Critical Collapse

of

Collisioness Matter

(in spherical symmetry)

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Outline

- Introduction: Critical Phenomenae
  & Collisionless matter.

- Equations: Matter & Geometry.

- The Code: Particle Mesh methods.

- Results: Static Solution?, Scaling law?,
  Universality?

Introduction

Critical Phenomena

- Choose initial conditions (parametrized by one parameter p) in so that there exists a value, p^*, of this parameter such that if $p > p^*$ forms a black hole, and if $p < p^*$ the matter implodes and then completely disperses.

- Find the value of p^* and therefore the solution at the tresshold of BH formation choosing an initial bracket $[\mathrm{p}_{\mathrm{lo}}, \mathrm{p}_{\mathrm{hi}}]$ and narrowing it by dynamical evolutions.

Introduction (cont.)

Type II – Infinitesimal BH Mass at Threshold:

- CSS or DSS.
- Scaling Law: $\ln M_{BH} = \gamma \ln |p - p^*|$

Type I – Finite mass-gap:

- Static or periodic.
- Scaling Law: $\tau = \sigma \ln |p - p^*|$

Introduction (cont.)

Collisionless Matter

- Einstein: Static Clusters.
- Oppenhaimer-Snyder: Homogeneous Gravitational Collapse.
- Tolman-Bondi: Non-homogeneous Collapse.
- Shapiro-Teukolsky: Collapse with Angular Momentum.
- Rein-Rendall-Schaeffer: Type I Critical Phenomena.

Equations

Geometry:

Maximal-Areal Coordinates:

- slicing condition: $K = 0$
- radial condition: $b = 1$

$$
\begin{aligned}
ds^2 \; = \; & \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2\, a^2\beta \, dt \, dr + a^2 dr^2 \\
& + r^2 \, d\Omega^2
\end{aligned}
\tag{1}
$$

These coordinates can penetrate apparent horizons.

Equations (cont.)

Einstein equations:

$$G_{ab} = 8\,\pi\,T_{ab} \tag{2}$$

- *Hamiltonian Constraint:*

$$\frac{a'}{a} = \frac{3}{2}a^2 r {K^\theta}_\theta^2 + \frac{1}{2r}\left(1 - a^2\right) + 4\pi r a^2 \rho \tag{3}$$

- *Momentum Constraint:*

$${K^\theta}_\theta{}' = -\frac{3}{r}{K^\theta}_\theta - 4\pi j_r \tag{4}$$

- *Slicing Condition:*

$$\begin{aligned} \alpha'' = \quad & \alpha'\left(\frac{a'}{a} - \frac{2}{r}\right) + \frac{2\alpha}{r}\left(2r\frac{a'}{a} + a^2 - 1\right) \\ & + 4\pi a^2 \alpha\,(S - 3\rho) \end{aligned} \tag{5}$$

- *Evolution of the 3-metric:*

$$\dot{a} \;=\; 2\,\alpha\,a\,{K^\theta}_\theta + (a\,\beta)' \tag{6}$$

$$\beta \;=\; \alpha r {K^\theta}_\theta \tag{7}$$

Equations (cont.)

Matter:

$$\text{if } f(x^a, p_a) = \frac{dN}{dV_p} \quad \text{then} \quad \frac{df}{d\tau} = 0 \tag{8}$$

in spherical symmetry:

$$f(t, x^i, p_i) = f(t, R\, x^i, R\, p_i) \quad R \in SO(3) \tag{9}$$

Therefore:

$$p^t \frac{\partial f}{\partial t} + p^t \frac{\partial r}{\partial t} \frac{\partial f}{\partial r} + p^t \frac{\partial p_r}{\partial t} \frac{\partial f}{\partial p_r} = 0 \tag{10}$$

with:

$$\begin{aligned} \frac{\partial p_r}{\partial t} &= -\alpha \frac{\partial \alpha}{\partial r} p^t + \frac{\partial \beta}{\partial r} p_r + \frac{1}{a^3} \frac{\partial a}{\partial r} \frac{{p_r}^2}{p^t} + \frac{l^2}{p^t r^3} \\ \frac{\partial r}{\partial t} &= \frac{p_r}{a^2 p^t} - \beta \end{aligned}$$

Equations (cont.)

