1 System

The system we are going to study is a collisionless swarm of nonzero restmass particles in spherical symmetry. In this case the meaning of collisionless is that the particles are uncorrelated or that we only take into account long range interactions between them, say the gravitational interactions. This system one of the easiest matter model we can have in GR. It turns out that this model is useful for cosmology since the galaxies behave in these way (see proyect for PHY381C) and also in stellar dynamics (see ST 1985). In this case we use this system as a toy model to investigate critical phenomenae.

In general relativity we represent the gravitation by the curvature of the spacetime. So, the particles are going to evolve freely, since they suffer no other iteration, in the spacetime curved by themselves. The goal is to figer out the metric coefficients of the spacetime because they are going to store all the information about the geometry, or the gravitation iteraction and therefore all the dynamical information of the system.

2 Review of 3+1 ADM Formalism

(This part is just review from the notes for 387N)

If we consider a manifold with a metric g_{ab} defined on it,(M, g_{ab}), we can slice this manifold on the (hiper)surfaces with constant value of a scalar field t (time coordinate). These surfaces are then spacelike and are described by a closed one-form, Ω_a, wich is just the covariant form of the gradient of t:

$$\Omega_a = dt = \nabla_a t \tag{1}$$

where d is the exterior derivative, therefore we have the usual property:

$$d^2 t = \nabla_{[a}\Omega_{b]} = 0 \tag{2}$$

The norm of this one forms is just:

$$g^{ab}\Omega_a\Omega_b = -\alpha^{-2} \tag{3}$$

where α is the lapse function or redshift function that, since Ω is time like, is going to be stricktly positive. Now we can normalize the one-form such that:

$$n_a = -\alpha\Omega_a = -\alpha\nabla_a t \tag{4}$$

Its dual contravariant vector, $n^a = g^{ab}n_b$ can be viewed as the four velocity of an observer moving orthogonally to the slices. Such an observer will have a 4-acceleration:

$$a^b = n^a\nabla_a n^b \tag{5}$$

The full space-time is described by the four dimensional metric g_{ab}, in order to keep the full description but in terms of the folation we will need to split the different tensor quantities in one time-like part, and other which is just the tensor projected onto the hipersurfaces of constant t. To find the time-like part of a contravariant vector:

$$W^n = -W^a n_a \tag{6}$$

and for a covariant one:

$$W_n = W_a n^a \tag{7}$$

For the projection to the hipersurfaces we use the usual projector operator:

$$P^a_b = \delta^a_b + n^a n_b \tag{8}$$

We can find what is the projection of the space-time metric to the hipersurfaces:

$$\gamma_{ab} = P^c_a P^d_b g_{cd} = g_{ab} + n_a n_b \tag{9}$$

Note that γ^{ab} is not necessary the inverse of γ_{ab} because we still raise indexes with the four dimensional metric $g^a b$. Note also that from this three dimensional metric we can define a Riemann tensor $\mathcal{R}_{abc}{}^c$ that is going to represent the intrinsic curvature of the hipersurface. We can define also the spatial Ricc tensor,$\mathcal{R}_{ab}$ and the spatial Ricci scalar, R.
It is also useful to define a natural derivative on the spacelike surfaces, and we can do that projecting the usual four dimensional covariant derivative:

$$D_a = P\nabla_a \tag{10}$$

This new derivative D_a is the natural derivative on the spatial hipersurfaces because it commutes with the spatial metric γ_{ab}. We can define now the extrinsic curvature tensor:

$$K_{ab} = -D_a n_b = \nabla_a n_b - n_a a_b \tag{11}$$

The geometry of the spacetime of the follation is completely defined by the three dimensional metric,γ_{ab}, that tell us how is the space at each time slice, and the extrinsic curvature tensor that tell us how is each slice embeded on the four dimensional spacetime.
Our goal now is to express Einstein equations in terms of this geometric quantities and different projections of the stress energy tensor:

$$\rho = T^{ab} n_a n_b \tag{12}$$
$$j^a = P^a{}_b (T^{bc} n_c) \tag{13}$$
$$S^{ab} = P^a{}_c P^b{}_d T^{cd} \tag{14}$$

These are interpreted as the local energy density, momentum density and spatial stress tensor for an observer moving perpendicular to the space-like hipersurfaces. Now we can write the Einstein equations in terms of source terms, the extrinsic curvature and the 3 dimensional metric defined on each slice, these equations split themselves into the constraint equations at each slice, the evolution equations of the 3 dimensinal metric and the evolution equations of the extrinsic curvature.
We have the Hamiltonian constraint that is found contracting the Einstein equations with n^a twice, $G_{ab} n^a n^b = T_{ab} n^a n^b$:

$$\mathcal{R} + K^2 - K^a{}_b K^b{}_a = 16\pi\rho \tag{15}$$

where K is the trace of the extrinsic curvature and $\mathcal{R}$ is the spatial Ricci scalar.
If we contract the Einstein equation once with n^a and project them into the hipersurface, $P(G^{ab} n_b) = P(T^{ab})$, we get the momentum constraints:

$$D_b K^{ab} - D^a K = 8\pi j^a \tag{16}$$

This two constraint equations involve only spatial tensors and spatial derivatives of spatial tensors, this means that they are satisfied on each space slice independently of the embeding.
In order to write the evolution equations in a general covariant form, we need a way to express the time derivatives in a geometric way without making any reference to any specific frame of reference. The way we are going to

do it is using the Lie derivative along the vector field N^a, where the vector field $N^a = \alpha n^a + \beta^a$ (β^a is an arbitrary spatial vector field). Where we are considering the next normalization for the vector field N^a.

$$N^a \Omega_a = 1 \tag{17}$$

We have the next properties for $\mathcal{L}_{N^a}$

$$\mathcal{L}_{N^a} P \;=\; 0 \tag{18}$$

The Lie derivatie conmutes with the the projection operator. Taking the different projections of the Einstein equations we get:

$$\mathcal{L}_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab} \tag{19}$$

which is the evolution equation for the spatial metric and:

$$\mathcal{L}_t K^a{}_b = \mathcal{L}_\beta K^a{}_b - D^a D_b \alpha + \alpha(\mathcal{R}^a{}_b + K K^a{}_b + 8\pi(\frac{1}{2} P^a{}_b (S - \rho) - S^a{}_b)) \tag{20}$$

which is the evolution equation of the extrinsic curvature.

2.1 3+1 Formalism in spherical symmetry

In this part I will write the constraint equations for spherical coordinates in two different coordinates systems or gauge conditions. The first one with be the areal spatial condition with the polar slicing and the second one will be the again the areal coordinates with the maximal slicing condition.

The most general 3 dimensional metric with spherical symmetry has the form:

$$ds^{3^2} = a^2(t,r)dr^2 + b^2(t,r)d\Omega \tag{21}$$

In matrix form it takes the form:

$$g_{ij} = \begin{pmatrix} a^2(t,r) & 0 & 0 \\ 0 & b^2(t,r)r^2 & 0 \\ 0 & 0 & b^2(t,r)r^2 \sin\theta \end{pmatrix}$$

and its inverse g^{ij} ($g^{ik} g_{kj} = \delta^i_j$):

$$g^{ij} = \begin{pmatrix} \frac{1}{a^2(t,r)} & 0 & 0 \\ 0 & \frac{1}{b^2(t,r)r^2} & 0 \\ 0 & 0 & \frac{1}{b^2(t,r)r^2 \sin\theta} \end{pmatrix}$$

For this metric the connection coefficients of the derivative operator that anhilates the metric ($\nabla^i g_{ij} = 0$) have the usual form:

$$\Gamma^i_{jk} = \frac{1}{2} g^{in} (g_{nj,k} + g_{nk,j} - g_{jk,n}) \tag{22}$$

The non-zero components are:

$$\Gamma^r_{rr} = \tfrac{a'}{a} \quad \Gamma^r_{\theta\theta} = -\tfrac{(r^2b^2)'}{2a^2} \quad \Gamma^r_{\phi\phi} = -\sin^2\theta \tfrac{(r^2b^2)'}{2a^2}$$

$$\Gamma^\theta_{r\theta} = \tfrac{(r^2b^2)'}{2r^2b^2} \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{r\phi} = \tfrac{(r^2b^2)'}{2r^2b^2} \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$

The Ricci tensor would be also useful since it appears on the constraint equations:

$$R_{ij} = \Gamma^n_{ij,n} - \Gamma^n_{in,j} + \Gamma^n_{nm}\Gamma^m_{ij} - \Gamma^n_{jm}\Gamma^m_{in} \tag{23}$$

is the contraction of the Rieman tensor:

$$R_{ij} = R^k_{ikj} \tag{24}$$

where the Rieman tensor is defined as:

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) x^l = R^l_{ijk} x^k \tag{25}$$

for any vector x^k. The non zero components of the Ricci tensor are:

$$R_{rr} = -\left[\tfrac{(r^2b^2)'}{r^2b^2}\right]' + \tfrac{a'}{a}\tfrac{(r^2b^2)'}{r^2b^2} - \tfrac{1}{2}\left[\tfrac{(r^2b^2)'}{r^2b^2}\right]^2$$

$$R_{\theta\theta} = -\left[\tfrac{(r^2b^2)'}{2a^2}\right]' + 1 - \tfrac{a'}{a}\left[\tfrac{(r^2b^2)'}{2a^2}\right]$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$$

the (1,1) form of this tensor is:

$$R^r_r = -\tfrac{2}{arb}\left(\tfrac{(rb)'}{a}\right)'$$

$$R^\theta_\theta = R^\phi_\phi = \tfrac{1}{a(rb)^2}\left[a - \left(\tfrac{rb}{a}(rb)'\right)'\right]$$

Another quatity that is going to be useful is $\alpha_{|i}{}^{|j}$:

$$\alpha_{|i}{}^{|j} = g^{jk}(\alpha_{,ik} - \Gamma^n_{ik}\alpha_{,jn}) \tag{26}$$

