacobian elliptic functions will play an important role our calculations. Recall that the Jacobian elliptic functions are doubly periodic and meromorphic on $\mathbb{C}$. The Jacobian sine function, $\mathrm{sn}(u)$, inverts the incomplete elliptical integral of the first kind. If $u \in \mathbb{C}$ is an arbitrary number, then under a suitable path of integration:

$$u = \int_0^{\mathrm{sn}(u)} \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

The Jacobian amplitude can be defined by the equation $\mathrm{sn}(u) = \sin(\mathrm{am}(u))$, and the Jacobian cosine function is defined by $\mathrm{cn}(u) = \cos(\mathrm{am}(u))$. As usual the complementary sine function is given by $\mathrm{dn}(u) = \sqrt{1-k^2\mathrm{sn}^2(u)}$. Notice that every Jacobian elliptic function implicitly depends on k; this parameter k is called the elliptic modulus.

Following standard notation, we will denote the real one-quarter period of $\mathrm{sn}(u)$ by K. Since $\mathrm{sn}(K) = 1$, we may compute K from the usual formula

$$\begin{aligned} K := K(k) &= \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ &= \frac{\pi}{2} {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}\middle| k^2\right]. \end{aligned}$$

Let $K' = K(\sqrt{1-k^2})$, and finally define the elliptic nome by $q = e^{-\pi\frac{K'}{K}}$.

Proposition 5.6.1. *If $k \in (0,1)$, then we have the following integral:*

$$\int_0^K \mathrm{am}(u)\mathrm{cn}(u)\mathrm{d}u = \frac{\pi}{2}\frac{\sin^{-1}(k)}{k} - \left(\frac{\mathrm{Li}_2(k) - \mathrm{Li}_2(-k)}{2k}\right). \tag{5.6.1}$$

Proof. Taking the derivative of each side of Eq. (5.5.1), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{S}(k,1) = \int_0^1 \frac{\sin^{-1}(x)}{\sqrt{1-k^2x^2}}\mathrm{d}x = \frac{\pi}{2}\frac{\sin^{-1}(k)}{k} - \left(\frac{\mathrm{Li}_2(k) - \mathrm{Li}_2(-k)}{2k}\right). \tag{5.6.2}$$

Making the u-substitution $x = \mathrm{sn}(u)$ completes the proof. □

We will need the following two inversion formulas for the elliptic nome.

Lemma 5.6.2. *Let q be the usual elliptic nome. Suppose that $q \in (0,1)$, then q is invertible using either of the formulas:*

$$k = \sin\left(4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})}\right), \tag{5.6.3}$$

$$k = \tanh\left(4\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{q^{n+1/2}}{(1-q^{2n+1})}\right). \tag{5.6.4}$$

Proof. To prove Eq. (5.6.3) observe that

$$\begin{aligned}\sin^{-1}(k) &= k\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-k^2x^2}}\\ &= k\int_0^K \mathrm{cn}(u)\mathrm{d}u \end{aligned}\tag{5.6.5}$$

Recall the Fourier series expansion for $\mathrm{cn}(u)$ (see [106], page 916):

$$\mathrm{cn}(u) = \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\left(\frac{\pi(2n+1)}{2K}u\right). \tag{5.6.6}$$

Since $0 < q < 1$, this Fourier series converges uniformly. It follows that we may substitute Eq. (5.6.6) into Eq. (5.6.5), and then swap the order of summation and integration to obtain:

$$\begin{aligned}\sin^{-1}(k) &= \frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\int_0^K \cos\left(\frac{\pi(2n+1)}{2K}u\right)\mathrm{d}u\\ &= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})}. \end{aligned}\tag{5.6.7}$$

Eq. (5.6.3) follows immediately from taking the sine of both sides of the equation.

Eq. (5.6.4) can be proved in a similar manner when starting from the integral

$$\tanh^{-1}(k) = k\int_0^1 \frac{\mathrm{d}x}{1-k^2x^2}.$$

□

Next we will utilize the Fourier-series expansions for the Jacobian elliptic functions to prove the following theorem:

Theorem 5.6.3. *If q is the usual elliptic nome, then the following formula holds for the dilogarithm:*

$$\begin{aligned}\frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{8} =& \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})}\\ &+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{1}{(2n+1)^2-(2m)^2}\frac{q^{n+m+1/2}}{(1+q^{2m})(1+q^{2n+1})}\end{aligned} \tag{5.6.8}$$

Proof. We have already stated the Fourier series expansion for $\mathrm{cn}(u)$ in Eq. (5.6.6). We will also require the Fourier series [106] for $\mathrm{am}(u)$:

$$\mathrm{am}(u) = \frac{\pi}{2K}u + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sin\left(\frac{\pi n}{K}u\right). \tag{5.6.9}$$

Substituting Eq. (5.6.6) and Eq. (5.6.9) into the integral in Eq. (5.6.1), and then simplifying yields:

$$\begin{aligned}&\frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{8} - \frac{\pi}{8}\sin^{-1}(k)\\ &= -\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})}\\ &\quad + 4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{1}{(2n+1)^2-(2m)^2}\frac{q^{n+m+1/2}}{(1+q^{2m})(1+q^{2n+1})}.\end{aligned} \tag{5.6.10}$$

This proof is nearly complete, the final step is to substitute the identity

$$\sin^{-1}(k) = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})}$$

into Eq. (5.6.10). This formula for $\sin^{-1}(k)$ follows immediately from Lemma 5.6.2.