Coupling:

$$T^{\mu\nu} = \sum_i^N \frac{p_i^\mu \, p_i^\nu}{m_i} \delta(\vec{r} - \vec{r}_i(t)) \qquad (11)$$

Approximating $\delta(\vec{r} - \vec{r}_i(t)) \quad \longrightarrow \quad 1/V_i$,

where $V_i = \frac{p^t}{m_i} \; \Delta r \; \int \; \sqrt{-g} \; d\phi \; d\theta$.

Therefore:

$$\rho_i = \frac{1}{4\pi \Delta r a} \frac{\left[\bar{p}^t\right]_i}{r_i^2} \qquad (12)$$

$$\begin{aligned} S_i \equiv {S^r}_{ri} + {S^a}_{ai} &= \frac{1}{4\pi \Delta r a^3} \frac{{[p_r]_i}^2}{[\bar{p}^t]_i \, r_i^2} \\ &+ \frac{1}{4\pi \Delta r a} \frac{[l]_i^2}{r_i^4 \; [\bar{p}^t]_i} \end{aligned} \qquad (13)$$

$$j_{ri} = \frac{1}{4\pi \Delta r a} \frac{[p_r]_i}{r_i^2} \qquad (14)$$

Interpolation: $f = \sum_i^N f_i \; W(r - r_i)$

The Code

The N body problem in Newtonian gravity:

$$\vec{F}_j = - \sum_{i \neq j} \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|^3} (\vec{r}_i - \vec{r}_j) \tag{15}$$

+ Evolution Eqn.

Direct Integration is $O(N^2)$!!!

Particle-Mesh:

$$\begin{aligned} \{m_i, r_i\} &\longrightarrow \rho(r) & (16) \\ \rho(r) &\longrightarrow \nabla^2 \Phi = 4\pi\rho(r) & (17) \\ \Phi &\longrightarrow \vec{F}_j & (18) \end{aligned}$$

+ Evolution Eqn.

The Code (cont.)

- Calculate the set of particles to evolve from a given $f(x^i, p_i)$. Using Monte Carlo simulation.

- Calculate the Stress-Energy tensor on the grid.

- Calculate the geometry integrating the constraint equations.

- Interpolate the values of the Christoffel symbols from grid to the particles positions.

- Update the position of the particles inteegrating the geodesic equations.

The Code (cont.)

- Constraint equations $\longrightarrow$ Second order Finite Difference approximation to the equations. (with appropiate boundary conditions, i.e. regularity, etc,...)

- Evolution equations $\longrightarrow$ `LSODA` with coefficients evaluated at the previous time. FIRST ORDER.

The Code (cont.)

