Status Report

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1 Description of the Project

In this thesis we are going to numerically solve the hydrodynamic equations for a fluid coupled with the Einstein equations in axisymmetric spacetimes. There are multiple astrophysical situations that can be appropriately modeled using general relativistic hydrodynamics. Examples are: neutron/quark star structure, collapse of neutron stars into black holes, merger of neutron stars, neutron star/black hole collisions, supernovae explosions, etc,... These are only some of the possible applications.

Einstein equations can be written as:

$$G_{ab} = 8\pi T_{ab}{}^{1}, \tag{1}$$

where G_{ab} is the Einstein tensor, and T_{ab} is the stress energy tensor for a fluid (see below). For perfect fluids the hydrodynamical equations may be derived from

$$\nabla_b T^{ba} = 0, \tag{2}$$

and the conservation of baryon (particle) number

$$\nabla_b J^b = 0. \tag{3}$$

In the last equation $J^b = \rho_o u^b$ where ρ_o is the rest mass density and u^b is the four velocity of the fluid. Despite the innocent appearance of the equations (1), (2), (3) we have a set of highly nonlinear coupled partial differential equations that in general need to be solved on a computer. In addition, we will enforce a simplifying condition which is the existence of a rotational Killing vector field ξ^a (i.e. axisymmetric spacetimes). This condition is not overly restrictive, most of the previous applications can be studied in some of the scenarios mentioned above. These spacetimes are also general enough to allow the study of the gravitational radiation produced in these scenarios. With this assumption the equations simplify significantly. Moreover the computational resources to solve them are much less than in the general case which allow us to obtain more accurate solutions.

One of the aims of these thesis is to use realistic equations of state, and to incorporate the microphysics needed to appropriately model some of the astrophysical situations above. In the rest of these notes we introduce some of the previous work done in relativistic hydrodynamics and present some of the working notes we have found useful to understand the work so far.

[1] We will use units such that $G = 1$ and $c = 1$.

2 Previous work

Numerical relativistic hydrodynamics goes back to the pioneering work in spherical symmetry by May and White [1]. They developed a Lagrangian code (the coordinates follow the movement of the fluid) to study spherical collapse in general relativity. The discretization was done using finite differences. In order to control the spurious oscillations produced in the vicinity of shocks they added artificial viscosity [2]. Their formulation became very popular and it was used to study neutron star collapse and supernovae explosions in spherical symmetry. The main problem with this approach is that the Lagrangian character of the code makes it difficult to generalize the method to higher dimensions, where the matter follows complicated paths which can "tangle" the coordinates.

In [3] Wilson introduced a new Eulerian (the coordinates are fixed and the fluid moves with respect to them) formalism. The hydrodynamic equations were written as a set of advection equations and also were discretized using finite differences. As before the introduction of artificial viscosity was fundamental in order to keep the code stable. This formalism was used (and still is) widely and multiple astrophysical situations, including axisymmetric ones have been modeled: accretion onto Kerr black holes [3], [4], collapse of rotating neutron stars [5], [6], [7], [8], neutron star mergers [9] or [10] (which uses a certain approximation to the full set of Einstein equations), etc,...

Special mention should be made of the work of Nakamura [11], because it is particularly relevant to our present work. Nakamura uses the so-called 2+1+1 formalism for treatment of the geometry; (while still using Wilson formalism for the hydrodynamic variables); this approach was first introduced by Geroch [12] and Maeda et al. [13]. In this formalism the spacetime is first decomposed into pieces parallel and orthogonal to the rotational Killing vector field; this produces effective "matter" fields from the geometry in the same way that a Kaluza-Klein reduction does. Second, the remaining three-dimensional spacetime is similarly decomposed with respect to a timelike direction which results in a set of evolution and constraint equations for the metric coefficients. We will explain this reduction in more detail below (probably sometime in the future). Nakamura and collaborators used this formalism to study the collapse of axisymmetric stars [14], [15].