□

The fact that Eq. (5.6.8) follows easily from an integral of the form

$$\int_0^K \mathrm{am}(u)\varphi(u)\mathrm{d}u,$$

suggests that we should try to generalize Eq. (5.6.8) by allowing $\varphi(u)$ to equal one of the other eleven Jacobian elliptic functions. Theorem 5.6.5 proves that ten of these eleven integrals reduce to dilogarithms and elementary functions. First, Theorem 5.6.4 will prove that the one exceptional integral can be expressed as the Mahler measure of an elliptic curve.

Theorem 5.6.4. *The following formulas hold whenever $k \in (0,1]$:*

$$\begin{aligned} \mathrm{m}\left(\frac{4}{k}+x+\frac{1}{x}+y+\frac{1}{y}\right) \\ &= -\log\left(\frac{k}{1+\sqrt{1-k^2}}\right)+\frac{2}{\pi}\int_0^1 \frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\mathrm{d}x \qquad (5.6.11)\\ &= -\log\left(\frac{k}{1+\sqrt{1-k^2}}\right)+\frac{2}{\pi}\int_0^K \mathrm{am}(u)\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}\mathrm{d}u. \qquad (5.6.12)\end{aligned}$$

Proof. First observe that if $k \in \mathbb{R}$ and $0<k\le 1$, then

$$\mathrm{m}\left(\frac{4}{k}+x+\frac{1}{x}+y+\frac{1}{y}\right) = -\log\left(\frac{k}{4}\right)+\mathrm{m}\left(1+\frac{k}{4}\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right).$$

For brevity let $\varphi(k)=\mathrm{m}\left(1+\frac{k}{4}\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right)$. Making the change of variables $(x,y)\to(x/y,yx)$, we have:

$$\begin{aligned}\varphi(k) =&\mathrm{m}\left(1+\frac{k}{4}\left(x+x^{-1}\right)\left(y+y^{-1}\right)\right)\\ =&\mathrm{m}\left(\frac{k}{4}\left(y+y^{-1}\right)\right)+\mathrm{m}\left(x^2+\frac{4}{k}\left(\frac{1}{y+y^{-1}}\right)x+1\right)\\ =&\log\left(\frac{k}{4}\right)+\mathrm{m}\left(x^2+\frac{4}{k}\left(\frac{1}{y+y^{-1}}\right)x+1\right).\end{aligned}$$

Applying Jensen's formula with respect to x reduces $\varphi(k)$ to a pair of one-dimensional integrals:

$$\begin{aligned}\varphi(k) = \log\left(\frac{k}{4}\right) &+\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1+\sqrt{1-k^2\cos^2(\theta)}}{k\cos(\theta)}\right|\mathrm{d}\theta\\ &+\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1-\sqrt{1-k^2\cos^2(\theta)}}{k\cos(\theta)}\right|\mathrm{d}\theta.\end{aligned} \qquad (5.6.13)$$

The right-hand integral vanishes under the assumption that $0<k\le 1$. Therefore, it follows that Eq. (5.6.13) reduces to

$$\varphi(k) = \log\left(\frac{k}{4}\right)+\frac{2}{\pi}\int_0^{\pi/2}\log\left(\frac{1+\sqrt{1-k^2\cos^2(\theta)}}{k\cos(\theta)}\right)\mathrm{d}\theta.$$

With the observation that $\int_0^{\pi/2} \log(\cos(\theta))\,\mathrm{d}\theta = -\frac{\pi}{2}\log(2)$, this formula becomes:

$$\varphi(k) = \frac{2}{\pi}\int_0^{\pi/2} \log\left(\frac{1+\sqrt{1-k^2\cos^2(\theta)}}{2}\right)\mathrm{d}\theta. \tag{5.6.14}$$

Making the u-substitution of $x = \cos(\theta)$, we obtain

$$\varphi(k) = \frac{2}{\pi}\int_0^1 \log\left(\frac{1+\sqrt{1-k^2x^2}}{2}\right)\frac{1}{\sqrt{1-x^2}}\mathrm{d}x.$$

Integrating by parts to eliminate the logarithmic term yields:

$$\begin{aligned}\varphi(k) &= \log\left(\frac{1+\sqrt{1-k^2}}{2}\right) + \frac{2}{\pi}\int_0^1 \frac{\sin^{-1}(x)}{x}\left(\frac{1-\sqrt{1-k^2x^2}}{\sqrt{1-k^2x^2}}\right)\mathrm{d}x\\ &= \log\left(\frac{1+\sqrt{1-k^2}}{2}\right) + \frac{2}{\pi}\int_0^1 \frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\mathrm{d}x - \frac{2}{\pi}\int_0^1 \frac{\sin^{-1}(x)}{x}\mathrm{d}x.\end{aligned}$$

Since $\int_0^1 \frac{\sin^{-1}(x)}{x}\mathrm{d}x = \frac{\pi}{2}\log(2)$, it follows that

$$\varphi(k) = \log\left(\frac{1+\sqrt{1-k^2}}{4}\right) + \frac{2}{\pi}\int_0^1 \frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\mathrm{d}x,$$

from which we obtain

$$\mathrm{m}\left(\frac{4}{k}+x+\frac{1}{x}+y+\frac{1}{y}\right) = -\log\left(\frac{k}{1+\sqrt{1-k^2}}\right) + \frac{2}{\pi}\int_0^1 \frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\mathrm{d}x.$$

To prove Eq. (5.6.12) simply make the u-substitution $x = \mathrm{sn}(u)$. $\square$

The elliptic curve defined by the equation $4/k+x+1/x+y+1/y = 0$ was one of the simplest curves that Boyd studied in [103]. Rodriguez Villegas derived q-series expansions for a wide class of functions defined by the Mahler measures of elliptic curves in [113]. We can recover one of his results by substituting the Fourier series expansions for $\mathrm{am}(u)$ and $\mathrm{cn}(u)/\mathrm{sn}(u)$ into Eq. (5.6.12).