The non zero components are:

$$\alpha_{|r}{}^{|r} = \tfrac{1}{a}\left(\tfrac{\alpha'}{a}\right)' \quad \alpha_{|\theta}{}^{|\theta} = \tfrac{\alpha'}{r^2b^2}\tfrac{(r^2b^2)'}{2a^2} \quad \alpha_{|\phi}{}^{|\phi} = \tfrac{(rb)'\alpha'}{rba^2}$$

The Ricci scalar $R = g^{ij}R_{ij}$:

$$R = \frac{2}{arb}\left[-2\left(\frac{(rb)'}{a}\right)' + \frac{a}{rb}\left(1 - \frac{(rb)'^2}{a^2}\right)\right] \tag{27}$$

Embeding the 3 spherical symmetric space into the most generic spacetime result in the next metric:

$$ds^2 = (\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2dr^2 + r^2b^2d\Omega^2 \tag{28}$$

where note that the shift vector only has the radial component different from zero. This is so because it not will break the spherical symmetry. In matrix form:

$$g_{\mu\nu} = \begin{pmatrix} (-\alpha^2 + a^2\beta^2) & a^2\beta & 0 & 0 \\ a^2\beta & a^2 & 0 & 0 \\ 0 & 0 & r^2b^2 & 0 \\ 0 & 0 & 0 & r^2b^2\sin^2\theta \end{pmatrix}$$

and the inverse:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta}{\alpha^2} & 0 & 0 \\ \frac{\beta}{\alpha^2} & (\frac{1}{a^2} - \frac{\beta^2}{\alpha^2}) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2b^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2b^2\sin^2\theta} \end{pmatrix}$$

For this spacetime the components of $n_\mu = (-\alpha, 0, 0, 0)$ and $n^\mu = (\frac{1}{\alpha}, -\frac{\beta}{\alpha}, 0, 0)$.

The extrinsic curvature can be writen in terms of the metric coefficients:

$$-2\alpha K_{ij} = \frac{\partial g_{ij}}{\partial t} - g_{ij,k}\beta^k - g_{ik}\beta^k_{,j} - g_{kj}\beta^k_{,i} \tag{29}$$

which result in the next non zero components:

$$K_{rr} = \tfrac{a}{\alpha}\left((a\beta)' - \dot{a}\right) \quad K_{\theta\theta} = \tfrac{rb}{\alpha}(\beta(rb)' - r\dot{b}) \quad K_{\phi\phi} = \sin^2\theta K_{\theta\theta}$$

and in the (1,1) form:

$$K^r_r = \frac{1}{a\alpha}\left((a\beta)' - \dot{a}\right) \quad K^\theta_\theta = \frac{1}{rb\alpha}(\beta(rb)' - r\dot{b}) \quad K^\phi_\phi = K^\theta_\theta$$

the trace of the extrinsic curvature is then:

$$K = \frac{1}{a\alpha}\left((a\beta)' - \dot{a}\right) + \frac{2}{\alpha}\left(\beta\frac{(rb)'}{rb} - \frac{\dot{b}}{b}\right) \tag{30}$$

the non zero components of the covariant derivative restricted to the hipersurface are:

$$K_{|r} = K_{,r} = -\frac{1}{a\alpha}\left(\frac{a'}{a} + \frac{\alpha'}{\alpha}\right)(a\beta' - \dot{a}) + \frac{1}{a\alpha}\left((a\beta)'' - \dot{a}'\right) - \frac{2}{\alpha^2}\alpha'\left(\beta\frac{(rb)'}{rb} - 1\right)$$

and we could calculate $K^j_{i\,|k} = K^j_{i\,,k} - \Gamma^n_{ik}K^j_n + \Gamma^j_{nk}$but since it only appears on the evolution equations and we will pursuit a constraint evoulation we leave it for some other time (Je ...)

2.2 Polar-Areal Coordinate System.

In this section we will write the equations for an specific choose of coordinates. This choose is areal for the radial coordinate which will imply that the proper surface of the spheres centered at r=0 and with radius r will be $4\pi r^2$ and polar slicing for the time coordinate. This sclicing condition is fixed by impsing that the trace of the extrinsic curvature is equal to the K^r_r component (in other words $K^\theta_\theta = K^\phi_\phi = 0$). This condition fix the way we are going to slice the spacetime. In particular will fix the shift vector to zero:

$$K^\theta_\theta = \frac{1}{rb\alpha}(\beta(rb)' - r\dot{b}) = 0 \tag{31}$$

then:

$$\beta(rb)' = r\dot{b} \tag{32}$$

choosing the areal coordinate set $b = 1$ therefore $\dot{b} = 0$ and the above equation results in:

$$\beta = 0 \tag{33}$$

The resulting metric is:

$$ds^2 = -\alpha^2(t,r)dt^2 + a^2(t,r)dr^2 + r^2 d\Omega \tag{34}$$

and the different geometrical objects:

$$R = \tfrac{2}{ar}\left[-2\left(\tfrac{1}{a}\right)' + \tfrac{a}{r}\left(1 - \tfrac{1}{a^2}\right)\right]$$

$$R^r_r = \tfrac{2}{r}\tfrac{a'}{a^3} \qquad R^\theta_\theta = R^\phi_\phi = \tfrac{1}{ar^2}\left(a - \left(\tfrac{r}{a}\right)'\right)$$

$$K^r_r = K = -\tfrac{\dot{a}}{a\alpha} \qquad K^\theta_\theta = K^\phi_\phi = 0$$

$$K^i_{r|i} = -\left(\tfrac{\dot{a}}{a\alpha}\right)' - \tfrac{2}{r}\left(\tfrac{\dot{a}}{a\alpha}\right) \quad K_{|r} = K^r_{r\,,r} = -\left(\tfrac{\dot{a}}{a\alpha}\right)$$

$$\alpha_{|r}{}^{|r} = \tfrac{1}{a}\left(\tfrac{\alpha'}{a}\right)' \qquad \alpha_{|\theta}{}^{|\theta} = \alpha_{|\phi}{}^{|\phi} = \tfrac{\alpha'}{a^2 r}$$

Momentum Constraint:

$$\dot{a} = -4\pi r a \alpha j_r \tag{35}$$

Hamiltonian Constraint:

$$a' = 4\pi r a^3 \rho - \frac{a}{2r}(a^2 - 1) \tag{36}$$

Evolution of the 3-metric

$$\dot{g}_{ij} = -2\alpha g_{ik} K^k_j \tag{37}$$

without information. (i.e. 0=0)
Evolution of the extrinsic curvature

$$\dot{K}^r_r = -\frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \frac{2\alpha a'}{ra^4} + 4\pi\alpha(S - \rho) - 8\pi\alpha S^r_r \tag{38}$$

$$\dot{K}^\theta_\theta = 0 = -\frac{\alpha'}{a^2 r} + \frac{\alpha}{ar^2}\left(a - \frac{1}{a} + r\frac{a'}{a^2}\right) + 4\pi\alpha(S - \rho) - 8\pi\alpha S^\theta_\theta \tag{39}$$

$$\dot{K}^\phi_\phi = 0 = -\frac{\alpha'}{a^2 r} + \frac{\alpha}{ar^2}\left(a - \frac{1}{a} + r\frac{a'}{a^2}\right) + 4\pi\alpha(S - \rho) - 8\pi\alpha S^\phi_\phi \tag{40}$$

where the two last equations are zero by the choice of slicing.

2.3 Maximal-Areal Coordinate System.

The spatial coordinate condition is, as before, $b = 1$ so the coordinates are going to be areal. For the slicing condition the choice is now $K = 0$ which is going to give us the 1 parameter family of 3 hipersurface with maximum volume [1] when we exhaust the rest of the coordinate freedom. The parameter of this family is going to be our time coordinate. WE have the next condition on the coefficients of the extrinsic curvature: $K^r_r = -2K^\theta_\theta$. We have the next metric:

$$ds^2 = -(\alpha^2 + a^2\beta^2)dt^2 + a^2\beta dtdr + a^2dr^2 + r^2d\Omega \tag{41}$$

where a, α and β are function both of t and r. The important geometrical objects in this coordinates take the next form:

$$R = \tfrac{2}{ar}\left[-2\left(\tfrac{1}{a}\right)' + \tfrac{a}{r}\left(a - \tfrac{1}{a^2}\right)\right]$$

$$R^r_r = -\tfrac{2}{ar}\left(\tfrac{1}{a}\right)' \qquad R^\theta_\theta = R^\phi_\phi = \tfrac{1}{ar^2}\left[a - \left(\tfrac{r}{a}\right)'\right]$$

$$K^r_r = \tfrac{1}{a\alpha}\left((a\beta)' - \dot{a}\right) \qquad K^\theta_\theta = K^\phi_\phi = \tfrac{\beta}{r\alpha}$$

$$K^i_{r|i} = K^r_{r,r} + \tfrac{2}{r}(K^r_r - K^\theta_\theta) \qquad K_{|r} = 0$$

$$\alpha_{|r}{}^{|r} = \tfrac{1}{a}\left(\tfrac{\alpha'}{a}\right)' \qquad \alpha_{|\theta}{}^{|\theta} = \alpha_{|\phi}{}^{|\phi} = \tfrac{\alpha'}{ra^2}$$

Momentum Constraint:

$$\frac{a'}{a} = \frac{3}{2}a^2rK^{\theta\,2}_\theta + 4\pi ra^2\rho + \frac{1}{2r}(1 - a^2) \tag{42}$$

Hamiltonian Constraint:

$$K^{\theta\,\prime}_\theta = -\frac{3}{r}K^\theta_\theta - 4\pi j_r \tag{43}$$

[1] Actually for slices with infinite volume there's no much measning in saying there is one which maximizes the volume since it is not bounded. More exactly the slices choosen by the maximum slicing condition are such that perturbations around the slice in the spacetime leave the volume of the slice unchanged since the volume of the infinitesimaly perturbed slice depends on the trace of extrinsic curvature.

