Flatspace cartesian coordinates [1]

Metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\nu dx^\mu \tag{1}$$

Stress-Energy Tensor

$$T^{\mu\nu} = \rho_o h u^\mu u^\nu + P\eta^{\mu\nu} \tag{2}$$

where $h = 1 + \epsilon + P/\rho_o$. The 3 velocity is defined as:

$$v^i = \frac{u^i}{u^t}. \tag{3}$$

The primitive variables are:

$$\mathbf{w} = \begin{bmatrix} \rho_o \\ v_i \\ \epsilon \end{bmatrix} \tag{4}$$

Equations of Motion

$$\frac{\partial \mathbf{F}^\mu(\mathbf{w})}{\partial x^\mu} = \boldsymbol{\psi}(\mathbf{w}) \tag{5}$$

Where:

$$\mathbf{F^0}(\mathbf{w}) = \begin{bmatrix} D \\ S_j \\ \tau \end{bmatrix} \tag{6}$$

$$\begin{aligned} D &= \rho_o W & (7) \\ S_j &= \rho_o h W^2 v_j & (8) \\ E &= \rho_o h W^2 - P & (9) \\ \tau &= E - D & (10) \\ W &= (1 - v^2)^{-1/2} = u^t & (11) \\ h &= 1 + \epsilon + P/\rho_o & (12) \end{aligned}$$

$$\mathbf{F^i}(\mathbf{w}) = \begin{bmatrix} D - v^i \\ S_j v^i + P\delta^i_j \\ \tau v^i + P v^i \end{bmatrix} \tag{13}$$

and

$$\boldsymbol{\psi}(\mathbf{w}) = \begin{bmatrix} 0 \\ T^{\mu\nu}\left(-\Gamma^j_{\nu\mu}\right) \\ -T^{\mu\nu}\Gamma^0_{\nu\mu} \end{bmatrix} \tag{14}$$

$$\boldsymbol{\psi}(\mathbf{w}) = \mathbf{0} \tag{15}$$

Jacobian Matrices

$$\mathbf{B^i} = \frac{\partial \mathbf{F^i}}{\partial \mathbf{F^0}} \tag{16}$$

$$
\begin{aligned}
\frac{\partial F^{i(0)}}{\partial D} &= 1 + \frac{S^i}{(\tau + P + D)^2}\left[1 + \frac{\partial P}{\partial D}\right] && (17)\\
\frac{\partial F^{i(j)}}{\partial D} &= -\frac{S_j S^i}{(\tau + P + D)^2}\left[1 + \frac{\partial P}{\partial D}\right] + \frac{\partial P}{\partial D}\delta^i_j && (18)\\
\frac{\partial F^{i(4)}}{\partial D} &= \frac{\partial P}{\partial D}v^i - \frac{S^i}{(\tau + P + D)^2}\left[1 + \frac{\partial P}{\partial D}\right](\tau + P) && (19)\\
\frac{\partial F^{i(0)}}{\partial S_k} &= -\frac{\delta^i_k}{\tau + P + D} + \frac{S^i}{(\tau + P + D)^2}\frac{\partial P}{\partial S_k} && (20)\\
\frac{\partial F^{i(j)}}{\partial S_k} &= \delta^k_j v^i + S_j\left[\frac{\delta^i_k}{\tau + P + D} - \frac{S^i}{(\tau + P + D)^2}\frac{\partial P}{\partial S_k}\right] + \frac{\partial P}{\partial S_k} && (21)\\
\frac{\partial F^{i(4)}}{\partial S_k} &= \frac{\partial P}{\partial S_k}v^i + (\tau + P)\left[\frac{\delta^i_k}{\tau + P + D} - \frac{S^i}{(\tau + P + D)^2}\frac{\partial P}{\partial S_k}\right] && (22)\\
\frac{\partial F^{i(0)}}{\partial \tau} &= \frac{S^i}{(\tau + P + D)^2}\left[1 + \frac{\partial P}{\partial \tau}\right] && (23)\\
\frac{\partial F^{i(j)}}{\partial \tau} &= -\frac{S_j S^i}{(\tau + P + D)^2}\left[1 + \frac{\partial P}{\partial \tau}\right] + \frac{\partial P}{\partial \tau}\delta^i_j && (24)\\
\frac{\partial F^{i(4)}}{\partial \tau} &= \left(1 + \frac{\partial P}{\partial \tau}\right)v^i - (\tau + P)\frac{S^i}{(P + \tau + D)^2}\left(\frac{\partial P}{\partial \tau} + 1\right) && (25)
\end{aligned}
$$