If we let $k = \sin(\theta)$, and then integrate Eq. (5.6.11) from $\theta = 0$ to $\theta = \frac{\pi}{2}$, we can prove that

$$\mathrm{m}\left(8+\left(z+\frac{1}{z}\right)\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right) = \frac{4}{\pi}G + \frac{4}{\pi^2}\int_0^1 \frac{\sin^{-1}(x)}{x}K(x)\mathrm{d}x. \tag{5.6.15}$$

Using Mathematica, we can reduce the right-hand integral to a rather complicated expression involving balanced hypergeometric functions evaluated at one.

Theorem 5.6.5. *We will assume that* $0 < k < 1$ *and that each Jacobian elliptic function has modulus* k*. Let* $p = \sqrt{\frac{1-k}{1+k}}$, $r = \frac{k}{1+\sqrt{1-k^2}}$, *and* $s = \frac{k}{\sqrt{1-k^2}}$, *then*

$$\begin{aligned}\int_0^K \operatorname{am}(u)\operatorname{sn}(u)\mathrm{d}u &= \int_0^1 \frac{u\sin^{-1}(u)}{\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u \\ &= \frac{\mathrm{Li}_2(is)-\mathrm{Li}_2(-is)}{2ki}\end{aligned} \tag{5.6.16}$$

$$\begin{aligned}\int_0^K \operatorname{am}(u)\operatorname{cn}(u)\mathrm{d}u &= \int_0^1 \frac{\sin^{-1}(u)}{\sqrt{1-k^2u^2}}\mathrm{d}u \\ &= \frac{\pi}{2}\frac{\sin^{-1}(k)}{k} - \frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{2k}\end{aligned} \tag{5.6.17}$$

$$\int_0^K \operatorname{am}(u)\operatorname{dn}(u)\mathrm{d}u = \frac{\pi^2}{8} \tag{5.6.18}$$

$$\begin{aligned}\int_0^K \operatorname{am}(u)\frac{1}{\operatorname{sn}(u)}\mathrm{d}u &= \int_0^1 \frac{\sin^{-1}(u)}{u\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u \\ &= -\frac{\pi}{2}\log(p) + \frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{i}\end{aligned} \tag{5.6.19}$$

$$\int_0^K \operatorname{am}(u)\frac{1}{\operatorname{cn}(u)}\mathrm{d}u = \infty \tag{5.6.20}$$

$$\begin{aligned}\int_0^K \operatorname{am}(u)\frac{1}{\operatorname{dn}(u)}\mathrm{d}u &= \int_0^1 \frac{\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\mathrm{d}u \\ &= \frac{1}{\sqrt{1-k^2}}\left(\frac{\pi^2}{8} + \frac{\mathrm{Li}_2(r^2)-\mathrm{Li}_2(-r^2)}{2}\right)\end{aligned} \tag{5.6.21}$$

$$\int_0^K \operatorname{am}(u)\frac{\operatorname{sn}(u)}{\operatorname{cn}(u)}\mathrm{d}u = \infty \tag{5.6.22}$$

$$\begin{aligned}\int_0^K \operatorname{am}(u)\frac{\operatorname{sn}(u)}{\operatorname{dn}(u)}\mathrm{d}u &= \int_0^1 \frac{u\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\mathrm{d}u \\ &= \frac{\mathrm{Li}_2(r)-\mathrm{Li}_2(-r)}{k\sqrt{1-k^2}}\end{aligned} \tag{5.6.23}$$

$$\begin{aligned}\int_0^K \mathrm{am}(u)\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}\mathrm{d}u &= \int_0^1 \frac{\sin^{-1}(u)}{u\sqrt{1-k^2u^2}}\mathrm{d}u \\ &= \frac{\pi}{2}\log(r) + \frac{\pi}{2}\mathrm{m}\left(\frac{4}{k}+x+\frac{1}{x}+y+\frac{1}{y}\right)\end{aligned} \tag{5.6.24}$$

$$\begin{aligned}\int_0^K \mathrm{am}(u)\frac{\mathrm{cn}(u)}{\mathrm{dn}(u)}\mathrm{d}u &= \int_0^1 \frac{\sin^{-1}(u)}{1-k^2u^2}\mathrm{d}u \\ &= -\frac{\pi}{2k}\log(p) - \frac{\mathrm{Li}_2(ir)-\mathrm{Li}_2(-ir)}{ki}\end{aligned} \tag{5.6.25}$$

$$\int_0^K \mathrm{am}(u)\frac{\mathrm{dn}(u)}{\mathrm{sn}(u)}\mathrm{d}u = 2G \tag{5.6.26}$$

$$\int_0^K \mathrm{am}(u)\frac{\mathrm{dn}(u)}{\mathrm{cn}(u)}\mathrm{d}u = \infty \tag{5.6.27}$$

Proof. First observe that Eq. (5.6.20), Eq. (5.6.22), and Eq. (5.6.27) all follow from the fact that $\mathrm{cn}(K) = 0$. Similarly, Eq. (5.6.18) and Eq. (5.6.26) both follow from the formula $\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{am}(u) = \mathrm{dn}(u)$.

We already proved Eq. (5.6.17) in Proposition 5.6.1, and Eq. (5.6.24) was proved in Theorem 5.6.4. This leaves a total of five formulas to prove.