Evolution of the 3-metric:

$$\begin{aligned}\dot{a} &= 2\alpha a K^{\theta}_{\theta} + (a\beta)' && (44)\\ \beta &= \alpha r K^{\theta}_{\theta} && (45)\end{aligned}$$

Evolution of the extrinsic curvature:

$$\alpha'' = \alpha'\left(\frac{a'}{a} - \frac{2}{r}\right) + \frac{2\alpha}{r^2}\left(2r\frac{a'}{a} + a^2 - 1\right) + 4\pi a^2\alpha(S - 3\rho) \qquad (46)$$

3 Coordinate System and Geometric Equations

The most general metric in spherical symmetry is:

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dtdr + a^2dr^2 + r^2b^2d\Omega^2 \qquad (47)$$

In this general coordinate system the field equations are [2]:

Evolution equations for metric coefficients a and b:

$$\begin{aligned}\dot{a} &= -a\alpha K^{r}_{r} + (a\beta)' && (48)\\ \dot{b} &= -\alpha b K^{\theta}_{\theta} + \frac{\beta}{r}(rb)' && (49)\end{aligned}$$

Evolution equations for the extrinsic curvature:

$$\begin{aligned}\dot{K}^{r}_{r} &= \beta(K^{r}_{r})' + \alpha K^{r}_{r}K - \frac{1}{a}(\frac{\alpha'}{a})' - \frac{2\alpha}{arb}(\frac{(rb)'}{a})' + 4\pi\alpha(S - \rho - 2S^{r}_{r})\\ \dot{K}^{\theta}_{\theta} &= \beta(K^{\theta}_{\theta})' + \alpha K^{\theta}_{\theta}K + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}(\frac{\alpha rb}{a}(rb)')' + 4\pi\alpha(S - \rho - 2S^{\theta}_{\theta})\end{aligned}$$

The Hamiltonian constraint:

$$16\pi\rho = -\frac{2}{arb}[(\frac{(rb)'}{a})' + \frac{1}{rb}((\frac{rb}{A}(rb)')' - a)] + 4K^{r}_{r}K^{\theta}_{\theta} + 2{K^{\theta}_{\theta}}^{2} \qquad (50)$$

And the momentum constraint:

$$4\pi j_r = -\frac{(rb)'}{rb}(K^{\theta}_{\theta} - K^{r}_{r}) - {K^{\theta}_{\theta}}' \qquad (51)$$

[2]See Matt's PhD thesis

We are going to work with maximal slicing condition ($K = 0$) and areal coordinates ($b = 1$). This means that in this coordinates the set of points that are at equal distance r of one point will have a surface of $4\pi r^2$. The effect of using $K = 0$ is that the lapse funtion, α, tends to zero rapidly close to the singularity, avoiding it.

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2 d\Omega^2 \tag{52}$$

The equations we get when we set $b = 1$ are:

$$\dot{a} = -a\alpha K_r^r + (a\beta)' \tag{53}$$

$$0 = -\alpha K_\theta^\theta + \frac{\beta}{r} \tag{54}$$

$$\dot{K}_r^r = \beta(K_r^r)' + \alpha K_r^r K - \frac{1}{a}(\frac{\alpha'}{a})' - \frac{2\alpha}{ar}(\frac{1}{a})' + 4\pi\alpha(S - \rho - 2S_r^r) \tag{55}$$

$$\dot{K}_\theta^\theta = \beta(K_\theta^\theta)' + \alpha K_\theta^\theta K + \frac{\alpha}{r^2} - \frac{1}{ar^2}(\frac{\alpha r}{a})' + 4\pi\alpha(S - \rho - 2S_\theta^\theta) \tag{56}$$

$$16\pi\rho = -\frac{2}{ar}[(\frac{1}{a})' + \frac{1}{r}((\frac{r}{A})' - a)] + 4K_r^r K_\theta^\theta + 2K_\theta^{\theta 2} \tag{57}$$

$$4\pi j_r = -\frac{1}{r}(K_\theta^\theta - K_r^r) - K_\theta^{\theta\prime} \tag{58}$$

So we have the evolution equation for a:

$$\dot{a} = -a\alpha K_r^r + (a\beta)' \tag{59}$$

and from the evolution equation for b we get:

$$\beta = -\frac{1}{2} r\alpha K_r^r \tag{60}$$

Now, from the maximal slicing condition we have $K = 0$ which means that $K_r^r + K_\theta^\theta + K_\phi^\phi = 0$ and from the spherical symmetry we have that $K_\theta^\theta = K_\phi^\phi$ therefore $K_\theta^\theta = K_\phi^\phi = -\frac{1}{2}K_r^r$. Plugging all this in the previous equations, we get that the momentum constraint is:

$$K_\theta^{\theta\prime} = -\frac{3}{r}K_\theta^\theta - 4\pi j_r \tag{61}$$

The Hamiltonian constraint is:

$$a' = \frac{a}{2r}(1 - a^2) + \frac{3}{2}a^3 r K_\theta^{\theta 2} + 4\pi a^3 r\rho \tag{62}$$

And finally the slicing condition, or the condition that $K = 0$ at all times (in other words $\dot{K} = \dot{K}^r_r + 2\dot{K}^\theta_\theta = 0$):

$$\alpha'' = \alpha'(\frac{a'}{a} - \frac{2}{r}) + \frac{2\alpha}{r^2}(a^2 - 1 + 2r\frac{a'}{a}) + 4\pi\alpha a^2(S - 3\rho) \tag{63}$$

In summary, the next equations form a complete set of equations to solve for the metric coefficients:

$$\frac{a'}{a} = \frac{1-a^2}{2r} + \frac{3}{2}ra^2K^{\theta 2}_\theta + 4\pi a^2 r\rho \tag{64}$$

$$K^{\theta\prime}_\theta = -\frac{3}{r}K^\theta_\theta - 4\pi j_r \tag{65}$$

$$\alpha'' = \alpha'(\frac{a'}{a} - \frac{2}{r}) + \frac{2\alpha}{r^2}(a^2 - 1 + 2r\frac{a'}{a}) + 4\pi a^2\alpha(S - 3\rho) \tag{66}$$

$$\beta = \alpha r K^\theta_\theta \tag{67}$$

where the source terms are (since $n_\mu = (-\alpha, 0, 0, 0)$):

$$\rho = \alpha^2 T^{tt} \tag{68}$$

$$S = a^2T^{rr} + r^2T^{\theta\theta} + r^2 sin^2\theta T^{\phi\phi} \tag{69}$$

$$j_r = \alpha a^2 T^{tr} \tag{70}$$

It would be useful to compute also de ADM mass of the system. One of the reasons this would be useful is because the conservation of the mass at infinity gives us a strong checking of the reliability of the code and also gives us a stright forward way to compute the apparent horizon. The ADM mass is calculated from the formula:

$$M = \frac{r}{2}(1 + \frac{\beta^2}{\alpha^2} - \frac{1}{a^2}) \tag{71}$$

That is calculated comparing the metric (6) with the Schwarzschild metric:

$$ds^2_s = -(1 - \frac{2M}{r_s})dt^2_s + (1 - \frac{2M}{r_s})^{-1}dr^2_s + r^2_s d\Omega \tag{72}$$

where the 's' subscript stands for Schwarzschild and we write it to stress the point that the Schwarzschild time and our time coordiante are different, note also that the radial coordiantes are the same. Following the same phylosophy as in Shapiro and Teukolsky 1985, we assumed that:

$$dt_s = C(dt + Ddr) \tag{73}$$
$$dr_s = dr \tag{74}$$

Since both radial coordinates are areal, and therefore equal.

Introducing this in the metric (26) we get:

$$ds_s^2 = -(1-\frac{2M}{r})C^2dt^2 - 2(1-\frac{2M}{r})C^2Ddtdr - (1-\frac{2M}{r})C^2D^2dr^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2d\Omega \tag{75}$$

Now comparing equation (29) with the metric (6) we get that:

$$C^2 = -\frac{-\alpha^2 + a^2\beta^2}{1 - \frac{2M}{r}} \tag{76}$$
$$D = \frac{a^2\beta}{-\alpha^2 + a^2\beta^2} \tag{77}$$

comparing the dt^2 and the $dtdr$ term respectively. Now from the dr^2 term we get the equation:

$$\frac{a^4\beta^2}{-\alpha^2 + a^2\beta^2} + \frac{1}{1 - \frac{2M}{r}} = a^2 \tag{78}$$

that after some manipulation can be rewritten as:

$$\frac{2M}{r} = 1 - \frac{1}{a^2} + \frac{\beta^2}{\alpha^2} \tag{79}$$

or equation (25). This last equation is interesting in its own, because when this quatity, $\frac{2M}{r}$, gets equal to 1 we have an apparent horizon.

4 Boundary and Regularity conditions for the Geometric Equations

We are going to consider that the origin $r = 0$ is locally flat. This means that:

$$a(t, r = 0) = 1 \tag{80}$$

Since we required to have regularity at the center of symmetry a(t,r) is going to be an even function which means:

$$\frac{\partial a}{\partial r}(t, r = 0) = 0 \tag{81}$$

This is also true for the lapse function α:

$$\frac{\partial \alpha}{\partial r}(t, r = 0) = 0 \tag{82}$$

K_θ^θ vanishes by regularity of equation (19):

$$K_\theta^\theta(t, r = 0) = 0 \tag{83}$$

And we don't know anything about its derivative. There are still a couple of conditions to have a well pose problem. The next condition is that at infinity we want to have flatspace. Since we cannot specify the conditions at infinity unless we compactify, we can't imposse the condition of asymptotic flatness but what we can do is force that our solution macth Schwarzschild at a finite radius, $r = rmax$:

$$\alpha(t, r = rmax) = \frac{1}{a(t, r = rmax)} \tag{84}$$

Note that the condition in α is a problem with boundary condtions at two points. This can reduced to a problem with conditions at 1 point explotting the linearity of the slicing condition. Setting the value of the lapse function equal to one at the origin ($\alpha(t, r = 0) = 1$) and multiplying the resulting solution by a factor such that α holds (58).

