Ultrarelativistic Fluids

Polar-Areal Coordinates:

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2 \tag{1}$$

Hamiltonian Constraint:

$$\frac{a'}{a} = -\frac{1}{2r}\left(a^2 - 1\right) + 4\pi r \tau a^2 \tag{2}$$

Momentum Constraint:

$$\frac{\alpha'}{\alpha} = a^2\left[4\pi r\left(Sv + P\right) + \frac{M}{r^2}\right] \tag{3}$$

Mass aspect function:

$$M = \frac{r}{2}\left(1 - \frac{1}{a^2}\right) \tag{4}$$

Energy Equation:

$$\dot{\tau} + \frac{1}{r^2}\left(r^2 X S\right) = 0 \tag{5}$$

Momentum Equation:

$$\dot{S} + \frac{1}{r^2}\left[r^2 X\left(Sv + P\right)\right] = \Sigma \tag{6}$$

where:

$$\begin{aligned} X &= \frac{\alpha}{a} \\ \Sigma &= \Theta + \frac{2XP}{r} \qquad (7) \\ \Theta &= \alpha a\left[\left(Sv - \tau\right)\left(8\pi r P + \frac{M}{r^2} + P\frac{M}{r^2}\right)\right]. \qquad (8) \end{aligned}$$

In the non-gravitating case $\alpha = 1$, $a = 1$ and $\Theta = 0$. In conservative form:

$$\partial_t \mathbf{q} + \frac{1}{r^2}\partial_r\left(r^2 X \mathbf{f}\right) = \boldsymbol{\psi} \tag{9}$$

where:

$$\mathbf{q} = \begin{bmatrix} \tau \\ S \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} S \\ (Sv + P) \end{bmatrix}, \quad \boldsymbol{\psi} = \begin{bmatrix} 0 \\ \Sigma \end{bmatrix} \tag{10}$$

In order to discretizate (9) we use a finite volume approximation:

$$\bar{\mathbf{q}}_i^{n+1} = \bar{\mathbf{q}}_i^n - \frac{3\Delta t}{r_{i+1/2}^3 - r_{i-1/2}^3}\left[\left(r^2 X \mathbf{F}\right)_{i+1/2}^n - \left(r^2 X \mathbf{F}\right)_{i-1/2}^n\right] + \Delta t\ \bar{\boldsymbol{\psi}}_i^n \tag{11}$$

where:

$$\begin{aligned} \mathbf{F}_{i+1/2} &= \frac{1}{2}\left[\mathbf{f}\left(\tilde{\mathbf{q}}_{i+1/2}^R\right) + \mathbf{f}\left(\tilde{\mathbf{q}}_{i+1/2}^L\right) - \Sigma_\mu |\lambda_\mu| \omega_\mu \boldsymbol{\eta}_\mu\right] \qquad (12) \\ \bar{\boldsymbol{\psi}}_i^n &= \boldsymbol{\psi}\left(\bar{\mathbf{q}}_i^n\right). \qquad (13) \end{aligned}$$

In (12) we have introduced the right and left reconstructed variables ($\tilde{\mathbf{q}}^R$, $\tilde{\mathbf{q}}^L$) they are defined in the following way:

$$\tilde{\mathbf{q}}^L_{i+1/2} = \bar{\mathbf{q}}_i + \boldsymbol{\sigma}_i \left(r_{i+1/2} - r_i\right) \tag{14}$$

$$\tilde{\mathbf{q}}^R_{i+1/2} = \bar{\mathbf{q}}_{i+1} + \boldsymbol{\sigma}_{i+1} \left(r_{i+1/2} - r_{i+1}\right) \tag{15}$$

where $\boldsymbol{\sigma}_{i+1/2}$ is:

$$\boldsymbol{\sigma}_i = \text{minmod}\left(\mathbf{s}_{i-1/2}, \mathbf{s}_{i+1/2}\right) \tag{16}$$

with:

$$\mathbf{s}_{i+1/2} = \frac{\bar{\mathbf{q}}_{i+1} - \bar{\mathbf{q}}_i}{r_{i+1} - r_i} \tag{17}$$

and

$$minmod(a,b) = \begin{cases} 0 & \text{if} \quad ab < 0 \\ a & \text{if} \quad |a| < |b| \text{ and } ab > 0 \\ b & \text{if} \quad |a| > |b| \text{ and } ab > 0 \end{cases} \tag{18}$$

In equation (12) we also use the caracteristics of the Riemman problem at the $i+1/2$ interface (λ_μ, ω_μ, $\boldsymbol{\eta}_\mu$). Given the *jacobian* matrix:

$$\mathbf{A}\big|_{i+1/2} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right|_{\mathbf{q}=1/2\,(\tilde{\mathbf{q}}^L_{i+1/2}+\tilde{\mathbf{q}}^R_{i+1/2})} \tag{19}$$