To prove Eq. (5.6.16), observe that after letting $u = \sqrt{1-z^2}$, we have

$$\int_0^1 \frac{u\sin^{-1}(u)}{\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u = \frac{is}{ki}\int_0^1 \frac{\sin^{-1}(\sqrt{1-z^2})}{\sqrt{1-(is)^2z^2}}\mathrm{d}z.$$

If $0 < k \le 1/\sqrt{2}$, then $|s| \le 1$. With this restriction on k, we may expand the square root in a Taylor series to obtain:

$$\begin{aligned}&= \frac{1}{ki}\sum_{m=0}^{\infty}(-1)^m\binom{-1/2}{m}(is)^{2m+1}\int_0^1 \sin^{-1}(\sqrt{1-z^2})z^{2m}\mathrm{d}z \\ &= \frac{1}{ki}\sum_{m=0}^{\infty}\frac{(is)^{2m+1}}{(2m+1)^2} \\ &= \frac{\mathrm{Li}_2(is)-\mathrm{Li}_2(-is)}{2ki}.\end{aligned} \tag{5.6.28}$$

Notice that Eq. (5.6.28) extends to $0 < k < 1$, since both sides of the equation are analytic in this interval. Therefore, Eq. (5.6.16) follows immediately.

To prove Eq. (5.6.19) make the u-substitution $u = \frac{z}{\sqrt{1-k^2+z^2}}$. Recalling that $\sin^{-1}\left(\frac{z}{\sqrt{1-k^2+z^2}}\right) = \tan^{-1}\left(\frac{z}{\sqrt{1-k^2}}\right)$, we obtain

$$\int_0^1 \frac{\sin^{-1}(u)}{u\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u = \int_0^\infty \frac{\tan^{-1}\left(\frac{z}{\sqrt{1-k^2}}\right)}{z\sqrt{1+z^2}}\mathrm{d}z.$$

Using Mathematica to evaluate this last integral yields:

$$\begin{aligned}
&= -\frac{\pi}{2}\log(p) + \sqrt{1-k^2}\,{}_3F_2\left[\left.{\frac{1}{2},1,1 \atop \frac{3}{2},\frac{3}{2}}\right| 1-k^2\right] \\
&= -\frac{\pi}{2}\log(p) + \frac{1}{2}\sum_{n=0}^\infty \frac{\left(2\sqrt{1-k^2}\right)^{2n+1}}{(2n+1)^2\binom{2n}{n}} \\
&= -\frac{\pi}{2}\log(p) + 2\left(\frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{2i}\right),
\end{aligned}$$

where Eq. (5.4.4) justifies the final step.

To prove Eq. (5.6.21) observe that after the u-substitution $u = \sin(\theta)$ we have

$$\int_0^1 \frac{\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\mathrm{d}u = \int_0^{\pi/2} \frac{\theta}{1-k^2\sin^2(\theta)}\mathrm{d}\theta.$$

Now substitute the Fourier series

$$\frac{\sqrt{1-k^2}}{1-k^2\sin^2(\theta)} = 1 + 2\sum_{m=1}^\infty (-1)^m\left(\frac{k}{1+\sqrt{1-k^2}}\right)^{2m}\cos(2m\theta) \tag{5.6.29}$$

into the integral, and simplify to complete the proof.

The proof of Eq. (5.6.23) follows the same lines as the derivation of Eq. (5.6.21). Observe that

$$\int_0^1 \frac{u\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\mathrm{d}u = \int_0^{\pi/2} \frac{\theta\sin(\theta)}{1-k^2\sin^2(\theta)}\mathrm{d}\theta.$$

Now substitute the Fourier series

$$\frac{k\sqrt{1-k^2}\sin(\theta)}{1-k^2\sin^2(\theta)} = 2\sum_{m=0}^\infty (-1)^m\left(\frac{k}{1+\sqrt{1-k^2}}\right)^{2m+1}\sin\left((2m+1)\theta\right) \tag{5.6.30}$$

into the integral, and simplify to complete the proof.

Finally, we are left with Eq. (5.6.25). Expanding $1/(1-k^2u^2)$ in a geometric series yields:

$$\begin{aligned}\int_0^1 \frac{\sin^{-1}(u)}{1-k^2u^2}\mathrm{d}u &= \sum_{n=0}^{\infty} k^{2n}\int_0^1 \sin^{-1}(u)u^{2n}\mathrm{d}u \\ &= \sum_{n=0}^{\infty} k^{2n}\left(\frac{\pi/2}{2n+1}-\frac{2^{2n}}{(2n+1)^2\binom{2n}{n}}\right) \\ &= -\frac{\pi}{4k}\log\left(\frac{1-k}{1+k}\right)-\frac{h_2(ik)}{2ik}.\end{aligned}$$

Substituting the closed form for $h_2(ik)$ provided by Eq. (5.4.4) completes the proof. □

We can obtain each of the following q-series by applying the method from Theorem 5.6.3 to the formulas in Theorem 5.6.5.

Corollary 5.6.6. *Let* $p=\sqrt{\frac{1-k}{1+k}}$, *and let* $r=\frac{k}{1+\sqrt{1-k^2}}$. *The following formulas hold for the dilogarithm:*

$$\begin{aligned}\frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{8} =&\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})} \\ &+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{1}{(2n+1)^2-(2m)^2}\frac{q^{m+n+1/2}}{(1+q^{2m})(1+q^{2n+1})},\end{aligned} \tag{5.6.31}$$

$$\begin{aligned}\frac{\mathrm{Li}_2(r)-\mathrm{Li}_2(-r)}{4} =&\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})} \\ &+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{(-1)^m}{(2n+1)^2-(2m)^2}\frac{q^{m+n+1/2}}{(1+q^{2m})(1+q^{2n+1})},\end{aligned} \tag{5.6.32}$$

$$\begin{aligned}\frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{8i} =&\frac{G}{4}+\frac{\pi}{16}\log(p)+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{q^{2n+1}}{(1-q^{4n+2})} \\ &+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2-(2m)^2}\frac{q^{m+2n+1}}{(1+q^{2m})(1-q^{4n+2})}.\end{aligned} \tag{5.6.33}$$

Proof. As we have already stated, each of these formulas can be proved by substituting Fourier series expansions for the Jacobian elliptic functions into Theorem 5.6.5.