where we have used that:

$$
\begin{aligned}
\frac{\partial v^i}{\partial D} &= -\frac{S^i}{(\tau + P + D)^2}\left(1 + \frac{\partial P}{\partial D}\right) && (26)\\
\frac{\partial v^i}{\partial S_k} &= \frac{\delta^i_k}{\tau + P + D} - \frac{S^i}{(\tau + P + D)^2}\frac{\partial P}{\partial S_k} && (27)\\
\frac{\partial v^i}{\partial \tau} &= -\frac{S^i}{(\tau + P + D)^2}\left(1 + \frac{\partial P}{\partial \tau}\right) && (28)
\end{aligned}
$$

the specific forms of $(\partial P/\partial D)$, $(\partial P/\partial S_k)$, $(\partial P/\partial \tau)$ depend on the equations of state.

Eigenvalues and Eigenvectors

Eigenvalues for jacobian matrix $\mathbf{B^x}$:

$$\lambda_0 = v^x \quad \text{(triple)} \tag{29}$$

$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2}\left\{v^x\left(1 - c_s^2\right) \pm c_s\sqrt{\left(1 - v^2\right)\left[\left(1 - v^2 c_s^2\right) - v^x v^x \left(1 - c_s^2\right)\right]}\right\} \tag{30}$$

Eigenvectors:

$$\mathbf{r}_{0,1} = \left[\frac{\tilde{\kappa}}{hW\left(\tilde{\kappa} - c_s^2\right)}, v_x, v_y, v_z, 1 - \frac{\tilde{\kappa}}{hW\left(\tilde{\kappa} - c_s^2\right)}\right] \tag{31}$$

$$\mathbf{r}_{0,2} = \left[Wv^y, 2hW^2 v_x v^y, h\left(1 + 2W^2 v_y v^y\right), 2hW^2 v_z v^y, 2hW^2 v^y - Wv^y\right] \tag{32}$$

$$\mathbf{r}_{0,3} = \left[Wv^z, 2hW^2 v_x v^z, 2hW^2 v_y v^z, h\left(1 + 2W^2 v_z v^z\right), 2hW^2 v^z - Wv^z\right] \tag{33}$$

$$\mathbf{r}_\pm = \left[1, hW\left(v_x - \frac{v^x - \lambda_\pm}{1 - v^x \lambda_\pm}\right), hWv_y, hWv_z, \frac{hW\left(1 - v^x v^x\right)}{1 - v^x \lambda_\pm} - 1\right] \tag{34}$$

where $\tilde{\kappa} = \kappa/\rho_o$,

Flatspace axisymmetric coordinates

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2 \tag{35}$$

Non zero Christoffel Symbols:

$$\Gamma^\rho_{\phi\phi} = -\rho \qquad \Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = \frac{1}{\rho} \tag{36}$$

Equation of Motion

$$\frac{1}{\rho}\frac{\partial\left[\rho\mathbf{F}^\mu\left(\mathbf{w}\right)\right]}{\partial x^\mu} = \boldsymbol{\psi}(\mathbf{w}) \tag{37}$$

In full detail:

$$\frac{\partial\left[\mathbf{F}^t(\mathbf{w})\right]}{\partial t} + \frac{1}{\rho}\frac{\partial\left[\rho\mathbf{F}^\rho\left(\mathbf{w}\right)\right]}{\partial\rho} + \frac{\partial\left[\mathbf{F}^z(\mathbf{w})\right]}{\partial z} = \boldsymbol{\psi}(\mathbf{w}) \tag{38}$$

where:

$$\mathbf{w} = \begin{bmatrix}\rho_o \\ v^\rho \\ v^z \\ v^\phi \\ \epsilon\end{bmatrix} \qquad \mathbf{F}^t(\mathbf{w}) = \begin{bmatrix} D \\ S_\rho \\ S_z \\ S_\phi \\ \tau\end{bmatrix} \tag{39}$$