The equations (38)-(41) have terms that go as $O(\frac{1}{r})$ so we will have problems to use these equations at $r = 0$. In order to avoid these problems we can take Taylor series expansions of the metric coefficients, impose the conditions we have at $r = 0$ for these coefficients, and plug these expansions into the equations. The equations at $r = 0$ that we get in this way are:

$$\frac{\partial a}{\partial r} = 0 \tag{85}$$

$$\frac{\partial K_\theta^\theta}{\partial r} = -\pi j \tag{86}$$

$$\frac{\partial^2\alpha}{\partial r^2} = \frac{4}{3}\pi(S_r^r + \rho) \quad (87)$$

$$\frac{\partial\alpha}{\partial r} = 0 \quad (88)$$

$$\beta = 0 \quad (89)$$

where we have also impossed that $S_\theta^\theta = 0$ at the origin since the only particles that can reach the origin are the ones with angular momentum equal to zero.

5 Particle Approach

5.1 Source Terms

We are going to approach the problem in two different ways. In the first one we use a monte carlo to simulate the initial profile of the distribution function we want to evolve. Once we have the swarm of particle that simulates the initial profile we can compute what is the curvature created by these particles and evolved them on that backgroud. We can iteratively solve the problem. This approach has the advantage that we only need to solve ODE's: the geodesic equations and the constraint equations. The big disavantadge is that the error due to the monte carlo is $O(\frac{1}{\sqrt{N}})$, where N is the number of particles that we use to simulate the distribution function. If the critical solution has structure at all scales, and we want to we won't be able to see the finer structure unless we increase the number of particles. Moreover we will need to square the number of particles to have a linear decrease of the noise. This caling behaivour will be crucial in our investigation.

Now in order to compute the stress-energy tensor for the system we use the stress-energy tensor for each particle and sum over the number of particles, since the particles only interact to each other gravitationally:

$$T^{\mu\nu} = \sum_{i=1}^{N} T_i^{\mu\nu} \quad (90)$$

where

$$T_i^{\mu\nu} = p_i^\mu \frac{p_i^\nu}{m_i}\delta(\vec{r} - \vec{r}_i(t)) \quad (91)$$

where p_i^μ is the μ component of the four momentum for the i-th particle, m_i is the rest mass, and $\vec{r}_i(t)$ is the spatial position of the particle at time t.

We can define ρ, S and j_r for each individual particle:

$$\rho_i = \alpha^2 T_i^{tt} \tag{92}$$
$$S_i = a^2 T_i^{rr} + r^2 T_i^{\theta\theta} + r^2 sin^2\theta T_i^{\phi\phi} \tag{93}$$
$$j_r = \alpha a^2 T_i^{tr} \tag{94}$$

To get the finite diference approximation for this expression we aproximate the delta function by $\frac{1}{V_i}$ where V_i is the proper volume of a shell of width Δr,(mesh spacing of the grid), so each particle is assumed to be distributed along one cell of our grid, cloud in cell (CIC) approximation:

$$V_i = \frac{p^t}{m_i}\Delta r \int \sqrt{-g}\, d\phi d\theta = 4\pi \Delta r \alpha a \frac{r_i^2 p_i^t}{m_i} \tag{95}$$

this gives the next source terms:

$$\rho = \frac{\alpha}{4\pi\Delta r a}\sum_{i=1}^{N}\frac{p_i^t}{r_i^2} \tag{96}$$
$$S = \frac{a}{4\pi\Delta r\alpha}\sum_{i=1}^{N}\frac{p_i^{r\,2}}{p_i^t r_i^2} + \frac{1}{4\pi\Delta r\alpha a}\sum_{i=1}^{N}\frac{l_i^2}{r_i^4 p_i^t} \tag{97}$$
$$j_r = \frac{a}{4\pi\Delta r}\sum_{i=1}^{N}\frac{p_i^r}{r_i^2} \tag{98}$$

Using $\alpha p^t = \bar{p}^t$ and p_r, we have:

$$\rho = \frac{1}{4\pi\Delta r a}\sum_{i=1}^{N}\frac{\bar{p}_i^t}{r_i^2} \tag{99}$$
$$S = \frac{1}{4\pi\Delta r a^3}\sum_{i=1}^{N}\frac{{p_r}_i^{\,2}}{\bar{p}_i^t r_i^2} + \frac{1}{4\pi\Delta r a}\sum_{i=1}^{N}\frac{l_i^2}{r_i^4 \bar{p}_i^t} \tag{100}$$
$$j_r = \frac{1}{4\pi\Delta r a}\sum_{i=1}^{N}\frac{{p_r}_i}{r_i^2} \tag{101}$$

Defining the quantities:

$$\bar{\rho} = \frac{1}{4\pi\Delta r}\sum_{i=1}^{N}\frac{\bar{p}_i^t}{r_i^2} \tag{102}$$

$$\bar{S}^r_r = \frac{1}{4\pi\Delta r}\sum_{i=1}^{N}\frac{{p_r}_i^2}{\bar{p}^t_i r_i^2} \quad (103)$$

$$\bar{S}^\theta_\theta = \frac{1}{4\pi\Delta r}\sum_{i=1}^{N}\frac{l_i^2}{r_i^4\bar{p}^t_i} \quad (104)$$

$$\bar{j}_r = \frac{1}{4\pi\Delta r}\sum_{i=1}^{N}\frac{{p_r}_i}{r_i^2} \quad (105)$$

We can rewrite the equations to solve at each slice as:

$$\frac{a'}{a} = \frac{1-a^2}{2r} + \frac{3}{2}ra^2{K^\theta_\theta}^2 + 4\pi ar\bar{\rho} \quad (106)$$

$${K^\theta_\theta}' = -\frac{3}{r}K^\theta_\theta - 4\pi\frac{\bar{j}_r}{a} \quad (107)$$

$$\alpha'' = \alpha'(\frac{a'}{a} - \frac{2}{r}) + \frac{2\alpha}{r^2}(a^2 - 1 + 2r\frac{a'}{a}) + 4\pi a\alpha(\frac{\bar{S}^r_r}{a^2} + \bar{S}^\theta_\theta - 3\bar{\rho}) \quad (108)$$

$$\beta = \alpha r K^\theta_\theta \quad (109)$$

5.2 Interpolation Schemes

(See 'Computer Simulation Using Particles')
When we use equations (77)-(80) we figer out the source terms at the position of the particles. In order to solve the finite difference equations for (81)-(84) we need to find, interpolate, what are the source terms on the grid where we are going to solve these equations. In order to do this we have used two different schemes of interpolation of increasing accuracy: Cloud in Cell (CIC), and Triangular Shaped Cloud (TSC). The first one is an interpolation to two mesh points mean while the last one is to three.

5.2.1 Cloud In Cell

The kernel of this interpolation scheme is:

$$W(x - x_p) = \begin{cases} 1 - \frac{|x-x_p|}{h} & \text{if } |x - x_p| < h \\ 0 & \text{if } |x - x_p| > h \end{cases}$$

5.2.2 Triangular Shaped Cloud

$$W(x-x_p) = \begin{cases} \frac{3}{4} - (\frac{|x-x_p|}{h})^2 & \text{if } |x-x_p| < \frac{h}{2} \\ \frac{1}{2}(\frac{3}{2} - \frac{|x-x_p|}{h})^2 & \text{if } \frac{h}{2} < |x-x_p| < \frac{3h}{2} \\ 0 & \text{if } |x-x_p| > \frac{3h}{2} \end{cases}$$

The same scheme should be used both in the interpolation of the source terms to the grid and in the restriction of the accelerations to the particles in order to avoid 'aliasing' (spurious forces usually due to the breakdown of the ??translational?? symmetry in the system)

5.3 Geodesic Equations

The geodesic equations are, in terms of the four momentum of the particle:

$$p^a \nabla_a p^b = 0 \tag{110}$$

so in our coordinate system, and in not so nice covariant form:

$$\frac{dp^t}{d\tau} = -\Gamma^t_{tt} p^{t2} - 2\Gamma^t_{tr} p^r p^t - \Gamma^t_{rr} p^{r2} - \Gamma^t_{\theta\theta} \frac{l^2}{r^4} \tag{111}$$

$$\frac{dp^r}{d\tau} = -\Gamma^r_{tt} p^{t2} - 2\Gamma^r_{tr} p^r p^t - \Gamma^r_{rr} p^{r2} - \Gamma^r_{\theta\theta} \frac{l^2}{r^4} \tag{112}$$

$$\frac{dp^\theta}{d\tau} = -2\Gamma^\theta_{r\theta} p^r p^\theta - \Gamma^\theta_{\phi\phi} p^\phi \tag{113}$$

where Γ's are the Cristofel symbols. Where we are going to consider that the cuadrimometum as a function of our time coordinate then taking into account that $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}$ and $pt = \frac{dt}{d\tau}$:

$$\frac{dp^t}{dt} = -\Gamma^t_{tt} p^t - 2\Gamma^t_{tr} p^r - \Gamma^t_{rr} \frac{p^{r2}}{p^t} - \Gamma^t_{\theta\theta} \frac{l^2}{p^t r^4} \tag{114}$$

$$\frac{dp^r}{dt} = -\Gamma^r_{tt} p^t - 2\Gamma^r_{tr} p^r - \Gamma^r_{rr} \frac{p^{r2}}{p^t} - \Gamma^r_{\theta\theta} \frac{l^2}{p^t r^4} \tag{115}$$

$$\frac{dp^\theta}{dt} = -2\Gamma^\theta_{r\theta} \frac{p^r p^\theta}{p^t} - \Gamma^\theta_{\phi\phi} \frac{p^{\phi 2}}{p^t} \tag{116}$$