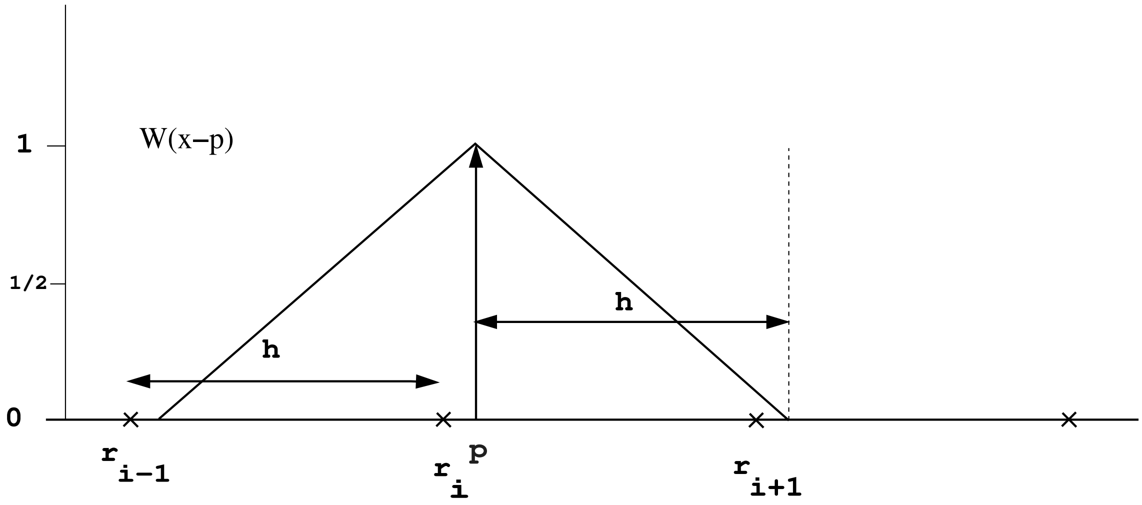
$$W(r-r_p) = \begin{cases} 1-|r-r_p|/h & : \quad |r-r_p| \leq h \\ 0 & : \quad \text{otherwise} \end{cases} \tag{19}$$

To find out the densities:

$$\rho(r) = \sum_{i=1}^{N} \rho_p W(r-r_p) \tag{20}$$

To find out the forces:

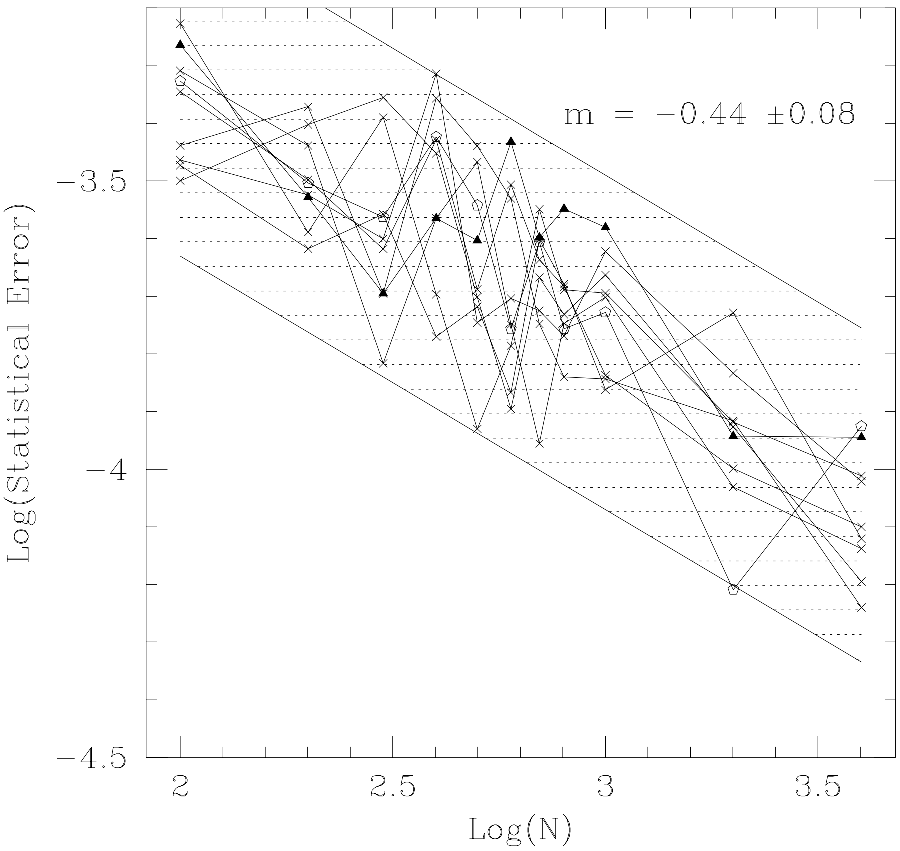
$$F(r_p) = \sum_{i=1}^{N_r} F_i W(r_i - r_p) \tag{21}$$