and

$$\mathbf{F}^\rho = \begin{bmatrix} Dv^\rho \\ S_\rho v^\rho + P \\ S_z v^\rho \\ S_\phi v^\rho \\ \left(\tau + P\right)v^\rho\end{bmatrix} \quad \mathbf{F}^z = \begin{bmatrix} Dv^z \\ S_\rho v^z \\ S_z v^z + P \\ S_\phi v^z \\ \left(\tau + P\right)v^z\end{bmatrix} \quad \mathbf{F}^\phi = \begin{bmatrix} Dv^\phi \\ S_\rho v^\phi \\ S_z v^\phi \\ S_\phi v^\phi + P \\ \left(\tau + P\right)v^\phi\end{bmatrix} \tag{40}$$

$$\boldsymbol{\psi}(\mathbf{w}) = \begin{bmatrix} 0 \\ T^{\phi\phi}\left(-\Gamma^\rho_{\phi\phi} g_{\rho\rho}\right) \\ 0 \\ T^{\rho\phi}\left(\frac{\partial g_{\phi\phi}}{\partial\rho} - 2\Gamma^\phi_{\rho\phi} g_{\phi\phi}\right) \\ 0\end{bmatrix} = \begin{bmatrix} 0 \\ T^{\phi\phi}\rho \\ 0 \\ 0 \\ 0\end{bmatrix} \tag{41}$$

Eigenvalues and Eigenvectors

Eigenvalues for jacobian matrix $\mathbf{B}^\rho$:

$$\lambda_0 = v^\rho \quad \text{(triple)} \tag{42}$$
$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2} \left\{ v^\rho \left(1 - c_s^2\right) \pm c_s \sqrt{\left(1 - v^2\right)\left[\left(1 - v^2 c_s^2\right) - v^{\rho 2}\left(1 - c_s^2\right)\right]} \right\} \tag{43}$$

where $v^2 = v^{\rho 2} + v^{z2} + \rho^2 v^{\phi 2}$, with eigenvectors:

$$\mathbf{r}_{0,1} = \left[\frac{\tilde{\kappa}}{hW\left(\tilde{\kappa} - c_s^2\right)}, v^\rho, v^z, \rho^2 v^\phi, 1 - \frac{\tilde{\kappa}}{hW\left(\tilde{\kappa} - c_s^2\right)} \right] \tag{44}$$
$$\mathbf{r}_{0,2} = \left[W v^z, 2hW^2 v^\rho v^z, h\left(1 + 2W^2 v^{z2}\right), 2hW^2 \rho^2 v^\phi v^z, 2hW^2 v^z - W v^z \right] \tag{45}$$
$$\mathbf{r}_{0,3} = \left[W v^\phi, 2hW^2 v^\rho v^\phi, 2hW^2 v^z v^\phi, h\left(1 + 2W^2 v^{\phi 2} \rho^2\right), 2hW^2 v^\phi - W v^\phi \right] \tag{46}$$
$$\mathbf{r}_\pm = \left[1, hW\left(v^\rho - \frac{v^\rho - \lambda_\pm}{1 - v^\rho \lambda_\pm} \right), hWv^z, hW\rho^2 v^\phi, \frac{hW\left(1 - v^{\rho 2}\right)}{1 - v^\rho \lambda_\pm} - 1 \right] \tag{47}$$

Eigenvalues for jacobian matrix $\mathbf{B}^z$:

$$\lambda_0 = v^z \quad \text{(triple)} \tag{48}$$
$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2} \left\{ v^z \left(1 - c_s^2\right) \pm c_s \sqrt{\left(1 - v^2\right)\left[\left(1 - v^2 c_s^2\right) - v^{z2}\left(1 - c_s^2\right)\right]} \right\} \tag{49}$$

with eigenvectors:

$$\mathbf{r}_{0,1} = \left[W v^\rho, h\left(1 + 2W^2 v^{\rho 2}\right), 2hW^2 v^z v^\rho, 2hW^2 \rho^2 v^\phi v^\rho, 2hW^2 v^\rho - W v^\rho \right] \tag{50}$$
$$\mathbf{r}_{0,2} = \left[\frac{\tilde{\kappa}}{hW\left(\tilde{\kappa} - c_s^2\right)}, v^\rho, v^z, \rho^2 v^\phi, 1 - \frac{\tilde{\kappa}}{hW\left(\tilde{\kappa} - c_s^2\right)} \right] \tag{51}$$
$$\mathbf{r}_{0,3} = \left[W v^\phi, 2hW^2 v^\rho v^\phi, 2hW^2 v^z v^\phi, h\left(1 + 2W^2 v^{\phi 2} \rho^2\right), 2hW^2 v^\phi - W v^\phi \right] \tag{52}$$
$$\mathbf{r}_\pm = \left[1, hWv^\rho, hW\left(v^z - \frac{v^z - \lambda_\pm}{1 - v^z \lambda_\pm} \right), hW\rho^2 v^\phi, \frac{hW\left(1 - v^{z2}\right)}{1 - v^z \lambda_\pm} - 1 \right] \tag{53}$$

Ultrarelativistic Equation of State

The EOS takes the simple form:

$$P = (\Gamma - 1)\,\rho_F \tag{54}$$

where $\rho_F = (1+\epsilon)\,\rho_o \approx \epsilon\rho_o$

$$\chi = \partial P/\partial\rho_o|_\epsilon = (\Gamma - 1)\,(1+\epsilon) \qquad \kappa = \partial P/\partial\epsilon|_{\rho_o} = (\Gamma - 1)\,\rho_o \tag{55}$$

The next relationships apply:

$$D = \rho_o W \tag{56}$$
$$S_\rho = \rho_o h W^2 v^\rho = (\rho_F + P)\,W^2 v^\rho \tag{57}$$
$$S_z = \rho_o h W^2 v^z = (\rho_F + P)\,W^2 v^z \tag{58}$$
$$S_\phi = \rho_o h W^2 \rho^2 v^\phi = (\rho_F + P)\,W^2 \rho^2 v^\phi \tag{59}$$
$$E = \rho_o h W^2 - P = (\rho_F + P)\,W^2 - P \tag{60}$$
$$\tau = E \tag{61}$$
$$W = \left(1 - v^2\right)^{-1/2} \tag{62}$$
$$W^2 = \frac{(E+P)^2}{(E+P)^2 - S_i S^i} \tag{63}$$
$$h = 1 + \epsilon + P/\rho_o = (1+\epsilon)\,\Gamma \tag{64}$$
$$\rho_o h = \rho + P \tag{65}$$
$$v^2 = v^{\rho 2} + v^{z2} + \rho^2 v^{\phi 2} \tag{66}$$
$$h{c_s}^2 = \chi + \frac{P}{\rho_o^2}\kappa \quad \rightarrow \quad c_s^2 = (\Gamma - 1) \tag{67}$$
$$T^{\phi\phi} = \rho_o h W^2 v^{\phi 2} + P \tag{68}$$
$$0 = (\tau + P)^2 - (\rho + P)\,(\tau + P) - S_i S^i \tag{69}$$
$$P = -2\beta\tau + \left[4\beta^2\tau^2 + (\Gamma - 1)\left(\tau^2 - S_i S^i\right)\right]^{1/2} \tag{70}$$
$$\beta = \frac{1}{4}\,(2 - \Gamma) \tag{71}$$
$$S^2 = {S_\rho}^2 + {S_z}^2 + \frac{1}{\rho^2}{S_\phi}^2 \tag{72}$$

section*Equation of Motion

$$\mathbf{w} = \begin{bmatrix} v^\rho \\ v^z \\ v^\phi \\ \rho_F \end{bmatrix} \qquad \mathbf{F^t}\,(\mathbf{w}) = \begin{bmatrix} S_\rho \\ S_z \\ S_\phi \\ \tau \end{bmatrix} \tag{73}$$

$$\mathbf{F}^\rho\,(\mathbf{w}) = \begin{bmatrix} S_\rho v^\rho + P \\ S_z v^\rho \\ S_\phi v^\rho \\ (\tau + P)\,v^\rho \end{bmatrix} \qquad \mathbf{F^z}\,(\mathbf{w}) = \begin{bmatrix} S_\rho v^z \\ S_z v^z + P \\ S_\phi v^z \\ (\tau + P)\,v^z \end{bmatrix} \tag{74}$$

$$\boldsymbol{\psi}(\mathbf{w}) = \begin{bmatrix} T^{\phi\phi}\rho \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{75}$$