An important property of the hydrodynamic equations is that they can be cast into what is called conservation law form, i.e. they can be written as

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = \mathbf{S}, \tag{4}$$

where $\mathbf{Q}$ is a state vector for the fluid, $\mathbf{F}$ are the corresponding fluxes and $\mathbf{S}$ are the source vector, in which no derivatives of the fluid variables appear. This was firstly exploited by the *Valencia group* in [16] (who also use an Eulerian formalism). Casting the equations in this form allows us to use the Godunov approach (see [17] for a summary about conservative methods) which ensures the correct jumps across the discontinuities of the variables (for that reason these methods are sometimes called High Resolution Shock Capturing, HRSC, methods). One advantage with respect to Wilson formulation is that these methods *don't* require the use of artificial viscosity for code stability. The methods also allow one to simulate fluids with higher γ factors ($\gamma = 1/\sqrt{1-v^2}$). These methods have been used very successfully in spherical symmetry ([18], [19] are only a couple of examples) and are now being used in fully 3-dimensional simulations [20].

In our case we have decided to adapt the Valencia formulation to the 2+1+1 decomposition of the spacetime to treat the hydrodynamic equations. To date this has not been done.

3 Hydrodynamic Equations in Conservation form

In this section we derive the hydrodynamical equations in such a way that they can easily be written in conservation form. Starting from the stress-energy tensor in the form ${T^\mu}_\nu$ and knowing that the divergence has to vanish by (2), we have

$$\begin{aligned}
&\nabla_\mu \left({T^\mu}_\nu\right) = \\
&\nabla_\mu \left(g_{\delta\nu} T^{\mu\delta}\right) = \\
&T^{\mu\delta}\nabla_\mu g_{\delta\nu} + g_{\delta\nu}\nabla_\mu T^{\mu\delta} = \\
&g_{\delta\nu}\left[\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}T^{\mu\delta}\right) + \Gamma^\delta_{\mu\lambda}T^{\mu\lambda}\right] = \\
&g_{\delta\nu}\left[\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\xi\delta}{T^\mu}_\xi\right) + \Gamma^\delta_{\mu\lambda}T^{\mu\lambda}\right] = \\
&g_{\delta\nu}\left[\partial_\mu\left(g^{\xi\delta}\right){T^\mu}_\xi + g^{\xi\delta}\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}{T^\mu}_\xi\right) + \Gamma^\delta_{\mu\lambda}T^{\mu\lambda}\right] = 0.
\end{aligned} \tag{5}$$

In the above expressions ∇_μ is the covariant derivative associated with the metric $g_{\mu\nu}$, g is the determinant of that metric, $\Gamma^\delta_{\mu\lambda}$ are the Christoffel symbols and ∂_μ denotes partial differentiation with respect to the coordinate x^μ. Further manipulation yields:

$$\begin{aligned}
g_{\delta\nu}g^{\xi\delta}\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}{T^\mu}_\xi\right) &= -g_{\delta\nu}\partial_\mu\left(g^{\xi\delta}\right){T^\mu}_\xi - \Gamma^\delta_{\mu\lambda}T^{\mu\lambda}g_{\delta\nu} \\
g_{\delta\nu}g^{\xi\delta}\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}{T^\mu}_\xi\right) &= \left[-\partial_\mu\left(g_{\delta\nu}g^{\xi\delta}\right) + g^{\xi\delta}\partial_\mu\left(g_{\delta\nu}\right)\right]{T^\mu}_\xi - \Gamma^\delta_{\mu\lambda}T^{\mu\lambda}g_{\delta\nu} \\
{\delta^\xi}_\nu\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}{T^\mu}_\xi\right) &= g^{\xi\delta}\partial_\mu\left(g_{\delta\nu}\right){T^\mu}_\xi - \Gamma^\delta_{\mu\lambda}T^{\mu\lambda}g_{\delta\nu} \\
\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}{T^\mu}_\nu\right) &= \partial_\mu\left(g_{\delta\nu}\right)T^{\mu\delta} - \Gamma^\delta_{\mu\lambda}T^{\mu\lambda}g_{\delta\nu},
\end{aligned} \tag{6}$$

or, after relabelling the dummy indexes,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}{T^\mu}_j\right) = \partial_\mu\left(g_{\nu j}\right)T^{\mu\nu} - \Gamma^\delta_{\mu\nu}T^{\mu\nu}g_{\delta j}. \tag{7}$$

Choosing the fluid variables in the correct way, we can see that this expression corresponds to the equations of the 3+1 Valencia formalism (see [21]). Note also that in the right hand side we don't have derivatives of the stress energy momentum tensor, only derivatives of the metric coefficients, and therefore we don't have derivative of the fluid variables. Equation (3) is almost trivial to write in conservation form:

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}J^\mu\right) = 0 \tag{8}$$

4 Roe Solver

In this section we explain how the Roe solver (the Godunov type solver that we use to integrate the hydrodynamic equations) works for a scalar conservation equation. This easy example will help us understand the more difficult case of a *system* of conservation laws. Consider the conservation law

$$\frac{\partial q(t,x)}{\partial t} + \frac{\partial F(t,x)}{\partial x} = 0 \tag{9}$$

Note that in this case the state vector is just a scalar and the problem is one dimensional. We can rewrite (9) in integral form:

$$\int_\Omega \frac{\partial q}{\partial t} d\Omega + \int_\Omega \frac{\partial F}{\partial x} d\Omega = 0 \tag{10}$$

where Ω is some small region of the spacetime, that will be one of the cells in our discretization of the spacetime. Writing the discretization of this equation in terms of cell averages we get:

$$(\bar{q}_i \Delta x)_{t+\Delta t} - (\bar{q}_i \Delta x)_t = -\left(\int F_{i+1/2} dt - \int F_{i-1/2} dt\right), \tag{11}$$

where the cell averages are defined by

$$\bar{q}_i = \frac{1}{\Delta x} \int_{i-1/2}^{i+1/2} q(x) dx. \tag{12}$$

In order to find the integrals of time for the fluxes, we use an idea due to Godunov. We can think about our discretization $\bar{q}_i$ being a piecewise constant reconstruction of the solution function $q(x)$ (in practice we use a piecewise linear reconstruction). Then at every cell boundary we have a Riemann problem (discontinuity in the fluid variable). In order to estimate the flux in equation (11) we can write the flux as $F(q^*)$ where q^* is the solution (at the cell boundary) to the approximate problem given by:

$$\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = 0 \tag{13}$$

where the Jacobian, $A = \partial F / \partial q$, is constant (i.e. we linearize the full equation). The solution of equation (13) with initial data given in Figure 1 is straightforward. The characteristics of equation (13) are given by $x - At = k$ where k is a constant. Therefore q^*:

$$q^* = \begin{cases} q_R & \text{if} A < 0 \\ q_L & \text{if} A > 0 \end{cases} \tag{14}$$

and we note that it is a constant on the time interval under consideration. We can write (14) as:

$$q^* = \begin{cases} q_L + \omega & \text{if} A < 0 \\ q_R - \omega & \text{if} A > 0 \end{cases} \tag{15}$$

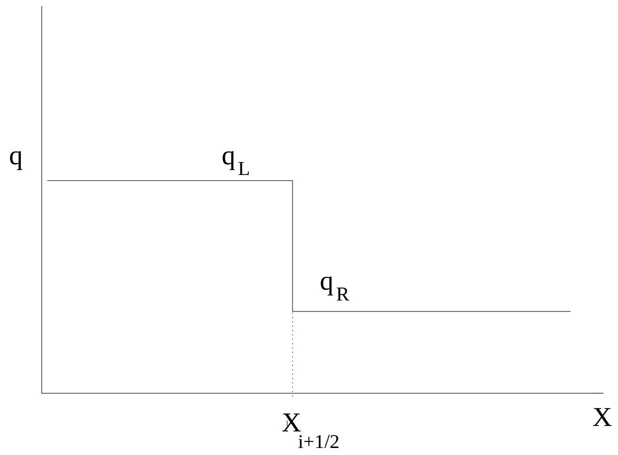

Figure 1: Discontinuity in q at $x_{i+1/2}$

where $\omega = q_R - q_L$. The flux is then given by:

$$F(q^*) = \begin{cases} A\, q_L + A\, \omega & \text{if} A < 0 \\ A\, q_R - A\, \omega & \text{if} A > 0 \end{cases} = \begin{cases} F(q_L) + A\, \omega & \text{if} A < 0 \\ F(q_R) - A\, \omega & \text{if} A > 0 \end{cases}, \tag{16}$$

or in a more compact form:

$$F_{i+1/2} = F(q^*_{i+1/2}) = \frac{1}{2}\left(F(q_{Li+1/2}) + F(q_{Ri+1/2}) - |A|_{i+1/2}\, \omega_{i+1/2}\right) \tag{17}$$

Using the fact that q^* is constant in the integration interval, we can calculate the integrals in (11) to get:

$$\frac{\bar{q}_{t+\Delta t} - \bar{q}_t}{\Delta t} = -\frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} \tag{18}$$

with the fluxes given by equation (17). In the general case in which the fluid is described by a state vector we will have to rotate to the basis in which the Jacobian A becomes diagonal, calculate the fluxes, then rotate back to the set of variables that interest us. If the equation has sources we also include them in equation (18).