Using the method described, we have already proved Eq. (5.6.31) in Theorem 5.6.3. Eq. (5.6.32) follows in a similar manner from Eq. (5.6.23).

Eq. (5.6.33) is a little trickier to prove. Expanding Eq. (5.6.26) in a q-series yields the identity

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sum_{j=0}^{n-1}\frac{(-1)^j}{2j+1}\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{q^{2n+1}}{1+q^{2n+1}}\\ &\quad+4\sum_{\substack{n=0\\ m=1}}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2-(2m)^2}\frac{q^{m+2n+1}}{(1+q^{2m})(1+q^{2n+1})}.\end{aligned}\tag{5.6.34}$$

Next expand Eq. (5.6.19) in the q-series

$$\begin{aligned}\frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{4i}=&\frac{G}{2}+\frac{\pi}{8}\log(p)\\ &+\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sum_{j=0}^{n-1}\frac{(-1)^j}{2j+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{q^{2n+1}}{1-q^{2n+1}}\\ &+4\sum_{\substack{n=0\\ m=1}}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2-(2m)^2}\frac{q^{m+2n+1}}{(1+q^{2m})(1-q^{2n+1})},\end{aligned}$$

and then combine it with Eq. (5.6.34) to complete the proof of Eq. (5.6.33). □

It is important to notice that the nine convergent integrals in Theorem 5.6.5 only produce three interesting q-series for the dilogarithm. The other q-series we may obtain from Theorem 5.6.5 really just restate known facts about the elliptic nome. For example, if we expand Eq. (5.6.21) in a q-series, we will obtain Eq. (5.6.31) with q replaced by q^2 and k replaced by r^2. This is equivalent to the fact that $q\left(\left(\frac{k}{1+\sqrt{1-k^2}}\right)^2\right)=q^2(k)$. If we let $\ell=\left(\frac{k}{1+\sqrt{1-k^2}}\right)^2$, then clearly k and ℓ satisfy a second degree modular equation [101].

5.7 A closed form for $\mathrm{T}(v,w)$, and Mahler measures for $\mathrm{T}\left(v,\frac{1}{v}\right)$

Recall that we defined $\mathrm{T}(v,w)$ using the following integral:

$$\mathrm{T}(v,w)=\int_0^1\frac{\tan^{-1}(vx)\tan^{-1}(wx)}{x}\mathrm{d}x. \tag{5.7.1}$$

Since this integral involves two arctangents, rather than one or two arcsines, $\mathrm{T}(v,w)$ possesses a number of useful properties that $\mathrm{S}(v,w)$ and $\mathrm{TS}(v,w)$ appear to lack.

First observe that $\mathrm{T}(v,w)$ obeys an eight term functional equation. If we let $\mathrm{T}(v)=\int_0^v\frac{\tan^{-1}(x)}{x}\mathrm{d}x$, then we can use properties of the arctangent function to prove the following formula:

$$\begin{aligned}\mathrm{T}(v,w)+\mathrm{T}\left(\frac{1}{v},\frac{1}{w}\right)-\mathrm{T}\left(\frac{w}{v},1\right)-\mathrm{T}\left(\frac{v}{w},1\right)&\\=\frac{\pi}{2}\left(\mathrm{T}(v)+\mathrm{T}\left(\frac{1}{w}\right)-\mathrm{T}\left(\frac{v}{w}\right)-\mathrm{T}\left(1\right)\right).&\end{aligned} \tag{5.7.2}$$

If $|v|<1$ and $|w|<1$, we can substitute arctangent Taylor series expansions into Eq. (5.7.1) to obtain:

$$\begin{aligned}\mathrm{T}(v,w)=&\sum_{n=0}^{\infty}\frac{(-1)^n w^{2n+2}}{(2n+2)^2}\sum_{m=0}^{n}\frac{(v/w)^{2m+1}}{2m+1}\\&+\sum_{n=0}^{\infty}\frac{(-1)^n v^{2n+2}}{(2n+2)^2}\sum_{m=0}^{n}\frac{(w/v)^{2m+1}}{2m+1}.\end{aligned} \tag{5.7.3}$$

Eq. (5.7.3) immediately reduces $\mathrm{T}(v,w)$ to multiple polylogarithms. Theorem 5.7.1 improves upon this result by expressing $\mathrm{T}(v,w)$ in terms of standard polylogarithms.