In terms of the primitive variables:

$$\mathbf{F}^{\rho}(\mathbf{w}) = \begin{bmatrix} (\rho_F + P) W^2 v^{\rho 2} + P \\ (\rho_F + P) W^2 v^{z2} \\ (\rho_F + P) W^2 \rho^2 v^{\phi 2} \\ (\rho_F + P) W^2 v^{\rho} \end{bmatrix} \qquad \mathbf{F}^{\mathbf{z}}(\mathbf{w}) = \begin{bmatrix} (\rho_F + P) W^2 v^{\rho 2} \\ (\rho_F + P) W^2 v^{z2} + P \\ (\rho_F + P) W^2 \rho^2 v^{\phi 2} \\ (\rho_F + P) W^2 v^{z} \end{bmatrix} \tag{76}$$

$$\frac{\partial W^2}{\partial v^i} = -\frac{2\gamma_{ij} v^j}{(1-v^2)^2} = -2W^4 \gamma_{ij} v^j \qquad \frac{\partial P}{\partial v^i} = 0 \tag{77}$$

$$\frac{\partial (F^{\rho})^{\rho}}{\partial v^{\rho}} = -2(\rho_F + P) W^4 v^{\rho 3} + 2(\rho_F + P) W^2 v^{\rho} = 2(\rho_F + P) W^2 v^{\rho}\left(1 - W^2 v^{\rho 2}\right) \tag{78}$$

$$\frac{\partial (F^{\rho})^{z}}{\partial v^{\rho}} = -2(\rho_F + P) W^4 v^{\rho} v^{z2} \tag{79}$$

$$\frac{\partial (F^{\rho})^{\phi}}{\partial v^{\rho}} = -2(\rho_F + P) W^4 v^{\rho} \rho^2 v^{\phi 2} \tag{80}$$

$$\frac{\partial (F^{\rho})^{\tau}}{\partial v^{\rho}} = -2(\rho_F + P) W^4 v^{\rho 2} + (\rho_F + P) W^2 = (\rho_F + P) W^2 \left(1 - 2W^2 v^{\rho 2}\right) \tag{81}$$

$$\frac{\partial (F^{\rho})^{\rho}}{\partial v^{z}} = -2(\rho_F + P) W^4 v^{z} v^{\rho 2} \tag{82}$$

$$\frac{\partial (F^{\rho})^{z}}{\partial v^{z}} = 2(\rho_F + P) W^2 v^{z}\left(1 - W^2 v^{z2}\right) \tag{83}$$

$$\frac{\partial (F^{\rho})^{\phi}}{\partial v^{z}} = -2(\rho_F + P) W^4 v^{z} \rho^2 v^{\phi 2} \tag{84}$$

$$\frac{\partial (F^{\rho})^{\tau}}{\partial v^{z}} = -2(\rho_F + P) W^4 v^{\rho} v^{z} \tag{85}$$

$$\frac{\partial (F^{\rho})^{\rho}}{\partial v^{\phi}} = -2(\rho_F + P) W^4 v^{\rho 2} \rho^2 v^{\phi} \tag{86}$$

$$\frac{\partial (F^{\rho})^{z}}{\partial v^{\phi}} = -2(\rho_F + P) W^4 v^{z2} \rho^2 v^{\phi} \tag{87}$$

$$\frac{\partial (F^{\rho})^{\phi}}{\partial v^{\phi}} = 2(\rho_F + P) W^2 \rho^2 v^{\phi}\left(1 - W^2 \rho^2 v^{\phi 2}\right) \tag{88}$$

$$\frac{\partial (F^{\rho})^{\tau}}{\partial v^{\phi}} = -2(\rho_F + P) W^4 \rho^2 v^{\phi} v^{\rho} \tag{89}$$

$$\frac{\partial (F^{\rho})^{\rho}}{\partial \rho_F} = \Gamma W^2 v^{\rho 2} + (\Gamma - 1) \tag{90}$$

$$\frac{\partial (F^{\rho})^{z}}{\partial \rho_F} = \Gamma W^2 v^{z2} \tag{91}$$

$$\frac{\partial (F^{\rho})^{\phi}}{\partial \rho_F} = \Gamma W^2 \rho^2 v^{\phi 2} \tag{92}$$

$$\frac{\partial (F^{\rho})^{\tau}}{\partial \rho_F} = \Gamma W^2 v^{\rho} \tag{93}$$