The Code (cont.)

$$\Delta s_N(t_o, r) = {}^{(1)}\left(2M(t_o, r)/r\right)_N - {}^{(2)}\left(2M(t_o, r)/r\right)_N . \tag{22}$$

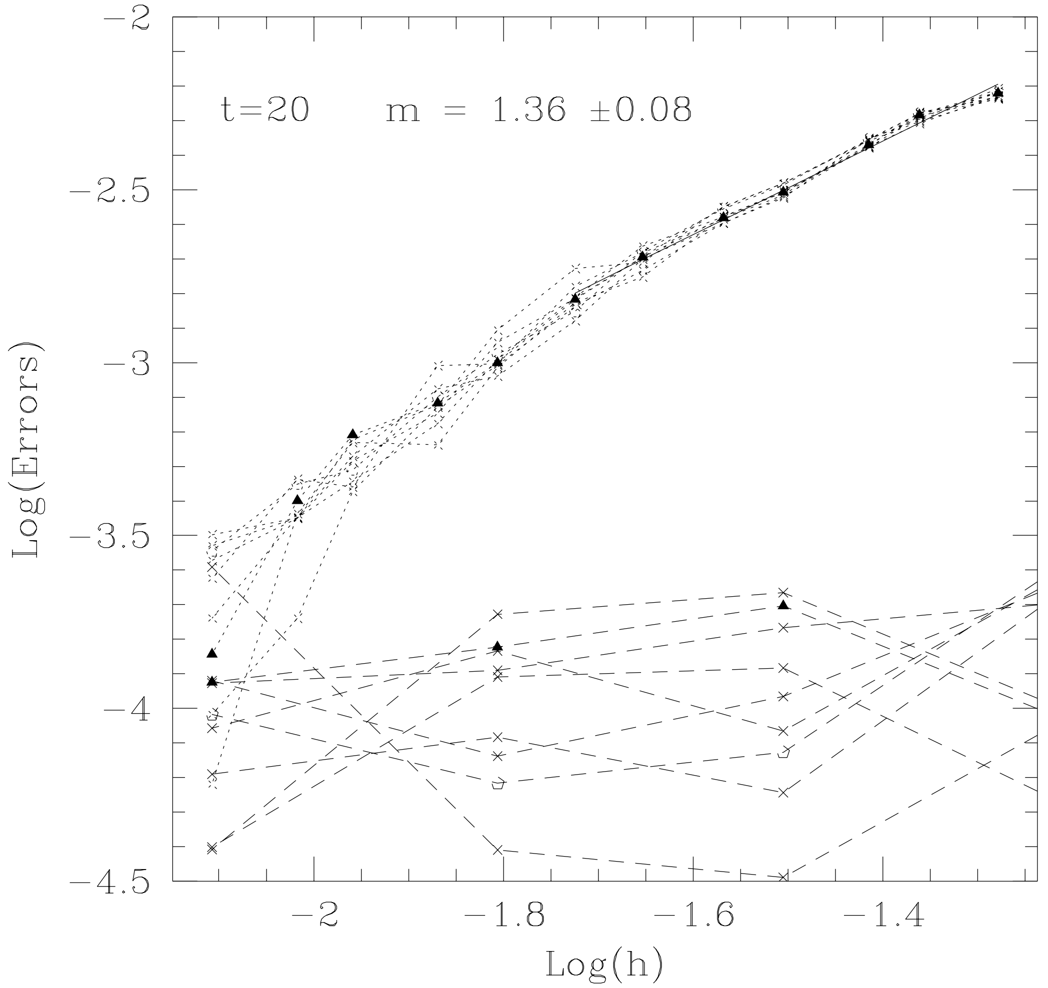
$$\Delta s_N(t_o) = \sqrt{\sum_{i=1}^{N_r} \Delta s_N(t_o, r_i)^2 / N_r} \tag{23}$$



The Code (cont.)

$$\Delta e_h(t_o, r) = q^h(t_o, r) - q^{h'}(t_o, r) \tag{24}$$

$$\Delta e_h(t_o) = \sqrt{\sum_{i=1}^{N_r} \Delta e_h(t_o, r_i)^2 / N_r} \tag{25}$$



Results

Static Solution?