5 2+1+1 Hydrodynamic Equations

Here we present the hydrodynamic equations in 2+1+1 conservative form. In 2+1+1 form, the general axisymmetric metric (with rotation) can be written

$$\begin{aligned} ds^2 = \quad & \left(-\alpha^2 + \frac{1}{s^2}\xi_t^2\right) dt^2 + H_{AB}\left(dx^A + \beta^A dt\right)\left(dx^B + \beta^B dt\right) + \frac{1}{s^2}\xi_A \xi_B dx^A dx^B \\ & + \frac{2}{s^2}\xi_t \xi_A dt dx^A + 2\xi_t dt d\phi + 2\xi_A dx^A d\phi + s^2 d\phi^2 \end{aligned} \tag{19}$$

where $\xi^\mu = \left(\frac{\partial}{\partial\phi}\right)^\mu$ is the Killing vector field and $s^2 = \xi^\mu \xi_\mu$ is its norm. In matrix form:

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + H_{AB}\beta^A\beta^B + \xi_t^2/s^2 & H_{AB}\beta^B + \xi_t\xi_A/s^2 & \xi_t \\ H_{AB}\beta^B + \xi_A & H_{AB} + \xi_A\xi_B/s^2 & \xi_A \\ \xi_t & \xi_A & s^2 \end{bmatrix}. \tag{20}$$

Here upper-case roman indexes, $A, B, \ldots$ take values 1,2 where $x^1 = \rho$ and $x^2 = z$ and the angle ϕ is the angle associated with the rotational symmetry. We now define some fluid variables that are useful in recasting equations (7) and (8):

$$\begin{aligned}
D &= \rho_o W && (21)\\
S_A &= \rho_o h W^2 v_A && (22)\\
S_\phi &= \rho_o h W^2 v_\phi && (23)\\
E &= \rho_o h W^2 - P && (24)\\
\tau &= E - D && (25)\\
v^A &= \frac{u^A}{\alpha u^0} + \frac{\beta^A}{\alpha} && (26)\\
v_\phi &= \frac{u_\phi}{\alpha u^0} \qquad \text{and} \qquad v^\phi = \frac{1}{s^2}\frac{u_\phi}{\alpha u^0} && (27)\\
W &= \alpha u^0 = \left(1 - v^A v_A - v^\phi v_\phi\right)^{-1/2}. && (28)
\end{aligned}$$

Here, ρ_o is the rest mass density, W is the Lorentz factor, $h = 1 + \epsilon + P/\rho_o$ is the relativistic specific enthalpy, ϵ is the specific internal energy, u^μ is the fluid four velocity, P is the pressure, v^A is the two-velocity, v^ϕ is the rotational velocity, E is the total energy, and τ is another energy variable that is useful in what follows. The stress-energy momemtum tensor as:

$$T_{ab} = \rho_o h u_a u_b + P g_{ab}, \tag{29}$$

and using the variables defined above, we can now re-write equation (8):

Conservation of Baryon Number:

$$\frac{\partial}{\partial t}\left\{s\sqrt{H}D\right\} + \frac{\partial}{\partial x^B}\left\{\alpha s\sqrt{H}D\left[v^B - \frac{\beta^B}{\alpha}\right]\right\} = 0, \tag{30}$$

where H is the determinant of the two metric H_{AB}.