$$
\mathbf{B}^{\rho} = \begin{bmatrix} 2\left(\rho_F + P\right) W^2 v^{\rho} \left(1 - W^2 v^{\rho 2}\right) & -2\left(\rho_F + P\right) W^4 v^{\rho} v^{z2} & -2\left(\rho_F + P\right) W^4 v^{\rho} \rho^2 v^{\phi 2} & 2\left(\rho_F + P\right) W^2 \left(1 - W^2 v^{\rho 2}\right) \\ -2\left(\rho_F + P\right) W^4 v^{z} v^{\rho 2} & 2\left(\rho_F + P\right) W^4 v^{z} \left(1 - W^2 v^{z2}\right) & -2\left(\rho_F + P\right) W^4 v^{z} \rho^2 v^{\phi 2} & -2\left(\rho_F + P\right) W^4 v^{\rho} v^{z} \\ -2\left(\rho_F + P\right) W^4 v^{\rho 2} \rho^2 v^{\phi} & -2\left(\rho_F + P\right) W^4 v^{z2} \rho^2 v^{\phi} & 2\left(\rho_F + P\right) W^2 \rho^2 v^{\phi} \left(1 - W^2 \rho^2 v^{\phi 2}\right) & -2\left(\rho_F + P\right) W^4 \rho^2 v^{\phi} v^{\rho} \\ \Gamma W^2 v^{\rho 2} + \left(\Gamma - 1\right) & \Gamma W^2 v^{z2} & \Gamma W^2 \rho^2 v^{\phi 2} & \Gamma W^2 v^{\rho} \end{bmatrix} \tag{94}
$$

See maple script in

`/d2/home/inaki/fluids/2d/docs/ultra.mws`

For $\mathbf{B}^r$

$$
\lambda_o = v^{\rho} \qquad \text{(double)} \tag{95}
$$

$$
\lambda_{\pm} = \frac{1}{1 - v^2 c_s^2} \left\{ v^{\rho} \left(1 - c_s^2\right) \pm c_s \sqrt{\left(v^2 - 1\right)\left(c_s^2 v^2 - 1 + v^{\rho 2}\left(1 - c_s^2\right)\right)} \right\} \tag{96}
$$

$$
\mathbf{r}_{o,1} = \begin{bmatrix} \frac{2v^z}{1+v^2-2v^{\rho 2}} \\ \frac{2v^z v^{\rho}}{1+v^2-2v^{\rho 2}} \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{r}_{o,2} = \begin{bmatrix} \frac{2v^{\phi}}{1+v^2-2v^{\rho 2}} \\ \frac{2v^{\phi} v^{\rho}}{1+v^2-2v^{\rho 2}} \\ 0 \\ 1 \end{bmatrix} \tag{97}
$$

$$
\mathbf{r}_{\pm} = \begin{bmatrix} \left(1 - v^{\rho 2}\right)\left(v^{\rho} - \lambda_{\pm}\right)\left(1 - c_s^2 v^2\right) \\ \left(1 - v^{\rho 2}\right)\left[\left(\lambda_{\pm} v^{\rho} - 1\right) c_s^2 \left(1 - v^2\right) + v^{\rho}\left(1 - c_s^2\right)\left(v^{\rho} - \lambda_{\pm}\right)\right] \\ v^z \left[c_s^2 \left(\lambda_{\pm}(1 + v^{\rho 2}) v^2 - v^{\rho 2}\lambda_{\pm} - 2v^{\rho} v^2\right) + \left(1 - c_s^2\right) v^{\rho 2}\left(\lambda_{\pm} - v^{\rho}\right) + \left(c_s^2 + 1\right) v^{\rho} - \lambda_{\pm}\right] \\ \rho^2 v^{\phi} \left[c_s^2 \left(\lambda_{\pm}(1 + v^{\rho 2}) v^2 - v^{\rho 2}\lambda_{\pm} - 2v^{\rho} v^2\right) + \left(1 - c_s^2\right) v^{\rho 2}\left(\lambda_{\pm} - v^{\rho}\right) + \left(c_s^2 + 1\right) v^{\rho} - \lambda_{\pm}\right] \end{bmatrix} \tag{98}
$$