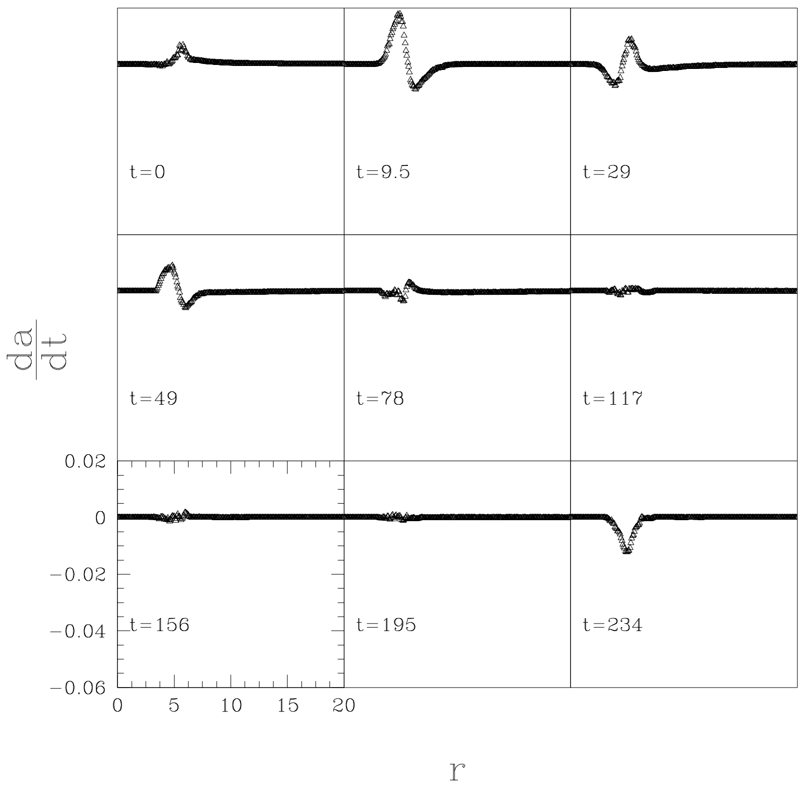
$$f(x^i, p_i) = R(r)\, P(p_r)\, L(l) \tag{26}$$

$$R(r) \propto r^2 e^{(-(r-r_o)^2/\Delta_r^2)}\, \Theta(r) \tag{27}$$

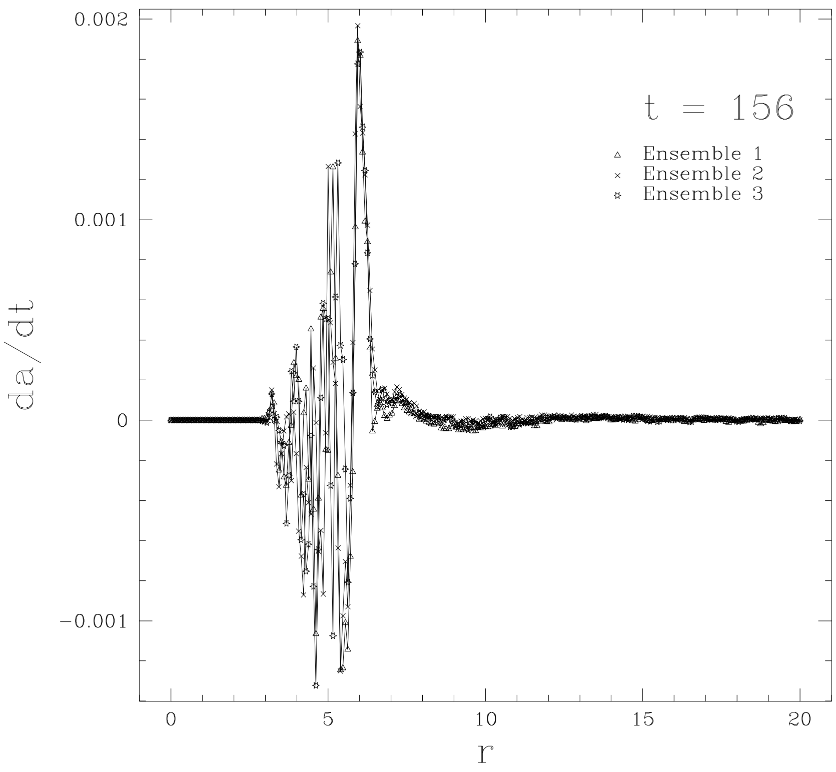
$$P(p_r) \propto e^{\left(-(\bar{p}_r-\bar{p}_{ro})^2/\Delta_{\bar{p}r}{}^2\right)} \tag{28}$$