Theorem 5.7.1. *If v and w are real numbers such that $|w/v| \leq 1$, then*

$$
\begin{aligned}
-4\mathrm{T}(v,w) =& 2\mathrm{Li}_3\left(\frac{w}{v}\right) - 2\mathrm{Li}_3\left(-\frac{w}{v}\right) + \mathrm{Li}_3\left(\frac{1-vi}{1-wi}\right) + \mathrm{Li}_3\left(\frac{1+vi}{1+wi}\right) \\
& - \mathrm{Li}_3\left(\frac{1+vi}{1-wi}\right) - \mathrm{Li}_3\left(\frac{1-vi}{1+wi}\right) \\
& - \mathrm{Li}_3\left(\frac{w(1-vi)}{v(1-wi)}\right) - \mathrm{Li}_3\left(\frac{w(1+vi)}{v(1+wi)}\right) \\
& + \mathrm{Li}_3\left(-\frac{w(1+vi)}{v(1-wi)}\right) + \mathrm{Li}_3\left(-\frac{w(1-vi)}{v(1+wi)}\right) \\
& + \log\left(\frac{1+v^2}{1+w^2}\right)\left(\mathrm{Li}_2\left(\frac{w}{v}\right) - \mathrm{Li}_2\left(-\frac{w}{v}\right)\right) \\
& - 4\tan^{-1}(v)\left(\frac{\mathrm{Li}_2(wi) - \mathrm{Li}_2(-wi)}{2i}\right) \\
& - 4\tan^{-1}(w)\left(\frac{\mathrm{Li}_2(vi) - \mathrm{Li}_2(-vi)}{2i}\right) \\
& - \pi\log\left(\frac{1+v^2}{1+w^2}\right)\tan^{-1}(w) + 4\log(v)\tan^{-1}(v)\tan^{-1}(w).
\end{aligned}
\tag{5.7.4}
$$

Proof. Substituting logarithms for the inverse tangents, we obtain

$$
\begin{aligned}
-4\mathrm{T}(v,w) &= \int_0^1 \log\left(\frac{1+ivu}{1-ivu}\right)\log\left(\frac{1+iwu}{1-iwu}\right)\frac{\mathrm{d}u}{u} \\
&= \int_0^{iw} \log\left(\frac{1+\frac{v}{w}u}{1-\frac{v}{w}u}\right)\log\left(\frac{1+u}{1-u}\right)\frac{\mathrm{d}u}{u}.
\end{aligned}
$$

The identity then follows (more or less) immediately from four applications of Lewin's formula

$$\begin{aligned}\int_0^x \log(1-z)\log(1-cz)\,\frac{\mathrm{d}z}{z}\\ =&\mathrm{Li}_3\left(\frac{1-cx}{1-x}\right)+\mathrm{Li}_3\left(\frac{1}{c}\right)+\mathrm{Li}_3(1)\\ &-\mathrm{Li}_3(1-cx)-\mathrm{Li}_3(1-x)-\mathrm{Li}_3\left(\frac{1-cx}{c(1-x)}\right)\\ &+\log(1-cx)\left[\mathrm{Li}_2\left(\frac{1}{c}\right)-\mathrm{Li}_2(x)\right]\\ &+\log(1-x)\left[\mathrm{Li}_2(1-cx)-\mathrm{Li}_2\left(\frac{1}{c}\right)+\frac{\pi^2}{6}\right]\\ &+\frac{1}{2}\log(c)\log^2(1-x),\end{aligned} \tag{5.7.5}$$

which was proved in [110], page 270. Condon has discussed the intricacies of applying this equation in [105]. □

This closed form for $\mathrm{T}(v,w)$ is quite complicated. Notice that a slight change in the integrand in Eq. (5.7.1) produces a remarkably simplified formula:

$$\int_0^1 \frac{\tan^{-1}(vx)\tan^{-1}(wx)}{\sqrt{1-x^2}}\mathrm{d}x=\pi\sum_{n=0}^{\infty}\frac{\left(\frac{v}{1+\sqrt{1+v^2}}\frac{w}{1+\sqrt{1+w^2}}\right)^{2n+1}}{(2n+1)^2}. \tag{5.7.6}$$

To prove Eq. (5.7.6), make the u-substitution $x=\sin(\theta)$, and then apply Eq. (5.3.16) twice.

There are two special cases of Eq. (5.7.4) worth mentioning. First observe that $\mathrm{T}\left(v,\frac{1}{v}\right)$ reduces to a very simple expression. If we let $w\to 1/v$ in Eq. (5.7.4), and perform a few torturous manipulations, we can show that

$$\begin{aligned}\mathrm{T}\left(v,\frac{1}{v}\right)=&\frac{\pi}{2}\mathrm{Im}\left[\mathrm{Li}_2(iv)\right]-\frac{1}{2}\left(\mathrm{Li}_3(v^2)-\mathrm{Li}_3(-v^2)\right)\\ &+\frac{\log(v)}{2}\left(\mathrm{Li}_2(v^2)-\mathrm{Li}_2(-v^2)\right).\end{aligned} \tag{5.7.7}$$

Lalín obtained an equivalent form of Eq. (5.7.7) using a different method. (See Appendix 2 in [107]. Lalín's formula for $\mathrm{T}(v,1)+\mathrm{T}(1/v,1)$ reduces to Eq. (5.7.7) after applying Eq. (5.7.2) with $w=1/v$). Observe that when

$w = v$ in Eq. (5.7.4), we have

$$\begin{aligned} \mathrm{T}(v,v) =& \frac{1}{2}\mathrm{Re}\left[\mathrm{Li}_3\left(\frac{1+vi}{1-vi}\right) - \mathrm{Li}_3\left(-\frac{1+vi}{1-vi}\right)\right] - \frac{7}{8}\zeta(3) \\ &+ 2\tan^{-1}(v)\mathrm{Im}\left[\mathrm{Li}_2(iv)\right] - \log(v)\left(\tan^{-1}(v)\right)^2. \end{aligned} \tag{5.7.8}$$

Finally, it appears that $\mathrm{T}(v,1)$ does not reduce to any particularly simple expression. Letting $w \to 1$ fails to simplify Eq. (5.7.4) in any appreciable way. Expanding $\mathrm{T}(v,1)$ in a Taylor series results in an equally complicated expression:

$$\begin{aligned} \mathrm{T}(v,1) =& \frac{1}{2}\sum_{n=0}^{\infty}\frac{v^{2n+1}}{(2n+1)^2}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k} \\ &+ \frac{\pi}{4}\int_0^v \frac{\tan^{-1}(x)}{x}\mathrm{d}x - \frac{\log(2)}{4}\left(\mathrm{Li}_2(v) - \mathrm{Li}_2(-v)\right). \end{aligned} \tag{5.7.9}$$

Theorem 5.7.3 relates $\mathrm{T}(v,w)$ to three-variable Mahler measures, and generalizes one of Lalín's formulas. Once again, we will need a simple lemma before we prove our theorem.