It will be convinient to express these equation in terms of p_r rather than p^r:

$$p_r = g_{r\mu}p^\mu = a^2(t,r)\beta(t,r)p^t + a^2(t,r)p^r \tag{117}$$

Solving for p^r:

$$p^r(t) = \frac{p_r(t)}{a^2(t,r)} - \beta(t,r)p^t(t) \tag{118}$$

We can perform the total derivatives with respect to the coordinate time using the next expresion:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dr}{d\tau}\frac{d\tau}{dt}\frac{\partial}{\partial r} = \frac{\partial}{\partial t} + \frac{p^r}{p^t}\frac{\partial}{\partial r} \tag{119}$$

Applying the operator above to the equation (110):

$$\frac{dp^r}{dt} = \frac{1}{a^2}\frac{dp_r}{dt} - 2\frac{p^r}{a^3}\left(\frac{\partial a}{\partial t} + \frac{p^r}{p^t}\frac{\partial a}{\partial r}\right) - \beta\frac{dp^t}{dt} - p^t\left(\frac{\partial \beta}{\partial t} + \frac{p^r}{p^t}\frac{\partial \beta}{\partial r}\right) \tag{120}$$

Substituting equations (105) and (110) into equation (112) results in:

$$\frac{dp_r}{dt} = -\alpha\frac{\partial \alpha}{\partial r}p^t + \frac{\partial \beta}{\partial r}p_r + \frac{1}{a^3}\frac{\partial a}{\partial r}\frac{{p_r}^2}{p^t} + \frac{l^2}{p^t r^3} \tag{121}$$

we have also the equation for r which can be derivided from (110) taking into account that $p^r = \frac{dr}{d\tau}$ and that $p^t = \frac{dt}{d\tau}$:

$$\frac{dr}{dt} = \frac{p_r}{a^2p^t} - \beta \tag{122}$$

And we have the, Dominant Energy condition, that gives us:

$$\alpha p^t = \sqrt{m^2 + \frac{{p_r}^2}{a^2} + \frac{l^2}{r^2}} \tag{123}$$

It is also convinient, as we said before, use $\bar{p}^t = \alpha p^t$ the resulting geodesic equations in these variables are:

$$\frac{dp_r}{dt} = -\frac{\partial\alpha}{\partial r}\bar{p}^t + \frac{\partial\beta}{\partial r}p_r + \frac{\alpha}{a^3}\frac{\partial a}{\partial r}\frac{{p_r}^2}{\bar{p}^t} + \frac{l^2\alpha}{\bar{p}^t r^3} \tag{124}$$

$$\frac{dr}{dt} = \frac{\alpha p_r}{a^2\bar{p}^t} - \beta \tag{125}$$

$$\bar{p}^t = \sqrt{m^2 + \frac{{p_r}^2}{a^2} + \frac{l^2}{r^2}} \tag{126}$$

$$\frac{dp^\theta}{dt} = -\frac{2}{r}\left(\frac{\alpha p_r}{a^2\bar{p}^t} - \beta\right)p^\theta + \frac{1}{\tan\theta}\frac{\alpha}{\bar{p}^t}\left(\frac{l^2}{r^4} - {p^\theta}^2\right) \tag{127}$$

$$\frac{d\theta}{dt} = \frac{\alpha p^\theta}{\bar{p}^t} \tag{128}$$

$${p^\phi}^2 = \frac{1}{\sin^2\theta}\left(\frac{l^2}{r^4} - {p^\theta}^2\right) \tag{129}$$

$$\frac{d\phi}{dt} = \frac{\alpha p^\phi}{\bar{p}^t} \tag{130}$$

5.4 Initial Data

In order to calculate the stress-energy tensor (or better the source terms for the constraint equations) we need to know the components of the four momemtum for each particle.

We cannot freeely specify the four components of the four momentum and the rest mass of each particle (which we want to keep constant) because we have, (in our variables)

$$\bar{p}^t = \sqrt{m^2 + \frac{p_r^2}{a^2} + \frac{l^2}{r^2}} \tag{131}$$

but we don't know the value if a yet, that is what we want to calculate. In order to avoid that problem we specify $\bar{p}_r = \frac{p_r}{a}$ for which we can calculate $\bar{p}^t$ in a simple way:

$$\bar{p}^t = \sqrt{m^2 + \bar{p}_r^2 + \frac{l^2}{r^2}} \tag{132}$$

where note that now this expression is independent of the geometry itself. We need also to express the constraint equations in terms of $\bar{p}_r$:

$$\frac{a'}{a} = \frac{1-a^2}{2r} + \frac{3}{2} r a^2 K_\theta^{\theta\,2} + 4\pi a r \bar{\rho} \tag{133}$$

$$K_\theta^{\theta\,\prime} = -\frac{3}{r} K_\theta^\theta + 4\pi \bar{\bar{j}}_r \tag{134}$$

$$\alpha'' = \alpha'\left(\frac{a'}{a} - \frac{2}{r}\right) + \frac{2\alpha}{r^2}\left(a^2 - 1 + 2r\frac{a'}{a}\right) + 4\pi a\alpha(\bar{\bar{S}}_r^r + \bar{S}_\theta^\theta - 3\bar{\rho}) \tag{135}$$

$$\beta = \alpha r K_\theta^\theta \tag{136}$$

Where the double bar means that we use the expression (77)-(80) with $\bar{p}^t$ and $\bar{p}_r$. These equations are used only for the initial problem. For the evolution we calculate p^t using the geometry at the previous time.

5.5 Algorithm

- 1. Generate the particles. This means generate the initial position and initial four momentum. See the section about the initial data for more explanation.

- 2. Figer out the source terms,$\bar{\rho}$, $\bar{S}_r$, $\bar{S}_a$, and $\bar{j}_r$ on the grid. To this I use Cloud in Cell (CIC) interpolation that assings the values to the two closest grid points using 'the lever rule' [3].

- 3. Solve the equations (18-21) on the grid with initial conditions at $r = 0$. Right now I rescale the lapse function after I integrate both the hamiltonian constraint and the slicing condition (I use lsoda for the integration).

- 4. I calculate the Γ's at the particle positions. Should use the same CIC interpolation to avoid aliasing (production of unreal forces due to the breaking of translational symmetry).

- 5. Solve the geodesic equations (55-57). This is the most expensive part in CPU time, in order to simulate an insane number of particles ($> 10^6$) it seems clearly to me that it could be done in parallel (at least this part).

[3] The same as in 'the galaxy project'

- 6. Check if the particle has left the range of integration, if so kill it. Check also if the particle has $r < 0$, if so reflecte it.
- 7. Update time and go to step 2.

5.6 Subroutines:

- vlasov: Main driver of the program takes care of the argument parsing and perfoms the time loop. Computes the time step.
- par_gen: Generates a random realization of the particles from the distribution function at the initial time via Monte Carlo simulation. The distribution function is such that produces a gaussian energy density in the radial direction, i.e:

$$\rho(r) = \int f(r') 4\pi r'^2 dr' = e^{-(r-r_o)^2/\sigma_r^2} \tag{137}$$

  and for the other components of the phase space, linear radial momentum and angular momentum uses gaussian distributions in phase space.
- energy: Computes the energy density on the grid using a CIC scheme and outputs the result.
- geometry: Solve the constraint equation using LSODA.
- restriction: Computes the different accelerations on the particles and restric them to the particles using CIC scheme (backwards???)
- geodesic: Solve the geodesic equations using LSODA.
- par_killer: 'Kill' or erase a particle when it leaves the range of integration and correct the number of active particles.
- flatspace: Set the metric coefficients to those of flat spacetime (i.e. $\alpha(t,r) = 1$, $a(t,r) = 1$, $\beta(t,r) = 0$, to use instead of geometry.
- schwarzschild: Set the metric coefficients to those of Schwarzschild solution in these coordinates (i.e. $\alpha(t,r) = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}$, $a(t,r) = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$, $\beta(t,r) = 0$)

5.7 Checkings of the code

In order to properly check the code we have done several testings. The first check is compute the evolution of the swarn of particles moving in flatspace. This check will allow us to check the main driver, the geodesic subroutine, restriction and energy as well as the particle killer subroutine.

In flatspace and due to the fact that the spacetime is static we have a timelike Killing vector field. In particular $\xi_\nu = \frac{\partial}{\partial t}$ is a Killing vector field. If we contract this vector field with the stress energy tensor, $T_{\mu\nu}$ we get a vector field which divergence is zero:

$$\nabla_\mu(T^{\mu\nu}\xi_\nu) = \nabla_\mu(T^{\mu\nu})\xi_\nu + T^{\mu\nu}\nabla_\mu\xi_\nu = 0 \tag{138}$$

The first term of the right hand side is zero by the conservation of the stress-energy tensor and the second is zero due to the fact that ξ^ν satisfy the Killing equation[4] and $T^{\mu\nu}$ is symetric. Integrating this divergence over the four volume we get and applying Gauss Law:

$$\int_V \nabla_\mu(T^{\mu\nu}\xi_\nu)\sqrt{-g}d^4x = \int_{\partial V} T^{\mu\nu}\xi_{\nu\mu}\sqrt{-g^{(3)}}d^3x = 0 \tag{139}$$

where the second integral is an integral over the 3-hypersurface that forms the boundary of the space-time V that we integrate over the first integral. Let's choose this 3-hypersurface to be composed by two hypersurfase with t-constant, let's say t_1 and t_2 then $n_\nu = \frac{\partial}{\partial t}$ fot the surface with t_1 and $n_\nu = -\frac{\partial}{\partial t}$ for t_2 then:

$$\int_{\partial V_{t_1}} T^{tt}\sqrt{-g^{(3)}}d^3x = \int_{\partial V_{t_2}} T^{tt}\sqrt{-g^{(3)}}d^3x \tag{140}$$

and since t_1 and t_2 are arbitrarly this means that the integral is contant for any value of the time. For the case when the spacetime is flat:

$$C = \sum_{i=1}^{N} \frac{p_i^{t\,2}}{m_i} \tag{141}$$

is a conserved quatity which is not a surprise.