$$L(l) \propto e^{\left(-(l-l_o)^2/\Delta_l{}^2\right)}\, \Theta(l) \tag{29}$$

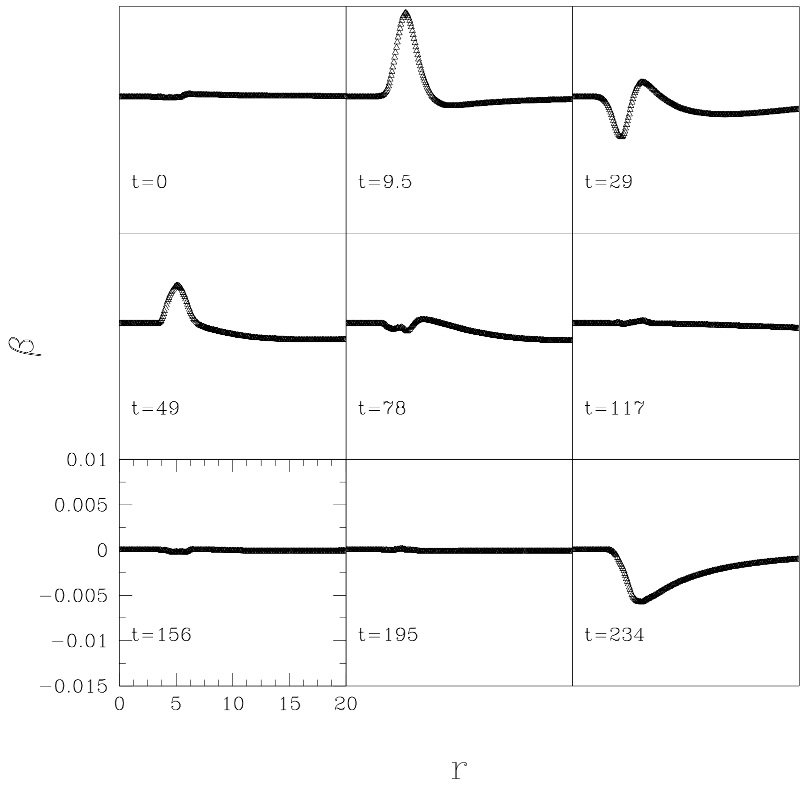
$r_o = 5,\ \Delta_r = 1,\ \bar{p_r}_o = 0,\ \Delta_{\bar{p}_r} = 2,\ l_o = 12,\ \Delta_l = 2$



