Introduction to an Introduction to Smoothed Particle Hydrodynamics

(SPH)

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http://laplace.physics.ubc.ca/People/inaki/sph.html

Outline

- Introduction: PROs and CONs.
- Basic SPH formalism.
- Example: Newtonian Equations.
- Implementation:
  - Neighbors Search.
  - Initial Conditions.
  - Varying Smoothing length (h's).
  - Artificial Viscosity.
  - Boundary Conditions.
- General Relativistic Work.
- Our Approach so far.

Introduction

- **PROs**
  - Alternative to HRSC, FD or PM methods.
  - Lagrangian Method $\longrightarrow$ self adapted.
  - Easy to implement. :)
  - Easy to go to 3D.
- **CONs**
  - Difficult Boundary Conditions.
  - Not so great handling Shocks: Artificial Viscosity.

Basic SPH formalism.

Given any field quantity $f(\vec{r})$:

$$< f(\vec{r}) >= \int f(\vec{r}\,')W(\vec{r}-\vec{r}\,'; h)d^3r' \qquad (1)$$

where the integration is over all space and the kernel should satisfy the next properties:

$$\int W(\vec{r}; h)d^3r = 1 \qquad \lim_{h\to 0} < f(\vec{r}) >= f(\vec{r}) \quad (2)$$

Monte-Carlo summation approximation:

$$\boxed{< f(\vec{r}) >= \sum_{b=1}^{N} f(\vec{r}_b)W(\vec{r}-\vec{r}_b; h)\frac{1}{n_b}} \qquad (3)$$

$$\boxed{< \nabla f(\vec{r}) >= \sum_{b=1}^{N} f(\vec{r}_b)\nabla W(\vec{r}-\vec{r}_b; h)\frac{1}{n_b}} \quad (4)$$

Example:
Newtonian Hydrodynamic Equations.

Continuous Equations

- Continuity Equation:(C.E.) $\frac{d\rho}{dt} = \rho \nabla_i v^i$
- Energy Conservation:(E.C.) $\frac{d\mathcal{U}}{dt} = -\frac{P}{\rho} \nabla_i v^i$
- Momentum Conservation:(M.C.) $\frac{dv_i}{dt} = -\frac{1}{\rho} \nabla_i P$

where $i = x, y, z$ and:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^i \nabla_i \tag{5}$$

Example (cont.)

We will use the following relations:

$$\rho\nabla_i v^i = \nabla_i(\rho v^i) - v^i\nabla_i\rho \tag{6}$$

$$\nabla_i P/\rho = \nabla_i(P/\rho) + P/\rho^2\nabla_i\rho \tag{7}$$

- C.E.:

$$\frac{d\rho}{dt} = \nabla_i(\rho v^i) - v^i\nabla_i\rho \tag{8}$$

- E.C.:

$$\frac{d\mathcal{U}}{dt} = \frac{1}{2}\left[-\nabla_i\left(\frac{Pv^i}{\rho}\right) + v^i\nabla_i\left(\frac{P}{\rho}\right) + \frac{P}{\rho^2}\nabla_i\left(\rho v^i\right) - \frac{Pv^i}{\rho^2}\nabla_i\rho\right] \tag{9}$$

- M.C.:

$$\frac{dv_i}{dt} = \nabla_i\left(\frac{P}{\rho}\right) + \frac{P}{\rho^2}\nabla_i\rho \tag{10}$$

Example (cont.)

Discrete Equations

- C.E.:

$$\frac{d\rho_a}{dt} = \sum_b m_b v^i_{ab} \nabla_{i_a} W_{ab} \tag{11}$$

$$\rho_a = \sum_b m_b W_{ab} \tag{12}$$

- E.C.:

$$\frac{d\mathcal{U}_a}{dt} = \frac{1}{2} \sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} \right) v^i_{ab} \nabla_{i_a} W_{ab} \tag{13}$$

- M.C.:

$$\frac{dv^i_a}{dt} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} \right) \nabla_{i_a} W_{ab} \tag{14}$$

where $v^i_{ab} = v^i_a - v^i_b$ and $W_{ab} = W(\vec{r}_a - \vec{r}_b; h)$.
Moving Particles:

$$\frac{dr^i_a}{dt} = v^i_a \tag{15}$$

Implementation

Initial Conditions

The problem (in 1D) consists in finding the coordinates positions $\{x_b\}$ for N particles such that:

$$< \rho(x) >= \sum_b m_b W(x - x_b; h) \qquad (16)$$

represents a good estimate (order $O(h^2)$) of the initial given density $\rho(x)$.

Some Solutions:

- Monte-Carlo.$\longrightarrow$ poor convergence with N, introduction statistical errors.
- Minimization process. $\longrightarrow$ very expensive.
- Varying Mass.$\longrightarrow$ Not all the particles are treated in the same way.
- Using the mass function $M(x) = \int \rho(x)dx$ and deposit a particle when M increases in $\Delta m = m_b$.

Implementation (cont.)

Neighbors Search.

Don't want to waste time checking for neighbors. A naive implementation will dropped the calculation to be N^2 per time step.

- 1D: sorted lists, and pointer.

- 3D: linked lists.

Implementation (cont.)

Varying Smoothing Length.

- Increases resolution.