λ_μ are the eigenvalues of the $\mathbf{A}$ matrix:

$$\lambda_\pm = \frac{1}{2}\left[A^1{}_1 + A^2{}_2 \pm \sqrt{(A^1{}_1 - A^2{}_2)^2 + 4A^1{}_2 A^2{}_1}\right]. \tag{20}$$

$\boldsymbol{\eta}_\mu$ are the left eigenvectors associated with the eigenvalues λ_μ:

$$\boldsymbol{\eta}_\pm = \begin{bmatrix} 1 \\ Y_\pm \end{bmatrix} \qquad T_\pm = \frac{\lambda - A^1{}_1}{A^1{}_2} \tag{21}$$

and the jump in the characteristic[1] variables:

$$\omega_+ = \frac{1}{d}\left[\eta_-^{(2)}\left(\tilde{q}^{R(1)} - \tilde{q}^{L(1)}\right) + \eta_-^{(1)}\left(\tilde{q}^{L(2)} - \tilde{q}^{R(2)}\right)\right] \tag{22}$$

$$\omega_- = \frac{1}{d}\left[\eta_+^{(1)}\left(\tilde{q}^{R(2)} - \tilde{q}^{L(2)}\right) + \eta_+^{(2)}\left(\tilde{q}^{L(1)} - \tilde{q}^{R(1)}\right)\right] \tag{23}$$

$$\tag{24}$$

where

$$d = \eta_+^{(1)}\eta_-^{(2)} - \eta_-^{(1)}\eta_+^{(2)}. \tag{25}$$

In equations (20) and (21) we have used the matrix components $A^i{}_j$ which are:

$$\begin{array}{ll} A^1{}_1 = 0 & A^1{}_2 = 1 \\ A^2{}_1 = -v^2 + (1-v^2)\,\frac{\partial P}{\partial \tau} & A^2{}_2 = 2v + (1-v^2)\,\frac{\partial P}{\partial S} \end{array} \tag{26}$$

where

$$\frac{\partial P}{\partial \tau} = -2\beta + \frac{(4\beta^2 + \Gamma - 1)\,\tau}{\left[4\beta^2\tau^2 + (\Gamma-1)\,(\tau^2 - S^2)\right]^{1/2}} \tag{27}$$

$$\frac{\partial P}{\partial S} = -\frac{(\Gamma - 1)\,S}{\left[4\beta^2\tau^2 + (\Gamma-1)\,(\tau^2 - S^2)\right]^{1/2}} \tag{28}$$

[1]In this case characteristic means diagonalized.

with $\beta = 1/4\,(2-\Gamma)$.

$$\begin{aligned}
S &= v\,(\tau + P) && (29)\\
\tau &= (\rho + P)\,W^2 - P && (30)\\
P &= (\Gamma - 1)\,\rho && (31)\\
\rho &= \frac{\tau + P}{W^2} - P && (32)\\
W^2 &= \frac{1}{1 - S^2} && (33)\\
0 &= \frac{(\Gamma - 1)}{(\tau + P)}\left[(\tau + P)^2 - S^2\right] - \Gamma P && (34)\\
P &= -\frac{1}{2}\,(2-\Gamma) + \frac{1}{2}\sqrt{\tau^2\,(2-\Gamma)^2 + 4\,(\Gamma - 1)\,(\tau^2 - S^2)} && (35)
\end{aligned}$$

Reconstruction of the primitive variables: [2]

$$\begin{aligned}
\tau &= \frac{1 + (\Gamma - 1)\,v^2}{1 - v^2}\rho && (36)\\
S &= \Gamma\rho\frac{v}{1 - v^2} && (37)
\end{aligned}$$

Jacobian:

$$\begin{array}{ll}
\frac{\partial \tau}{\partial \rho} = 1 + \Gamma\frac{v^2}{1-v^2} & \frac{\partial \tau}{\partial v} = \frac{2\Gamma v \rho}{(1-v^2)^2} \\
\frac{\partial S}{\partial \rho} = \Gamma\frac{v}{1-v^2} & \frac{\partial S}{\partial v} = \Gamma\rho\frac{1+v^2}{(1-v^2)^2}
\end{array} \qquad (38)$$

Update of the equations:

$$\begin{aligned}
\mathbf{q}^{n+1/2} &= \mathbf{q}^n + \frac{\Delta t}{2} L(\mathbf{q}^n) && (39)\\
\mathbf{q}^{n+1} &= \mathbf{q}^n + \Delta t L(\mathbf{q}^{n+1/2}) && (40)
\end{aligned}$$

[2]Note that this system can be set into only 1 non-linear equation for v