For our code this quantity is conserved up to the tolerance of the integrator of the geodesic.

---

[4] $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$

6 Statistical Approach

If we consider that all the particles have the same rest mass we can store all information about the dynamics in a distribution function f. This is a function in phase space. So in general will be function in a 6 dimensional space. In order to calculate any physical magnitude of the system we integrate f over the phase space, we will come to this point when we have to calculate the stress-energy tensor.

If the system is collisionless we have the Lioville theorem about the conservation of the volume in phase space, V, and we also know that the particle number N is also going to be a conserved quatity. Therefore we can write the conservation of the distribution function, $f = \frac{N}{V}$ as:

$$\frac{df}{d\tau} = 0 \tag{142}$$

This equation is known as the "collisionless Boltzmann equation" or the "kinetic equation". We can expand the total derivative which results in:

$$\frac{\partial x^{\alpha}}{\partial \tau}\frac{\partial f}{\partial x^{\alpha}} + \frac{\partial p_{\alpha}}{\partial \tau}\frac{\partial f}{\partial p_{\alpha}} = 0 \tag{143}$$

where x^{α} are the space-time variables and p^{α} are the momentum variables.

To figer out the energy momentum tensor we have to integrate over the mass hyperboloid of the momentum space. In this case we have a restriction over the momentum space because $p^{\alpha}p_{\alpha} = -m$, we want future directed momentum and all the particles have the same mass. So we cannot integrate over the whole momentum space but over the part which this equation holds.

6.1 Source Terms

Check mass factors, some are missing.

$$T^{\mu\nu} = \int p^{\mu}p^{\nu}f(t, r, p^{r}, l)\frac{1}{m}dV_{p} \tag{144}$$

where the volume element in momentum space is restricted by

$$-m^{2} = g_{\alpha\beta}p^{\alpha}p^{\beta} \tag{145}$$

to the next expression:

$$dV_p = m\frac{dp_r dp_\theta dp_\phi}{p^t\sqrt{-g}} \tag{146}$$

We can introduce new variables adapted to the symmetry in the momentum space following ST(1989):

$$\begin{aligned} l^2 &= p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} & (147)\\ \psi &= \tan^{-1}\frac{p_\theta\sin\theta}{p_\phi} & (148) \end{aligned}$$

And $\bar{p}^t$ that is given, as before, by:

$$\bar{p}^t = \sqrt{m^2 + \frac{p_r^2}{a^2} + \frac{l^2}{r^2}} \tag{149}$$

and the volume element in these new variables:

$$dV_p = m\frac{dp_r dl^2 d\psi}{2ar^2\bar{p}^t} \tag{150}$$

We are able now to construct the source terms:

$$\begin{aligned} \rho &= \frac{\pi}{ar^2}\int f(t,r,p_r,l)\sqrt{m^2 + \frac{p_r^2}{a^2} + \frac{l^2}{r^2}}dp_r dl^2 & (151)\\ j_r &= \frac{\pi}{ar^2}\int f(t,r,p_r,l)p_r dp_r dl^2 & (152)\\ S_r^r &= \frac{\pi}{a^3r^2}\int \frac{p_r^2 f(t,r,p_r,l)}{\sqrt{m^2 + \frac{p_r^2}{a^2} + \frac{l^2}{r^2}}}dp_r dl^2 & (153)\\ S_\theta^\theta &= \frac{\pi}{ar^4}\int \frac{l^2 f(t,r,p_r,l)}{\sqrt{m^2 + \frac{p_r^2}{a^2} + \frac{l^2}{r^2}}}dl^2 dp_r & (154) \end{aligned}$$

we could define as before bar quantities as:

$$\begin{aligned} \bar{\rho} &= \frac{\pi}{r^2}\int f(t,r,p_r,l)\sqrt{m^2 + \frac{p_r^2}{a^2} + \frac{l^2}{r^2}}dp_r dl^2 & (155)\\ \bar{j}_r &= \frac{\pi}{r^2}\int f(t,r,p_r,l)p_r dp_r dl^2 & (156) \end{aligned}$$

$$
\begin{aligned}
\bar{S}^r_r &= \frac{\pi}{r^2}\int \frac{p_r^2 f(t,r,p_r,l)}{\sqrt{m^2+\frac{p_r^2}{a^2}+\frac{l^2}{r^2}}}dp_r dl^2 && (157)\\
\bar{S}^\theta_\theta &= \frac{\pi}{r^4}\int \frac{l^2 f(t,r,p_r,l)}{\sqrt{m^2+\frac{p_r^2}{a^2}+\frac{l^2}{r^2}}}dl^2 dp_r && (158)
\end{aligned}
$$

And for these quatities we have that the same set of equations as before: but it is more conveniant to make a change of variables in the integral and define $\bar{p}_r = \frac{p_r}{a}$. With this new definition we get that:

$$
\begin{aligned}
\rho &= \frac{\pi}{r^2}\int f(t,r,\bar{p}_r,l)\sqrt{m^2+\bar{p}_r^2+\frac{l^2}{r^2}}d\bar{p}_r dl^2 && (159)\\
j_r &= a\frac{\pi}{r^2}\int f(t,r,\bar{p}_r,l)\bar{p}_r d\bar{p}_r dl^2 && (160)\\
S^r_r &= \frac{\pi}{r^2}\int \frac{\bar{p}_r^2 f(t,r,\bar{p}_r,l)}{\sqrt{m^2+\bar{p}_r^2+\frac{l^2}{r^2}}}d\bar{p}_r dl^2 && (161)\\
S^\theta_\theta &= \frac{\pi}{r^4}\int \frac{l^2 f(t,r,\bar{p}_r,l)}{\sqrt{m^2+\bar{p}_r^2+\frac{l^2}{r^2}}}d\bar{p}_r dl^2 && (162)
\end{aligned}
$$

And note that the integration limits don't change, since we are integrating over the whole momentum space and a is a positive quantity (This may be taken into account).

With this new variables the equations to solve at each slice are:

$$
\begin{aligned}
\frac{a'}{a} &= \frac{1-a^2}{2r}+\frac{3}{2}ra^2{K^\theta_\theta}^2+4\pi a^2 r\rho && (163)\\
{K^\theta_\theta}' &= -\frac{3}{r}K^\theta_\theta+4\pi a\tilde{j}_r && (164)\\
\alpha'' &= \alpha'(\frac{a'}{a}-\frac{2}{r})+\frac{2\alpha}{r^2}(a^2-1+2r\frac{a'}{a})+4\pi a^2\alpha(S-3\rho) && (165)\\
\beta &= \alpha r K^\theta_\theta && (166)
\end{aligned}
$$

where $\tilde{j}_r = \frac{j_r}{a}$ and it is independent of a.

6.2 Boltzmman Equation

As we mention before the dynamical equation is going to be the Boltzmman equation:

$$\frac{\partial x^\alpha}{\partial \tau}\frac{\partial f}{\partial x^\alpha}+\frac{\partial p_\alpha}{\partial \tau}\frac{\partial f}{\partial p_\alpha}=0 \tag{167}$$

In spherical simmetry this equation reduces to:

$$\frac{\partial f}{\partial t}+\frac{\partial r}{\partial t}\frac{\partial f}{\partial r}+\frac{\partial p_r}{\partial t}\frac{\partial f}{\partial p_r}=0 \tag{168}$$

where $\frac{\partial r}{\partial t}$ and $\frac{\partial p_r}{\partial t}$are given by equations (58) and (57). Resulting:

$$\frac{\partial f}{\partial t}+(\frac{\alpha p_r}{a^2\bar{p}^t}-\beta)\frac{\partial f}{\partial r}+(-\frac{\partial \alpha}{\partial r}\bar{p}^t+\frac{\partial \beta}{\partial r}p_r+\frac{\alpha}{a^3}\frac{\partial a}{\partial r}\frac{{p_r}^2}{\bar{p}^t}+\frac{\alpha l^2}{\bar{p}^t r^3})\frac{\partial f}{\partial p_r}=0 \tag{169}$$

d The distribution function f should follow some boundary conditions.

6.2.1 Conditions at $r = 0$

By simmetry f has to be an even function since it is a scalar field. Therefore:

$$\frac{\partial f}{\partial r}(r=0)=0 \tag{170}$$

Also the for the only value of l that the distribution function doesn't vanish is $l = 0$:

$$f(r=0, l\neq 0)=0 \tag{171}$$

For $r = 0$ the integrals are not well defined, so we need to take into account that.

6.2.2 Initial Value Problem

As in the previous case we have the typical problem in GR and the initial conditions. To find out what are the source terms that appear in the constraint equations we need to know the geometry but we only can know the geometry once we have integrated the constraint equations. In order to go around this problem we won't specify the distribution function by itself but a proper combination that allow us to solve the constraint equations.

Afterwards we can, and we indeed will, calculate the distribution function $f_o(r, p_r, l) = f(t = 0, r, p_r, l)$.