- Try to keep the number of neighbors constant

$$\frac{dh_a}{dt} = -\left(\frac{h_a}{\rho_a}\right)\frac{d\rho_a}{dt} \tag{17}$$

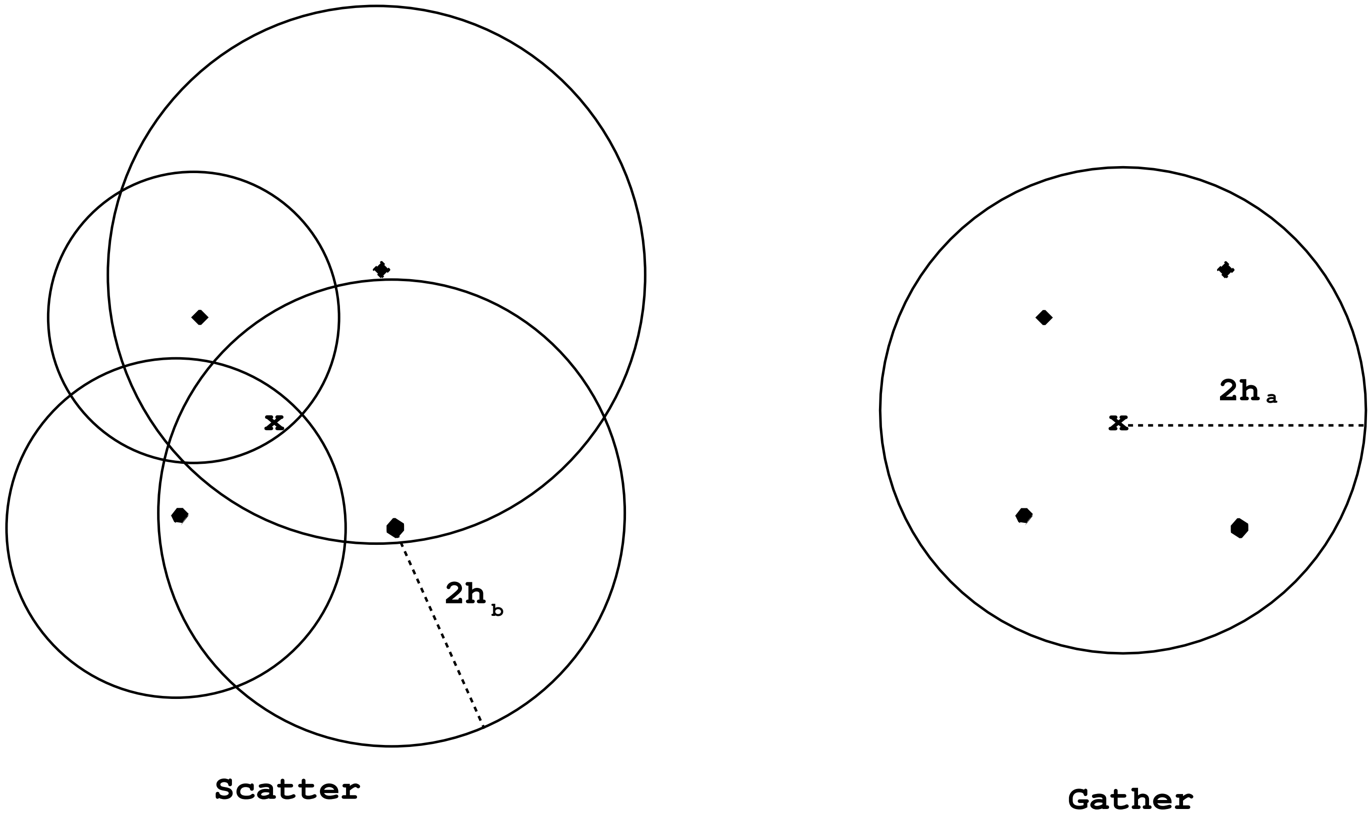
- Discrete Equations of motion should be corrected to keep $O(h^2)$ accuracy (Nelson et al. (1994))

  - $\nabla_i h$ terms.

  - dh/dt terms.

- Gathering & Scattering point of view to search for neighbors.

$$< f(r) >= \int f(r')W(h')dr' \qquad < f(r) >= \int f(r')W(h)dr'$$



$$< f(r) >= \frac{1}{2}\left[\int f(r')W(h')dr' + \int f(r')W(h)dr'\right] \quad (18)$$

Implementation (cont.)

Artificial Viscosity.

Monaghan and Gingold (1983)

$$P_a \longrightarrow P_a + Q_a \tag{19}$$

where

$$Q_a = \frac{1}{2} \sum_b m_b \left(\frac{q_{ab}}{\rho_a} + \frac{q_{ba}}{\rho_b} \right) W_{ab} \tag{20}$$

where:

$$q_{ab} = \begin{cases} \rho_a H_a \left[-\alpha c_a h_a \left(\nabla_i v^i \right)_{ab} + \beta h_a^2 \left(\nabla_i v^i \right)_{ab}^2 \right] & \text{if } \left(\nabla_i v^i \right) < 0 \\ 0 & \text{otherwise} \end{cases}$$

H_a is the enthalpy.
c_a is the speed of sound.
α and β are tunable parameters.

Implementation (cont.)

Boundary Conditions.

?

General Relativistic Work

- Spherical symmetric Collapse. P.J.Mann (1992). Fully relativistic calculation.

- Tidal disruption of stars by supermassive black holes. Laguna, *et al.* (1993). Background calculations. Kheyfets *et al.* (1990) formalism of covariant kernels.

- Equations in Conservation form. S.Siegler, *et al.* (1999). Taylor made for Background calculations. Not strictly 3+1. Flatspace kernels.

Our Approach so far

- Collapse in Spherical Symmetry.

- Romero conservative formulation for the fluid.

- Geometry solved in a non-regular mesh (particles positions)