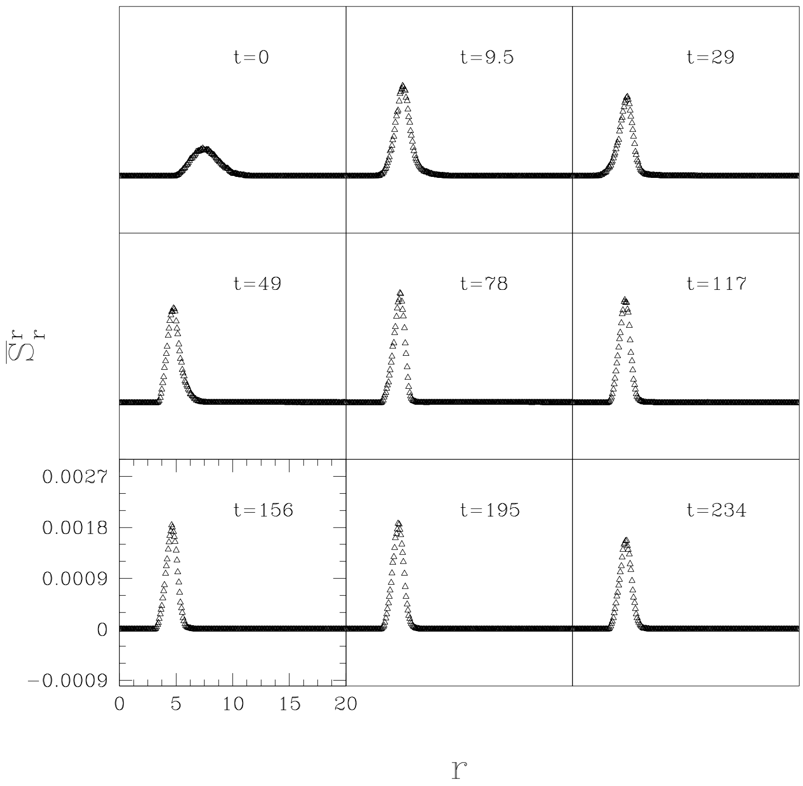
Results (cont.)



Results (cont.)



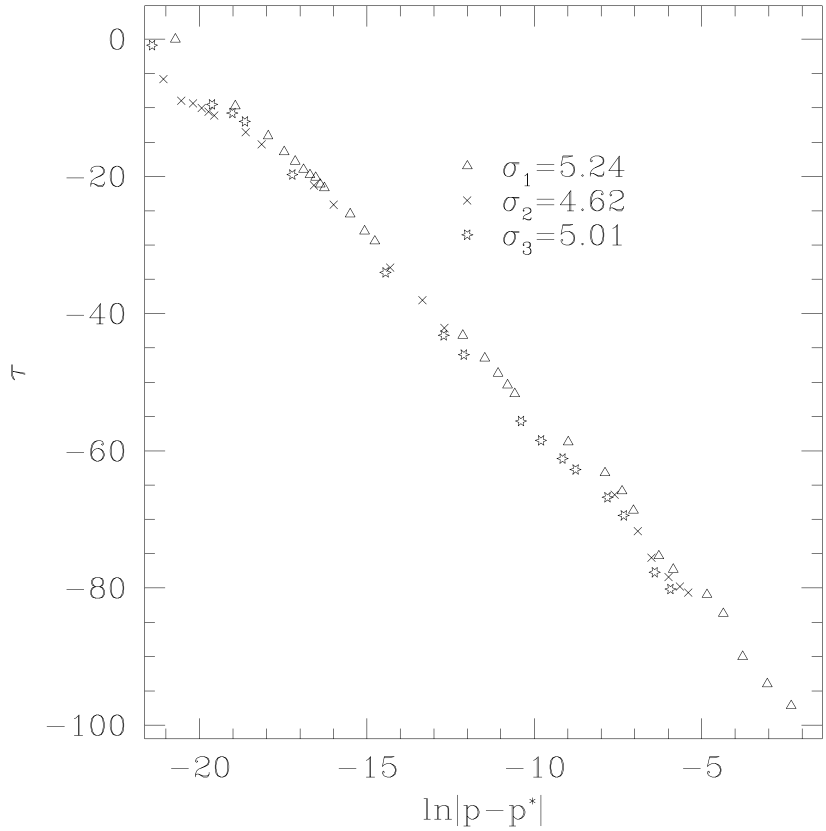
Results (cont.)



Results (cont.)