The rest of the equations derived from (7) are:

Euler Equations:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{s\sqrt{H}S_A\right\} + \frac{\partial}{\partial x^B}\left\{\alpha s\sqrt{H}\left[S_A\left(v^B - \frac{\beta^B}{\alpha}\right) + P\delta^B_A\right]\right\} = \\
&\alpha s\sqrt{H}\left\{T^{\mu\nu}\left(\frac{\partial g_{\nu A}}{\partial x^\mu} - \Gamma^\delta_{\nu\mu}g_{\delta A}\right) - S_\phi\left[\frac{1}{\alpha}\frac{\partial}{\partial t}\left(\frac{\xi_A}{s^2}\right) + \left(v^B - \frac{\beta^B}{\alpha}\right)\frac{\partial}{\partial x^B}\left(\frac{\xi_A}{s^2}\right)\right]\right\} \quad (31)
\end{aligned}$$

Energy Equation:

$$\frac{\partial}{\partial t}\left\{s\sqrt{H}\tau\right\}+\frac{\partial}{\partial x^B}\left\{\alpha s\sqrt{H}\left[\tau\left(v^B-\frac{\beta^B}{\alpha}\right)+Pv^B\right]\right\}=$$
$$\alpha s\sqrt{H}\left\{\alpha\left(T^{\mu 0}\frac{\partial\ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right\} \tag{32}$$

Conservation of angular Momentum:

$$\frac{\partial}{\partial t}\left\{s\sqrt{H}S_\phi\right\}+\frac{\partial}{\partial x^B}\left\{\alpha s\sqrt{H}S_\phi\left[v^B-\frac{\beta^B}{\alpha}\right]\right\}=0 \tag{33}$$

The above equations are completed by an equation of state:

$$P=P(\rho_o,\epsilon). \tag{34}$$

Note that equations (30)-(32) are in the same form as (4) with the state vector given by (D, S_A, S_ϕ, τ). These variables are referred to as *conservative* variables, and we note that we will have transform to and from a different set (known as *primitive* variables) (ρ_o, v^A, v^ϕ, P) on which the fluxes and the sources depend. The dependence of conservative variables on the primitive ones is highly non-linear and, usually, recovering the primitive variables needs to be done via a Newton-Raphson method. In the next section we will explain a procedure to recover the primitive variables for a general equation of state.

5.1 Characteristic structure of the equations

As we mentioned in section 4, in the case that the conservation equations depend on more that one variable, we need to change to a basis in which the Jacobians become diagonal. In order to do that we need to know the characteristic structure of the equations (meaning the eigenvalue/eigenvector structure of the Jacobians $\partial\mathbf{F}/\partial\mathbf{Q}$). Equations (30)-(32) have the same structure as the equations within the 3+1 Valencia formulation, therefore we can take their characteristic structure directly from [21]. The state vector is:

$$\mathbf{q}(\mathbf{w})=\left[D,S_A,S_\phi,\tau\right]^T \tag{35}$$

and the vector of fluxes is:

$$\mathbf{F}^B(\mathbf{w})=\left[D\left(v^B-\frac{\beta^B}{\alpha}\right),S_A\left(v^B-\frac{\beta^B}{\alpha}\right)+P\delta^B_A,S_\phi\left(v^B-\frac{\beta^B}{\alpha}\right),\tau\left(v^B-\frac{\beta^B}{\alpha}\right)+Pv^B\right]^T. \tag{36}$$

The corresponding sources $\mathbf{S}(\mathbf{w})$ are

$$\mathbf{S}(\mathbf{w})=\quad\left[0,T^{\mu\nu}\left(\frac{\partial g_{\nu A}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta A}\right)-S_\phi\left[\frac{1}{\alpha}\frac{\partial}{\partial t}\left(\frac{\xi_A}{s^2}\right)+\left(v^B-\frac{\beta^B}{\alpha}\right)\frac{\partial}{\partial x^B}\left(\frac{\xi_A}{s^2}\right)\right],\right.$$
$$\left.0,\alpha\left(T^{\mu 0}\frac{\partial\ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right]^T \tag{37}$$

With the aid of these vectors we can write the equations in the simple form:

$$\frac{1}{\alpha s\sqrt{H}}\frac{\partial}{\partial t}\left(s\sqrt{H}\mathbf{q}\right)+\frac{1}{\alpha s\sqrt{H}}\frac{\partial}{\partial x^B}\left(\alpha s\sqrt{H}\mathbf{F}^B\right)=\mathbf{S} \tag{38}$$