Lemma 5.7.2. *If v and w are positive real numbers, then*

$$\mathrm{T}(v,w) = \tan^{-1}(v)\int_0^w \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \int_0^{\tan^{-1}(v)}\int_0^{\frac{w}{v}\tan(\theta)}\frac{\tan^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta, \tag{5.7.10}$$

$$\mathrm{T}\left(v,\frac{1}{v}\right) = \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{1}{2}\int_0^{\pi/2}\int_0^{v^2\tan(\theta)}\frac{\tan^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta. \tag{5.7.11}$$

Proof. While we can verify Eq. (5.7.10) with a trivial integration by parts, the proof of Eq. (5.7.11) is slightly more involved.

To prove Eq. (5.7.11), first let $w = \frac{1}{v}$ in Eq. (5.7.10). This produces

$$\begin{aligned} \mathrm{T}\left(v,\frac{1}{v}\right) =& \tan^{-1}(v)\int_0^{1/v}\frac{\tan^{-1}(u)}{u}\mathrm{d}u \\ &- \int_0^{\tan^{-1}(v)}\int_0^{\frac{\tan(\theta)}{v^2}}\frac{\tan^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta. \end{aligned} \tag{5.7.12}$$

Letting $v \to 1/v$ in Eq. (5.7.12) gives

$$\mathrm{T}\left(\frac{1}{v},v\right) = \tan^{-1}\left(\frac{1}{v}\right)\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \int_0^{\tan^{-1}\left(\frac{1}{v}\right)}\int_0^{v^2\tan(\theta)}\frac{\tan^{-1}(z)}{z}\mathrm{d}z\mathrm{d}\theta$$

$$= \left(\frac{\pi}{2} - \tan^{-1}(v)\right) \int_0^v \frac{\tan^{-1}(u)}{u} \mathrm{d}u - \int_{\tan^{-1}(v)}^{\pi/2} \int_0^{\frac{v^2}{\tan(\theta)}} \frac{\tan^{-1}(z)}{z} \mathrm{d}z\mathrm{d}\theta$$

Now apply Eq. (5.2.4) twice, which transforms this last identity to

$$\begin{aligned} \mathrm{T}\left(\frac{1}{v}, v\right) &= \left(\frac{\pi}{2} - \tan^{-1}(v)\right) \left(\int_0^{\frac{1}{v}} \frac{\tan^{-1}(u)}{u} \mathrm{d}u + \frac{\pi}{2}\log(v)\right) \\ &\quad - \int_{\tan^{-1}(v)}^{\pi/2} \left(\int_0^{\frac{\tan(\theta)}{v^2}} \frac{\tan^{-1}(z)}{z} \mathrm{d}z - \frac{\pi}{2}\log\left(\frac{1}{v^2}\tan(\theta)\right)\right) \mathrm{d}\theta \end{aligned} \tag{5.7.13}$$

To complete the proof, simply add equations (5.7.12) and (5.7.13) together, and then simplify the resulting sum. □

Theorem 5.7.3. *If $v > 0$, then the following Mahler measures hold:*

$$\begin{aligned} &\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2 + \left(y + v^2\left(\frac{1-x}{1+x}\right)\right)^2 z\right) \\ &\qquad = \frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{8}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right) + \frac{1}{2}\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2\right), \end{aligned} \tag{5.7.14}$$

$$\begin{aligned} &\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2 + v^2\left(\frac{1-x}{1+x}\right)\left(\frac{1-y}{1+y}\right)\left(z - z^{-1}\right)\right) \\ &\qquad = \frac{8}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{16}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right), \end{aligned} \tag{5.7.15}$$

$$\begin{aligned} &\mathrm{m}\left(\left(y - y^{-1}\right) + v^2\left(\frac{1-x}{1+x}\right)\left(z - z^{-1}\right)\right) \\ &\qquad = \frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\mathrm{d}u - \frac{8}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right), \end{aligned} \tag{5.7.16}$$

$$\begin{aligned}
&\mathrm{m}\left(\begin{array}{l}\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)\left(1-v^4\left(\dfrac{1-x}{1+x}\right)^2\right)^2\\[2ex] +\left(z-z^{-1}\right)^2(1+y)^2\left(1+v^4\left(\dfrac{1-x}{1+x}\right)^2\right)^2\end{array}\right)\\
&\quad=\frac{8}{\pi}\int_0^v\frac{\tan^{-1}(u)}{u}\mathrm{d}u-\frac{16}{\pi^2}\mathrm{T}\left(v,\frac{1}{v}\right)+\frac{4}{\pi}\int_0^{\pi/2}\log\left(1+v^2\tan(\theta)\right)\mathrm{d}\theta\\
&\quad\quad+\log(2).
\end{aligned} \tag{5.7.17}$$