For $\mathbf{B}^z$

$$
\lambda_o = v^{z} \qquad \text{(double)} \tag{99}
$$

$$
\lambda_{\pm} = \frac{1}{1 - v^2 c_s^2} \left\{ v^{z} \left(1 - c_s^2\right) \pm c_s \sqrt{\left(v^2 - 1\right)\left(c_s^2 v^2 - 1 + v^{z2}\left(1 - c_s^2\right)\right)} \right\} \tag{100}
$$

$$
\mathbf{r}_{o,1} = \begin{bmatrix} \frac{2v^{\rho}}{1+v^2-2v^{z2}} \\ 1 \\ \frac{2v^z v^{\rho}}{1+v^2-2v^{z2}} \\ 0 \end{bmatrix} \qquad \mathbf{r}_{o,2} = \begin{bmatrix} \frac{2v^{\phi}}{1+v^2-2v^{z2}} \\ 0 \\ \frac{2v^{\phi} v^{z}}{1+v^2-2v^{z2}} \\ 1 \end{bmatrix} \tag{101}
$$

$$
\mathbf{r}_{\pm} = \begin{bmatrix} \left(1 - v^{z2}\right)\left(v^{z} - \lambda_{\pm}\right)\left(1 - c_s^2 v^2\right) \\ v^{\rho} \left[c_s^2 \left(\lambda_{\pm}(1 + v^{z2}) v^2 - v^{z2}\lambda_{\pm} - 2v^{z} v^2\right) + \left(1 - c_s^2\right) v^{z2}\left(\lambda_{\pm} - v^{z}\right) + \left(c_s^2 + 1\right) v^{z} - \lambda_{\pm}\right] \\ \left(1 - v^{z2}\right)\left[\left(\lambda_{\pm} v^{z} - 1\right) c_s^2 \left(1 - v^2\right) + v^{z}\left(1 - c_s^2\right)\left(v^{z} - \lambda_{\pm}\right)\right] \\ \rho^2 v^{\phi} \left[c_s^2 \left(\lambda_{\pm}(1 + v^{z2}) v^2 - v^{z2}\lambda_{\pm} - 2v^{z} v^2\right) + \left(1 - c_s^2\right) v^{z2}\left(\lambda_{\pm} - v^{z}\right) + \left(c_s^2 + 1\right) v^{z} - \lambda_{\pm}\right] \end{bmatrix} \tag{102}
$$

Left eigenvectors for $\mathbf{B}^\rho$:

$$\mathbf{l}_{o,1} = \left[-\frac{v^z\left(1+v^2-2v^{\rho 2}\right)}{\left(1-v^{\rho 2}\right)\left(1-v^2\right)}, \frac{v^z v^\rho\left(1+v^2-2v^{\rho 2}\right)}{\left(1-v^{\rho 2}\right)\left(1-v^2\right)}, \frac{1-v^2+2v^{z2}}{1-v^2}, \frac{2v^z v^\phi}{1-v^2}\right] \tag{103}$$

$$\mathbf{l}_{o,2} = \left[-\frac{\rho^2 v^\phi\left(1+v^2-2v^{\rho 2}\right)}{\left(1-v^{\rho 2}\right)\left(1-v^2\right)}, \frac{v^\rho \rho^2 v^\phi\left(1+v^2-2v^{\rho 2}\right)}{\left(1-v^{\rho 2}\right)\left(1-v^2\right)}, \frac{2\rho^2 v^\phi v^z}{1-v^2}, \frac{1-v^2+2\rho^2 v^{\phi 2}}{1-v^2}\right] \tag{104}$$