The eigenvalues for the Jacobian of the flux in the ρ direction $\mathbf{A}^\rho_{ij}=\partial\mathbf{F}^\rho_j/\partial\mathbf{q}^i$ [2] are :

$$\begin{aligned}
\lambda_0 &= \alpha v^\rho-\beta^\rho \qquad\qquad \text{(triple)}, && (39)\\
\lambda_\pm &= \frac{\alpha}{1-v^2c_s^2}\left\{v^\rho\left(1-c_s^2\right)\pm c_s\sqrt{(1-v^2)\left[H^{\rho\rho}\left(1-v^2c_s^2\right)-v^\rho v^\rho\left(1-c_s^2\right)\right]}\right\}-\beta^\rho, && (40)
\end{aligned}$$

The right eigenvectors (these are not for a general metric of the form (19) but for a metric with a diagonal 2 metric H_{AB} with both diagonal elements equal to ψ^4) [3] are needed in order to go back and forth from the basis where the Jacobian is diagonal to the basis of conservative variables.

$$\begin{aligned}
\mathbf{r}_{0,1} &= \left(\frac{\mathcal{K}}{hW},v_\rho,v_z,v_\phi,1-\frac{\mathcal{K}}{hW}\right)^T && (41)\\
\mathbf{r}_{0,2} &= \left(Wv_z,2hW^2v_\rho v_z,h\left(\psi^4+2W^2v_zv_z\right),2hW^2v_zv_\phi,Wv_z\left(2hW-1\right)\right) && (42)\\
\mathbf{r}_{0,3} &= \left(Wv_\phi,2hW^2v_\rho v_\phi,2hW^2v_zv_\phi,h\left(s^2+2W^2v_\phi v_\phi\right),Wv_\phi\left(2hW-1\right)\right) && (43)\\
\mathbf{r}_\pm &= \left(1,hW\mathcal{C}^\rho_\pm,hWv_z,hWv_\phi,hW\tilde{\mathcal{A}}^\rho_\pm-1\right) && (44)
\end{aligned}$$

where:

$$\begin{aligned}
\mathcal{K} &= \frac{\tilde{\kappa}}{\tilde{\kappa}-c_s^2} && (45)\\
\tilde{\kappa} &= \kappa/\rho_o && (46)\\
\mathcal{C}^\rho_\pm &= v_\rho-\mathcal{V}^\rho_\pm && (47)\\
\mathcal{V}^\rho_\pm &= \left(v^\rho-\Lambda^\rho_\pm\right)/\left(1/\psi^4-v^\rho\Lambda^\rho_\pm\right) && (48)\\
\tilde{\mathcal{A}}^\rho_\pm &= \left(1/\psi^4-v^\rho v^\rho\right)/\left(1/\psi^4-v^\rho\Lambda^\rho_\pm\right) && (49)\\
\Lambda^\rho_\pm &= \lambda_\pm/\alpha+\beta^\rho/\alpha && (50)
\end{aligned}$$

The left eigenvectors are also needed in order to rotate back to the conservative variable basis. This is done by inverting the matrix produced with the right eigenvectors. Analogous results can be found for the flux in the z direction.

6 Primitive Variable reconstruction for a general EOS

Here we will present a method to recover the primitive variables (ρ_o,v^A,v^ϕ,P) from a given set of conservative variables (D,S_A,S_ϕ,τ). This is a generalization of the method discussed in [22] for a general equation of state of the form[4]:

$$\epsilon=\epsilon(\rho_o,P) \tag{51}$$

[2]Note that the indexes i,j run from 1 to 5, the number of hydrodynamic equations.

[3]This is the actual metric we will use in our calculations.

[4]This form is equivalent to (34) assuming that the equation is invertible.

The following relationship will be useful:

$$\begin{aligned}
\epsilon' = \left.\frac{\partial \epsilon}{\partial P}\right|_{D,\tau,S_i} &= \left.\frac{\partial \epsilon}{\partial \rho_o}\right|_{P} \left.\frac{\partial \rho_o}{\partial P}\right|_{D,\tau,S_i} + \left.\frac{\partial \epsilon}{\partial P}\right|_{\rho_o} \\
&= -\frac{D}{W^2} \left.\frac{\partial W}{\partial P}\right|_{D,\tau,S_i} \left.\frac{\partial \epsilon}{\partial \rho_o}\right|_{P} + \left.\frac{\partial \epsilon}{\partial P}\right|_{\rho_o} \\
&= -\frac{D}{W^2} \left.\frac{\partial W}{\partial P}\right|_{D,\tau,S_i} (-1) \left.\frac{\partial P}{\partial \rho_o}\right|_{\epsilon} / \left.\frac{\partial P}{\partial \epsilon}\right|_{\rho_o} + \frac{1}{\kappa} \\
&= \frac{D}{W^2} \left.\frac{\partial W}{\partial P}\right|_{D,\tau,S_i} \frac{\chi}{\kappa} + \frac{1}{\kappa},
\end{aligned}$$