We will specify the funcion $\tilde{f}_o(r, \tilde{p_r}, l)$ where $\tilde{p_r} = \frac{p_r}{a}$. If we specify $\tilde{f}$ we can figer out the source terms by:

$$\rho_o(r) = \frac{\pi}{r^2} \int \tilde{f}_o(r, \tilde{p_r}, l) \sqrt{m^2 + \tilde{p_r}^2 + \frac{l^2}{r^2}} d\tilde{p_r} dl^2 \tag{172}$$

$$\mathrm{J}_{r\,o} = \frac{a\pi}{r^2} \int \tilde{f}_o(r, \tilde{p_r}, l) \sqrt{m^2 + \tilde{p_r}^2 + \frac{l^2}{r^2}} d\tilde{p_r} dl^2 \tag{173}$$

$$S^r_{r\,o} = \frac{\pi}{r^2} \int \frac{\tilde{f}_o(r, \tilde{p_r}, l) \tilde{p_r}^2}{\sqrt{m^2 + \tilde{p_r}^2 + \frac{l^2}{r^2}}} d\tilde{p_r} dl^2 \tag{174}$$

$$S^\theta_{\theta\,o} = \frac{\pi}{r^2} \int \frac{\tilde{f}_o(r, \tilde{p_r}, l) l^2}{\sqrt{m^2 + \tilde{p_r}^2 + \frac{l^2}{r^2}}} d\tilde{p_r} dl^2 \tag{175}$$

We can define, now the bar quatities as before, and change the equations as before in order to absorb all the factors of a that appear on the source terms. In order to find what is the distribution function associated with $\tilde{f}_o$ we can use:

$$f_o(r, p_r, l) = \frac{\tilde{f}_o(r, \tilde{p_r}, l)}{a(r)} \tag{176}$$

relation that can be derived knowing that:

$$N = \int dr \int dl \int dp_r f_o(r, p_r, l) = \int dr \int dl \int d\tilde{p_r} \tilde{f}_o(r, \tilde{p_r}, l) \tag{177}$$

and

$$\int dr \int dl \int d\tilde{p_r} \tilde{f}_o(r, \tilde{p_r}, l) = \int dr \int dl \int dp_r J \tilde{f}_o(r, \tilde{p_r}, l) \tag{178}$$

where J is the jacobian which is equal to $J = \frac{1}{a}$.

6.2.3 Finite Difference Aproximation

The previous equations are solved using RNPL (Rapid Numerical Prototyping language), LSODA and a tridiagonal solver from LAPACK. In order to solve the equations we need a finite diference aproximation to the equations.

Boltzmman Equation The Boltzmman equation is solved in a three plus one grid. Let me use the subindices (i, j, k) for the part of the phase space and the superindex n for the time step, then $g = g^n_{i,j,k}$ will be the value of the grid function $g(r, p_r, l)$ at the point label by (i, j, k) in phase space and at the time step n.

Now if we define the operators:

$$\mu^t_+ g = \frac{1}{2}\left(g^{n+1}_{i,j,k} + g^n_{i,j,k}\right) \quad (179)$$

$$\Delta^t_+ g = \frac{g^{n+1}_{i,j,k} - g^n_{i,j,k}}{\Delta t} \quad (180)$$

$$\Delta^r_o g = \frac{g^n_{i+1,j,k} - g^n_{i-1,j,k}}{2\Delta r} \quad (181)$$

$$\Delta^{p_r}_o g = \frac{g^n_{i,j+1,k} - g^n_{i,j-1,k}}{2\Delta p_r} \quad (182)$$

$$\Delta^l_o g = \frac{g^n_{i,j,k+1} - g^n_{i,j,k-1}}{2\Delta l} \quad (183)$$

we can write the equation in the bulk of the mesh as:

$$\Delta^t_+ f = \mu^t_+ \left[\left(\frac{\alpha_3 p_r}{a_3 \bar{p}^t} - \beta_3\right)\Delta^r_o f + \left(-(\Delta^r_o \alpha_3)\bar{p}^t + (\Delta^r_o \beta_3)p_r + \frac{\alpha_3}{{a_3}^3}(\Delta^r_o a_3)\frac{{p_r}^2}{\bar{p}^t} + \frac{\alpha_3 l^2}{r^3 \bar{p}^t}\right)\Delta^{p_r}_o f\right] \quad (184)$$

In principle we could apply this equations for $1 < i < Nr, 1 < j < Np_r$, $1 < k < Nl$, where Nr is the number of grid points in the r direction, Np_r in the p_r and Nl in the l direction of the phase space.

a_3, α_3, β_3 are the three dimensional extensions of the one dimensional metric functions to the phase space being constant in the p_r, l directions (i.e. $a_3(r, p_r, l) = a(r)\ \forall p_r, l$ and the same for α_3 and β_3). The definition of these extensions of one dimensional function is needed to be consistent with the RNPL syntax.

Boundary Conditions At the Boundaries we cannot use the previous equation either because they need special care analitically or because the grid point defined by $i-1, j, k$ or $i+1, j, k$ (the same in the other two directions) lye outside the rage allowed for the indexes. Now we will explain which are the boundary conditions we use at each of the planes which form the boundary of integration:

i=1 In this case we need to treat differently the case when the angular momentum is zero which is the only case when any case can reach the origin and therefore the only case when the distribution function can be different from zero. So we have that for $1 < k$ the condition $f^n_{1,j,k} = 0$

For the case $k = 1$ we have to split the line in two different parts. In the part with negative linear momentum we use the forward finite diferent approximation for $\frac{\partial}{\partial r}$:

$$D^r_+ \; g = \frac{-3g^n_{i,j,k} + 4g^n_{i+1,j,k} - g^n_{i+2,j,k}}{2\Delta r} \tag{185}$$

in equation (184). For the other half of the line we have the condition:

$$f(0, p_r, 0) = f(0, -p_r, 0) \tag{186}$$

So the distribution function remains with the same value but the linear momentum changes sign, this can be rewritten in finite diference form as:

$$f^n_{1,j,1} = f^n_{1,Nr-j,1} \tag{187}$$

which is set by hand once the RNPL subroutines have been created.

i=Nr This plane has also two different regions. The one with p_r negative and other one with p_r positive. Let us assume that the rage of integration for p_r is symetric (i.e. $\min p_r = -\max p_r$) then for $1 < j < \frac{Np_r}{2}$ the boundary condition is $f^n_{Nr,j,k} = 0$ which means that there are no more particles coming from infinite that the ones we already set within the range of integration.

For $\frac{Np_r}{2} + 1 < j < Np_r$ we use the backwards approximation for the derivative with respect to r

$$D^r_- \; g = \frac{3g^n_{i,j,k} - 4g^n_{i-1,j,k} + g^n_{i-2,j,k}}{2\Delta r} \tag{188}$$

instead of Δ^r_o on equation (184)

j=1 and j=Np$_r$ $f^n_{i,1,k} = 0$ and $f^n_{i,Np_r,k} = 0$ because we assume that the distribution function is of compact support. This ensures the integrability of f over the phase space.

k=1 This case doesn't differ from the case in the bulk of the grid, we use the equation (184) apart of course the line, already discussed, with i=1.

k=Nl $f^n_{i,j,Nl} = 0$ because f has compact support.

Source Terms The integrals for the source terms (172-175) are approximated by (for $1 < i$):

$$\begin{aligned} \bar{\rho}^n_i = \sum_{j=1,k=1}^{Np_r-1,Nl-1} \quad & \frac{\pi}{4}\left(f^n_{i,j,k} + f^n_{i,j+1,k} + f^n_{i,j,k+1} + f^n_{i,j+1,k+1}\right) \\ & \frac{1}{{r_i}^2}\sqrt{1 + \left(\frac{p_{r\,j+1} + p_{r\,j}}{2}\right)^2 \frac{1}{\left(a^{n-1}_i\right)^2} + \left(\frac{l_{k+1} + l_k}{2}\right)^2 \frac{1}{{r_i}^2}} \\ & \left(p_{r\,j+1} - p_{r\,j}\right)\left({l_{k+1}}^2 - {l_k}^2\right) \end{aligned} \tag{189}$$

$$\begin{aligned} \bar{j_r}^n_i = \sum_{j=1,k=1}^{Np_r-1,Nl-1} \quad & \frac{\pi}{4}\left(f^n_{i,j,k} + f^n_{i,j+1,k} + f^n_{i,j,k+1} + f^n_{i,j+1,k+1}\right) \\ & \frac{1}{{r_i}^2}\frac{p_{r\,j+1} + p_{r\,j}}{2} \\ & \left(p_{r\,j+1} - p_{r\,j}\right)\left({l_{k+1}}^2 - {l_k}^2\right) \end{aligned} \tag{190}$$

$$\begin{aligned} \bar{S_r}^{r\,n}_{r\,i} = \sum_{j=1,k=1}^{Np_r-1,Nl-1} \quad & \frac{\pi}{4}\left(f^n_{i,j,k} + f^n_{i,j+1,k} + f^n_{i,j,k+1} + f^n_{i,j+1,k+1}\right) \\ & \frac{\left(\frac{1}{2}(p_{r\,j+1} + p_{r\,j})\right)^2}{{r_i}^2\sqrt{1 + \left(\frac{p_{r\,j+1} + p_{r\,j}}{2}\right)^2 \frac{1}{\left(a^{n-1}_i\right)^2} + \left(\frac{l_{k+1} + l_k}{2}\right)^2 \frac{1}{{r_i}^2}}} \\ & \left(p_{r\,j+1} - p_{r\,j}\right)\left({l_{k+1}}^2 - {l_k}^2\right) \end{aligned} \tag{191}$$

$$\bar{S_{\theta i}^{\theta}}^n = \sum_{j=1,k=1}^{Np_r-1,Nl-1} \frac{\pi}{4}\left(f_{i,j,k}^n + f_{i,j+1,k}^n + f_{i,j,k+1}^n + f_{i,j+1,k+1}^n\right) \frac{\left(\frac{1}{2}(l_{k+1}+l_k)\right)^2}{{r_i}^4\sqrt{1+\left(\frac{{p_r}_{j+1}+{p_r}_j}{2}\right)^2\frac{1}{\left(a_i^{n-1}\right)^2}+\left(\frac{l_{k+1}+l_k}{2}\right)^2\frac{1}{{r_i}^2}}} \left({p_r}_{j+1}-{p_r}_j\right)\left({l_{k+1}}^2-{l_k}^2\right) \quad (192)$$

at $i = 1$ all these quantities are set equal to 0. Note also that the term a_i^{n-1} which appears on the square root inside the integral is not actually the value of a at the previous time step but at the previous iteration.