$$\mathbf{l}_\pm = \frac{\pm}{\Delta}\left[\begin{array}{c} -c_s^2\lambda_\mp v^{\rho 3}v^2 + 3c_s^2 v^\rho\left(v^{z2}+\rho^2 v^{\phi 2}\right)\left(v^\rho-\lambda_\mp\right) + \\ \left(c_s^2+1\right)\left(1-v^{\rho 2}\right) + \left(c_s^2+1\right)\left(v^{z2}+\rho^2 v^{\phi 2}\right) + \left(c_s^2+1\right)\lambda_\mp v^\rho\left(3v^{\rho 2}-2\right) + \\ \lambda_\mp v^\rho\left(3-4v^{\rho 2}\right) + v^{\rho 4} - \left(v^{z2}+\rho^2 v^{\phi 2}\right) - 1 \\ \left[1+\left(2c_s^2-1+2v^{\rho 2}c_s^2\right)v^{\rho 2}\right]v^\rho + \left(-3c_s^2-v^{\rho 2}c_s^2\right)v^\rho v^2 + \\ \left[-1+\left(1-4v^{\rho 2}c_s^2\right)v^{\rho 2}+\left(3v^{\rho 2}c_s^2+c_s^2\right)v^2\right]\lambda_\mp \\ -2c_s^2 v^z\left(1-v^{\rho 2}\right)\left(1-2\lambda_\mp v^\rho+v^{\rho 2}\right) \\ -2c_s^2 v^\phi\left(1-v^{\rho 2}\right)\left(1-2\lambda_\mp v^\rho+v^{\rho 2}\right) \end{array}\right] \tag{105}$$

$$\Delta = \left(1-v^{\rho 2}\right)^3 c_s^2\left(\lambda_- - \lambda_+\right)\left(1-v^2\right)\left(1-c_s^2 v^2\right) \tag{106}$$

Left Eigenvectors for $\mathbf{B}^z$:

$$\mathbf{l}_{o,1} = \left[-\frac{v^\rho\left(1+v^2-2v^{z2}\right)}{\left(1-v^{z2}\right)\left(1-v^2\right)}, \frac{1-v^2-2v^{\rho 2}}{1-v^2}, \frac{v^\rho v^z\left(1+v^2-2v^{z2}\right)}{\left(1-v^{z2}\right)\left(1-v^2\right)}, \frac{2v^\rho v^\phi}{1-v^2}\right] \tag{107}$$

$$\mathbf{l}_{o,2} = \left[-\frac{\rho^2 v^\phi\left(1+v^2-2v^{z2}\right)}{\left(1-v^{z2}\right)\left(1-v^2\right)}, \frac{2\rho^2 v^\rho v^\phi}{1-v^2}, \frac{\rho^2 v^z v^\phi\left(1+v^2-2v^{z2}\right)}{\left(1-v^{z2}\right)\left(1-v^2\right)}, \frac{1-v^2+2\rho^2 v^{\phi 2}}{1-v^2}\right] \tag{108}$$

$$\mathbf{l}_\pm = \frac{\pm}{\Delta_z}\left[\begin{array}{c} -c_s^2\lambda_\mp v^{z3}v^2 + 3c_s^2 v^z\left(v^{\rho 2}+\rho^2 v^{\phi 2}\right)\big)\left(v^z-\lambda_\mp\right) + \\ \left(c_s^2+1\right)\left(1-v^{z2}\right) + \left(c_s^2+1\right)\left(v^{\rho 2}+\rho^2 v^{\phi 2}\right) + \left(c_s^2+1\right)\lambda_\mp v^z\left(3v^{z2}-2\right) + \\ \lambda_\mp v^z\left(3-4v^{z2}\right) + v^{z4} - \left(v^{\rho 2}+\rho^2 v^{\phi 2}\right) - 1 \\ -2c_s^2 v^\rho\left(1-v^{z2}\right)\left(1-2\lambda_\mp v^z+v^{z2}\right) \\ \left[1+\left(2c_s^2-1+2c_s^2 v^{z2}\right)v^{z2}\right]v^z + \left(-3c_s^2-c_s^2 v^{z2}\right)v^z v^2 + \\ \left[-1+\left(1-4c_s^2 v^{z2}\right)v^{z2}+\left(3c_s^2 v^{z2}+c_s^2\right)v^2\right]\lambda_\mp \\ -2c_s^2 v^\phi\left(1-v^{z2}\right)\left(1-2\lambda_\mp v^z+v^{z2}\right) \end{array}\right] \tag{109}$$

$$\Delta_z = \left(1-v^{z2}\right)^3 c_s^2\left(\lambda_- - \lambda_+\right)\left(1-v^2\right)\left(1-c_s^2 v^2\right) \tag{110}$$

where in this case the λ's stand for the z eigenvalues. All but the 2nd component of $l_\pm$ compare compare with [4].

References

[1] ApJ, 476:221-231 (1997)

[2] `gr-qc/9811015`

[3] `astro-ph/9911034`

[4] `http://www.aei-potsdam.mpg.de/~hawke/thesis/thesis.html`