where in the second line we have used the fact that $D = \rho_o W$. To recover the pressure from the conservative variables we use a Newton-Raphson method to solve the equation $f(P) = 0$ (f is given by the expression below). In what follows a prime denotes a derivative with respect the pressure keeping the conservative variables fixed. In the iterative process we do the following steps (starting from a guess for the pressure P):

- $P = P_{n+1}$
- $\mathcal{Z} = \tau + D + P$
- $W(P) = \sqrt{\mathcal{Z}^2/(\mathcal{Z}^2 - S^2)} \quad \longrightarrow \quad W'(P) = (1 - W^2)/\sqrt{\mathcal{Z}^2 - S^2}$
- $f(P) = D\,(1+\epsilon)\,W + P\,(W^2 - 1) - D - \tau \quad \longrightarrow$
  $f'(P) = D\,(1+\epsilon)\,W' + DW\epsilon' + W^2 - 1 + 2PWW'$
  where to be more precise
  $$f'(P) = D\left(1 + \epsilon\left(\frac{D}{W}, P\right)\right) W' + DW\left(W'\frac{D}{W^2}\frac{\chi}{\kappa} + \frac{1}{\kappa}\right) + W^2 - 1 + 2PWW' \quad (52)$$
- $P_{n+1} = P_n - f(P)/f'(P)$

The process is repeated until $f(P)$ drops below the desired tolerance. Once P is recovered, the rest of the primitive variables are trivially calculated:

$$v_A = \frac{S_A}{\tau + D + P} \quad (53)$$

$$v_\phi = \frac{S_\phi}{\tau + D + P} \quad (54)$$

$$\rho_o = D\sqrt{1 - v^2} \quad (55)$$

7 The story so far...

We have developed a one dimensional (spherically symmetric) code that solve the hydrodynamical equations coupled with gravity in polar-areal coordinates [5] for the ideal equation

[5]these coordinates can be viewed as a generalization of the usual Schwarzschild coordinates to the case of time-dependent spacetimes

of state $P = (\Gamma - 1)\,\rho_o \epsilon$. The code seems to converge and to be stable for many of the distributions of matter that we have tested. This code also has been tested with previous work [22] giving satisfactory results. Another one dimensional code in maximal-isotropic coordinates, which can be viewed as the spherically-symmetric specialization of the axisymmetric coordinates used in [24], is underway. This code will help us to calibrate and check the results from the two dimensional code.

Using as a basis a two dimensional code developed by M. Choptuik, E. Hirschmann, S. Liebling and F. Pretorius [23], [24], we have implemented the hydro formulation described above in flat spacetime. We have observed some problems (instabilities) at evacuated regions (regions where the fluid density decreases to small values) for configurations with high γ's and adiabatic constant $\Gamma = 2$, and we are currently working to stabilize the code. We are also working to properly simulate full general relativistic hydrodynamics, coupling the hydrodynamic equations to those which govern the geometry. The geometry code uses the 2+1+1 decomposition we briefly discussed above, and its existence provides a key motivation for casting the hydrodynamic equations in 2+1+1 form.

8 Immediate goals

- Stabilize the code or at least properly delineate the range of parameters where they are valid.
- Finish the one dimensional code in maximal-isotropic coordinates and incorporate the hydrodynamic equations into the fully relativistic two dimensional code.
- Solve the initial data problem for star-like configurations in two dimensions.
- Introduce and study the effects of one or more *realistic* equations of state. This is something that, to our best knowledge, has never been published for a fully relativistic multidimensional code.
- Introduce the microphysics that is needed to simulate some of the astrophysical scenarios we have discussed, including some treatment of neutrino transport.
- Study the incorporation of some kind of adaptive mesh algorithms.

References

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