Proof. Each of these results follows, in order, from substituting Eq. (5.2.2), Eq. (5.2.5), Eq. (5.2.6), and Eq. (5.2.7), into Eq. (5.7.11). ☐

Corollary 5.7.4. *The following identities are true:*

$$\mathrm{m}\left((1+z)(1+y)+(1-z)(x-y)\right)=\frac{7}{2\pi^2}\zeta(3)+\frac{\log(2)}{2}, \tag{5.7.18}$$

$$\mathrm{m}\left(4(1+y)+(1-y)\left(x-x^{-1}\right)\left(z-z^{-1}\right)\right)=\frac{14}{\pi^2}\zeta(3) \tag{5.7.19}$$

$$\mathrm{m}\left((1+x)\left(y-y^{-1}\right)+(1-x)\left(z-z^{-1}\right)\right)=\frac{7}{\pi^2}\zeta(3) \tag{5.7.20}$$

$$\begin{aligned}
&\mathrm{m}\left(16(1+y)^2-4\left(z+z^{-1}\right)^2+(1+y)^2\left(z-z^{-1}\right)^2\left(x+x^{-1}\right)^2\right)\\
&\qquad=\frac{14}{\pi^2}\zeta(3)+\frac{4}{\pi}G
\end{aligned} \tag{5.7.21}$$

Proof. To prove Eq. (5.7.18), let $v=1$ in Eq. (5.7.14). From Eq. (5.7.7) we know that $\mathrm{T}(1,1)=\frac{\pi}{2}G-\frac{7}{8}\zeta(3)$, hence

$$\begin{aligned}
\frac{7}{\pi^2}\zeta(3)+\log(2)&=\mathrm{m}\left(1-\left(\frac{1-x}{1+x}\right)^2+\left(y+\frac{1-x}{1+x}\right)^2z\right)\\
&=\mathrm{m}\left(4x+\left((1+x)y+(1-x)\right)^2z\right).
\end{aligned}$$

Now let $(x,y,z)\to\left(x,\frac{y}{z},-xz^2\right)$ to obtain

$$\begin{aligned}
\frac{7}{\pi^2}\zeta(3)+\log(2)&=\mathrm{m}\left(4x-\left((1+x)y+(1-x)z\right)^2x\right)\\
&=\mathrm{m}\left(4-\left((1+x)y+(1-x)z\right)^2\right)
\end{aligned}$$

$$= 2\mathrm{m}\left(2 + (1+x)y + (1-x)z\right).$$

With the final change of variables $(x, y, z) \to \left(z, \frac{1}{yz}, \frac{x}{yz}\right)$, we have

$$\begin{aligned}\frac{7}{\pi^2}\zeta(3) + \log(2) &= 2\mathrm{m}\left(2 + \frac{(1+z)}{yz} + \frac{(1-z)x}{yz}\right)\\ &= 2\mathrm{m}\left((1+z)(1+y) + (1-z)(x-y)\right),\end{aligned}$$

completing the proof of Eq. (5.7.18).

The proof of Eq. (5.7.19) through Eq. (5.7.21) follows almost immediately from our evaluation of $\mathrm{T}(1,1)$. The proof Eq. (5.7.21) also requires the fairly easy fact that $\int_0^{\pi/2} \log\left(1 + \tan(\theta)\right) \mathrm{d}\theta = G + \frac{\pi}{4}\log(2)$

□

5.8 Conclusion

In principle, we should be able to apply the techniques in this paper to prove formulas for infinitely many three-variable Mahler measures. The main difficulty, which is significant, lies in the challenge of finding infinitely many Mahler measures for the arctangent and arcsine integrals. In Section 5.2 we proved one such formula for the arcsine integral, and four formulas for the arctangent integral.

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Chapter 6

Conclusion

6.1 Computational proofs?

We will conclude this thesis by raising the question as to whether or not computational proofs exist for formulas like (1.1.1) and (1.1.2). Indeed, algorithms such as the PSLQ algorithm and the WZ algorithm now make it possible for computers to both discover and *prove* interesting formulas with relatively little human assistance (see [119] and [116]). For example, Guillera recently proved the following hypergeometric series for $1/\pi^2$ using purely computational methods [118]:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n}} \binom{2n}{n}^5 \left(820n^2 + 180n + 13\right). \tag{6.1.1}$$

The novelty of Guillera's approach is that it applies to both Ramanujan's original formulas for $1/\pi$ [120], and to computationally discovered examples such as (6.1.1). At the very least, it seems extremely likely that a WZ proof should exist for equation (1.1.1), since that identity closely resembles formulas in [118]. Such an approach could also be used to eliminate the algebraic K theory from proofs of similar formulas in [121] and [117].

It seems more difficult to speculate on whether or not computational proofs should exist for identities such as (1.1.2). In Chapter 2 we reduced that identity to an equivalent relation between a lattice sum and a hypergeometric function:

$$\begin{aligned}\sum_{n=0}^{\infty} & \binom{2n}{n}^2 \frac{(1/16)^{2n+1}}{2n+1} \\ &\stackrel{?}{=} \frac{540}{\pi^2} \sum_{\substack{n_i=-\infty \\ i\in\{1,2,3,4\}}}^{\infty} \frac{(-1)^{n_1+n_2+n_3+n_4}}{\left((6n_1-1)^2+3(6n_2-1)^2+5(6n_3-1)^2+15(6n_4-1)^2\right)^2}.\end{aligned} \tag{6.1.2}$$

For reasons outlined in Section 2.4, it seems likely that equation (6.1.2) is really a special case of a more general formula involving Meijer G-functions. So far, we have been unable to prove or disprove this last statement.

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