Geometric Equations In this section we will explain how the equations (163-166) for the geometry are solve numerically. the first two equations, equation163 and 164 are integrated from the origin using the integrator LSODA. The initial conditions to integrate from $r = 0$ are $a(0) = 1$ and $\beta(0) = 0$ the output from the LSODA integrattor is asked to be at the intervals whcih coincide to the ones in the r direction in the grid where we solve for f. The $\bar{\rho}$ and $\bar{j}_r$ are averaged between the value at the current point to the next. i.e. For the integration between point i to point $i+1$ the $\bar{\rho}$ and $\bar{j}_r$ in the formulas are:

$$\bar{\rho} = \frac{1}{2}\left(\bar{\rho}_i + \bar{\rho}_{i+1}\right) \quad (193)$$

$$\bar{j}_r = \frac{1}{2}\left(\bar{j}_{ri} + \bar{j}_{ri+1}\right) \quad (194)$$

Once we have calculated a we can compute the coeficients of the equation for the lapse function.

$$\alpha'' = \alpha'(\frac{a'}{a} - \frac{2}{r}) + \frac{2\alpha}{r^2}(a^2 - 1 + 2r\frac{a'}{a}) + 4\pi a\alpha(\bar{\bar{S}}_r^r + \bar{S}_\theta^\theta - 3\bar{\rho}) \quad (195)$$

Now we can discretize for $1 < i < Nr$ this equation by:

$$
\begin{aligned}
\frac{\alpha_{i+1}-2\alpha_i+\alpha_{i-1}}{h^2} = \quad & \frac{\alpha_{i+1}-\alpha_{i-1}}{2h}\left(\frac{a_{i+1}-a_{i-1}}{2ha_i}-\frac{2}{r_i}\right)+ \\
& \frac{2\alpha_i}{{r_i}^2}\left({a_i}^2-1+2r_i\frac{a_{i+1}-a_{i-1}}{2ha_i}\right)+ \\
& 4\pi a_i\alpha_i\left(\frac{S^r_{r\,i}}{{a_i}^2}+S^\theta_{\theta\,i}-3\rho_i\right)
\end{aligned} \tag{196}
$$

rearraging this equation gives us:

$$
\left(\frac{1}{h^2}+\frac{f_i}{2h}\right)\alpha_{i-1}-\left(\frac{2}{h^2}+g_i\right)\alpha_i+\left(\frac{1}{h^2}-\frac{f_i}{2h}\right)\alpha_{i-1}=0 \tag{197}
$$

where:

$$
f_i = \frac{a_{i+1}-a_{i-1}}{2ha_i}-\frac{2}{r_i} \tag{198}
$$

$$
g_i = \frac{2}{{r_i}^2}\left({a_i}^2-1+2r_i\frac{a_{i+1}-a_{i-1}}{2ha_i}\right)+4\pi a_i\left(\frac{S^r_{r\,i}}{{a_i}^2}+S^\theta_{\theta\,i}-3\rho_i\right) \tag{199}
$$

which we can solve using a tridiagonal solver with the next boundary conditions:

$$
\left(-3+\frac{\frac{1}{h^2}+\frac{f_2}{2h}}{\frac{1}{h^2}-\frac{f_2}{2h}}\right)\alpha_1+\left(4+\frac{-\frac{2}{h^2}-g_2}{\frac{1}{h^2}-\frac{f_2}{2h}}\right)\alpha_2=0 \tag{200}
$$

this equation can be derived from the forward finite differene approximation to $\frac{\partial\alpha}{\partial r}=0$ at $r=0$:

$$
\frac{-3\alpha_1+4\alpha_2-\alpha_3}{2h}=0 \tag{201}
$$

and equation (197) with $i=2$. We also have the outwards boundary condition.

$$
\alpha_{N_r}=\frac{1}{a_{N_r}} \tag{202}
$$

or!!!!

$$
\alpha_{N_r}=\sqrt{1-\left(\frac{2m}{r}\right)^{n-1}_{N_r}} \tag{203}
$$

where the $n-1$ means the value calculated in the previous iteration.

At the first instant of time, I integrate all the geometric equations using LSODA. In order to make the equation for α first order we introduce the auxiliary variable $u = \frac{\partial \alpha}{\partial r}$ with boundary condition

$$\alpha(0) = 1 \tag{204}$$
$$u(0) = 0 \tag{205}$$

Once we have solved all the equation we rescale the value of $\alpha(r)$ to have $\alpha_{N_r} = \frac{1}{a_{N_r}}$, note that we can only do this rescaling since the slicing equation is linear in α. In addition to (193) and (194) we impossed also:

$$S^r_r = \frac{1}{2}(S^r_{r\,i+1} + S^r_{r\,i}) \tag{206}$$
$$S^\theta_\theta = \frac{1}{2}(S^\theta_{\theta\,i+1} + S^\theta_{\theta\,i}) \tag{207}$$

6.2.4 Dissipation

Usually in a finite diference approximation to an hyperbolic equation the stability is mainly dominated by the high frecuency components. That is the case of our finite diference approximation to the Boltzmman equation, above. Since this components of the solution can be troublesome we will follow the phylosophy of Kreiss and Oliger dissipation and try to kill, dump, these components of the solution. We will do the next substitution.

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \frac{\epsilon}{3}(\frac{\partial^4}{\partial^4 r}d^4 r + \frac{\partial^4}{\partial^4 x}d^4 x + \frac{\partial^4}{\partial^4 y}d^4 y) \tag{208}$$

with the aid of the operator:

$$\begin{aligned} Diss(g) = \quad & \frac{g^n_{i+2,j,k} - 4g^n_{i+1,j,k} + 6g^n_{i,j,k} - 4g^n_{i-1,j,k} + g^n_{i-2,j,k}}{\Delta t} \\ & \frac{g^n_{i,j+2,k} - 4g^n_{i,j+1,k} + 6g^n_{i,j,k} - 4g^n_{i,j-1,k} + g^n_{i,j-2,k}}{\Delta t} \\ & \frac{g^n_{i,j,k+2} - 4g^n_{i,j,k+1} + 6g^n_{i,j,k} - 4g^n_{i,j,k-1} + g^n_{i,j,k-2}}{\Delta t} \end{aligned} \tag{209}$$

we can write the final finite diference expresion for the bulk of the grid $(2 < i < Nr - 1, 2 < j < Np_r - 1, 2 < k < Nl - 1)$:

$$\Delta^t_+ f + \frac{\epsilon}{16}\mu^t_+(Diss(f)) =$$

$$\mu^t_+ \left[\left(\frac{\alpha_3 p_r}{a_3 \bar{p}^t} - \beta_3 \right) \Delta^r_o f + \left(-(\Delta^r_o \alpha_3)\bar{p}^t + (\Delta^r_o \beta_3) p_r + \frac{\alpha_3}{{a_3}^3} (\Delta^r_o a_3) \frac{{p_r}^2}{\bar{p}^t} + \frac{\alpha_3 l^2}{r^3 \bar{p}^t} \right) \Delta^{p_r}_o f \right] \tag{210}$$

On the point where we cannot apply this operator (point with either $i = 2, Nr - 1$ or $j = 2, Npr - 1$, or $k = 2, Nl - 1$) we will apply the equation (184).

6.2.5 Compactification(NOT BEING USED!!!)

The linear momentum in the radial direction and the angular mometum are unbounded. In order to allow to them a range from zero to infinity we are going to compactify infinity to a finite value. We need to have a one to one onto application that maps the infinite to a finite range:

$$\begin{aligned} x &= tanh(pr) & (211) \\ y &= tanh(l) & (212) \end{aligned}$$

where x range from -1 to 1 and y range from 0 to 1.
In the case of flatspace and m=1 for the particles, we have the next Boltzmman equation:

$$\frac{\partial f}{\partial t} + \frac{p_r}{\sqrt{1 + p_r^2 + \frac{l^2}{r^2}}} \frac{\partial f}{\partial r} + \frac{l^2}{r^3 \sqrt{1 + p_r^2 + \frac{l^2}{r^2}}} \frac{\partial f}{\partial p_r} = 0 \tag{213}$$

with the mapping previously defined:

$$\frac{\partial f}{\partial t} + \frac{atanh(x)}{\sqrt{1 + atanh(x)^2 + \frac{atanh(y)^2}{r^2}}} \frac{\partial f}{\partial r} + \frac{atanh(y)^2}{r^3 \sqrt{1 + atanh(x)^2 + \frac{atanh(y)^2}{r^2}}} \frac{1}{cosh(x)^2} \frac{\partial f}{\partial x} = 0 \tag{214}$$

Now we will discuss the boundary conditions in this case. We want in order to have f compact suport:

$$f(t, r, x = -1, y) = f(t, r, x = 1, y) = 0 \tag{215}$$

and possibly (NOT SURE YET):

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{216}$$

at $x = \pm 1$ and $y = 1$. The distribution function is an even function of the magnitude of the angular momentum l, so this implies:

$$\frac{\partial f}{\partial l} = \frac{1}{cosh(y)^2}\frac{\partial f}{\partial y} = 0 \tag{217}$$

at $l = 0$ or $y = 0$. These conditions in the momentum space plus the condition to have an even distribution function of r:

$$\frac{\partial f}{\partial r} = 0 \tag{218}$$

and r=rmax, we will use a backwards approximation of the equation.

7 From Particles to Distribution

I think I should explain a little bit how to go from particles to distribution using the Taylor's transport theorem. From the reference on

`http://www.fusion.kth.se/~jaun/`

the Taylor's transport theorem states that if a large number of deterministic orbits evolve according to:

$$dX(t) = v < X(t), t > dt \tag{219}$$

the evolution of particle density distribution $f(x,t)$ is given by the advection equation:

$$\frac{\partial f}{\partial t} + \nabla(vf) = 0 \tag{220}$$

A References