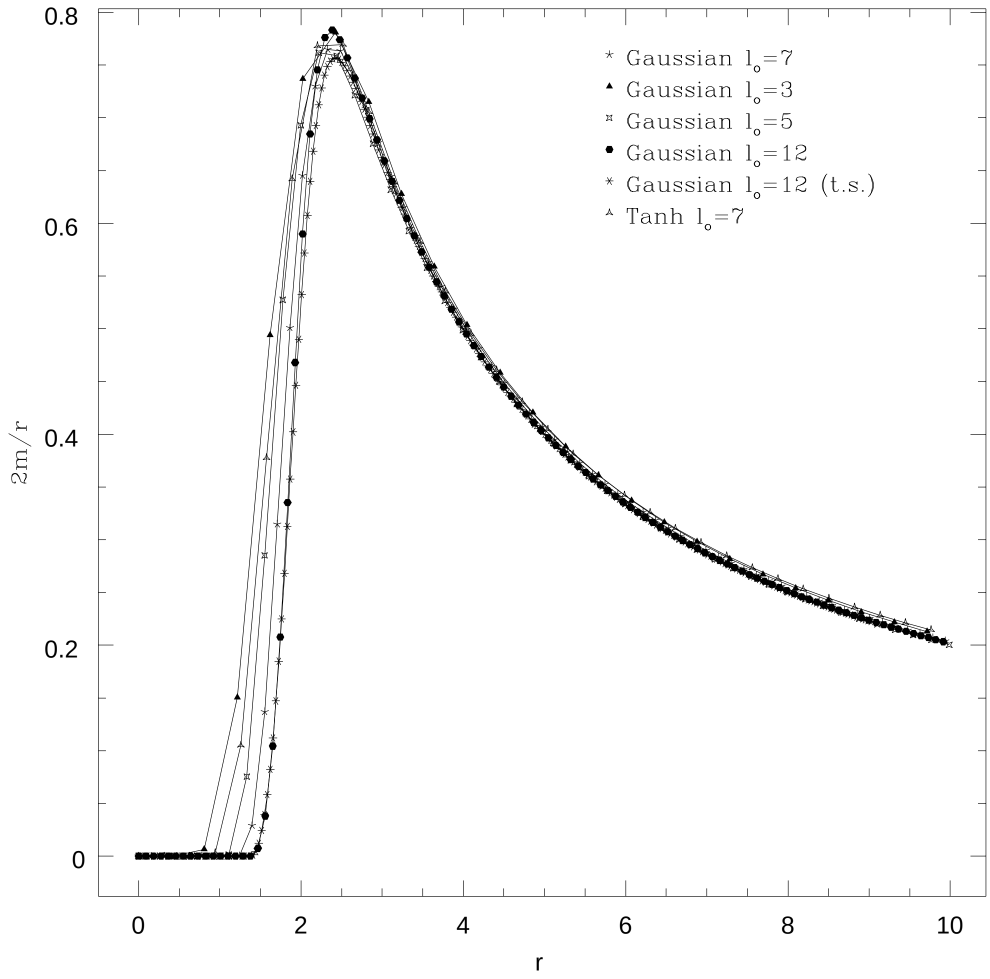
Scaling law

$$\tau = -\sigma \ln|p - p^*| \qquad (30)$$



Results (cont.)

Universality?



Results (cont.)

Universality?

Family	σ
Gaussian, $l_o = 7$ (set 1)	4.7
Gaussian, $l_o = 7$ (set 2)	4.6
Gaussian, $l_o = 3$	5.5
Gaussian, $l_o = 5$	5.0
Gaussian, $l_o = 12$	5.1
Gaussian, $l_o = 12$ t.s. (set 1)	5.2
Gaussian, $l_o = 12$ t.s. (set 2)	4.6
Gaussian, $l_o = 12$ t.s. (set 3)	5.0
Tanh, $l_o = 7$	5.9

$$\sigma = 5.0 \pm 0.5 \tag{31}$$