ENO Approximations in 1D. [1]

Newton divided Differences

$$
\begin{aligned}
V[x_{i-1/2}] &= V(x_{i-1/2}) && (1)\\
V[x_{i-1/2}, x_{i+1/2}] &= \frac{V[x_{i+1/2}] - V[x_{i-1/2}]}{x_{i+1/2} - x_{i-1/2}} = \\
&\quad \frac{V(x_{i+1/2}) - V(x_{i-1/2})}{x_{i+1/2} - x_{i-1/2}} && (2)\\
V[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}] &= \frac{V[x_{i+1/2}, x_{i+3/2}] - V[x_{i-1/2}, x_{i+1/2}]}{x_{i+3/2} - x_{i-1/2}} = \\
&\quad \frac{1}{x_{i+3/2} - x_{i-1/2}} \left(\frac{V(x_{i+3/2}) - V(x_{i+1/2})}{x_{i+3/2} - x_{i+1/2}} - \frac{V(x_{i+1/2}) - V(x_{i-1/2})}{x_{i+1/2} - x_{i-1/2}} \right)
\end{aligned}
$$

In general:

$$V[x_{i-1/2}, ..., x_{i+j-1/2}] = \frac{V[x_{i+1/2}, ..., x_{i+j-1/2}] - V[x_{i-1/2}, ..., x_{i+j-3/2}]}{x_{i+j-1/2} - x_{i-1/2}} \tag{3}$$

If the mesh is uniform[2]:

$$V[x_{i-1/2}, ..., x_{i+j-1/2}] = \frac{D^j V[x_{i-1/2}]}{j!\,(\Delta x)^j} \tag{4}$$

where D^j is the forward difference:

$$
\begin{aligned}
DV[x_{i-1/2}] &= V(x_{i+1/2}) - V(x_{i-1/2}) && (5)\\
D^2V[x_{i-1/2}] &= V(x_{i+3/2}) - 2V(x_{i+1/2}) + V(x_{i-1/2}) && (6)\\
D^3V[x_{i-1/2}] &= V(x_{i+5/2}) - 3V(x_{i+3/2}) + 3V(x_{i+1/2}) - V(x_{i-1/2}) && (7)\\
& && (8)
\end{aligned}
$$

Newton Undivided differences

$$V\langle x_{i-1/2}, x_{i+1/2}\rangle = V[x_{i-1/2}, x_{i+1/2}] = \bar{q}_i \tag{9}$$

$$V\langle x_{i-1/2}, ..., x_{i+j+1/2}\rangle = V\langle x_{i+1/2}, ..., x_{i+j+1/2}\rangle - V\langle x_{i-1/2}, ..., x_{i+j-1/2}\rangle \tag{10}$$

ENO recostruction

From the cell averages $\{\bar{q}_i\}$ construct the primitive function:

$$V(x_{i+1/2}) = \sum_{j=j_{min}}^{i} \bar{q}_j \Delta x_j \tag{11}$$

Calculate the Newton divided (undivided for uniform mesh) differences for V for the stencil:

$$\tilde{S}_2(i) = \{x_{i-1/2}, x_{i+1/2}\}. \tag{12}$$

For $l = 2, ..., k$, assuming

$$\tilde{S}_l(i) = \{x_{j+1/2}, ...x_{j+l-1/2}\} \tag{13}$$

is known, add one of the neighbouring points, $x_{j-1/2}$ or $x_{j+l+1/2}$, to the stencil:

- If

  $$|V\left[x_{j-1/2}, ..., x_{j+l-1/2}\right]| < |V\left[x_{j+1/2}, ..., x_{j+l+1/2}\right]| \tag{14}$$

  add $x_{j-1/2}$ to the stencil $\tilde{S}_l$

- Otherwise, add $x_{j+l+1/2}$

In cartesian coordinates use either a Newton interpolant:

$$p(x) = \sum_{j=1}^{k} V\left[x_{i-r-1/2}, ..., x_{i-r+j-1/2}\right] \sum_{m=0}^{j-1} \prod_{\substack{l=0 \\ l \neq m}}^{j-1} \left(x - x_{i-r+l-1/2}\right) \tag{15}$$

or a Lagrangian one:

$$p\left(x\right) = \sum_{m=0}^{k} \sum_{j=0}^{m-1} \bar{q}_{i-r+j} \Delta x_{i-r+j} \left(\frac{\sum^{k}_{\substack{l=0 \\ l \neq m}} \prod^{k}_{\substack{q=0 \\ q \neq m,l}} \left(x - x_{i-r+l-1/2}\right)}{\prod^{k}_{\substack{l=0 \\ l \neq m}} \left(x_{i-r+m-1/2} - x_{i-r+l-1/2}\right)} \right) \tag{16}$$

In spherical symmetry:

$$V\left(r_{i+1/2}\right) = \sum_{j=j_{min}}^{i} \bar{q}_j \left(r^3_{j+1/2} - r^3_{j-1/2}\right) \tag{17}$$

and the Lagrangian interpolant changes to:

$$p\left(r\right)=\frac{1}{3\ r^{2}}\sum_{m=0}^{k}\sum_{j=0}^{m-1}\bar{q}_{i-r+j}\left(r_{i-r+j+1/2}^{3}-r_{i-r+j-1/2}^{3}\right)\left(\frac{\sum^{k}_{\substack{l=0\\ l\neq m}}\prod^{k}_{\substack{q=0\\ q\neq m,l}}\left(r-r_{i-r+l-1/2}\right)}{\prod^{k}_{\substack{l=0\\ l\neq m}}\left(r_{i-r+m-1/2}-r_{i-r+l-1/2}\right)}\right) \tag{18}$$

Header for the subroutine: eno_interpolation.f

```
'
         subroutine  eno_interpolation(qi, xi,     Nxi,
     &                               q,  x_imh,  Nx,
     &                               size_sten,  geom,
     &                               direction, V_imh )

c-------------------------------------------------------------
c Fri Feb 22 09:22:57 PST 2002
c-------------------------------------------------------------

         implicit      none

         character*17  cdnm
         parameter   ( cdnm = 'eno_interpolation' )

         logical       ltrace

         parameter   ( ltrace = .false. )


c-------------------------------------------------------------
c  Interpolated values (output) at coordinates xi. (input)
c-------------------------------------------------------------
         integer       Nxi

         real*8        qi  ( Nxi )
         real*8        xi  ( Nxi )

c-------------------------------------------------------------
```

```
c  Cell averages and cell borders coordinates. (input)
c-------------------------------------------------------------

         integer       Nx

         real*8        q     ( Nx )
         real*8        x_imh ( Nx )

c-------------------------------------------------------------
c  Mass function. (internal)
c-------------------------------------------------------------

         real*8        V_imh ( Nx )
         real*8        aux   ( 1000 )

c-------------------------------------------------------------
c  Size of the eno stencil. (input)
c-------------------------------------------------------------
         integer       size_sten

c-------------------------------------------------------------
c   geom = 0 for cartesian       (input)
c        = 1 for spherical symmetry.
c-------------------------------------------------------------

         integer       geom

c-------------------------------------------------------------
c   direction =  1 for right reconstruction. (input)
c             = -1 for left  reconstruction.
c
c               (-1)       |      (1)
c                          |
c              =======>    |    <========
c                          |
c                          |
c  Only afects if xi is a border of a cell.
```

```
c-----------------------------------------------------------
        integer       direction

        integer       pt,  pt2
        integer       i,   s
```

References

[1] NASA/CR-97-206253 ICASE Report No. 97-65

[2] http://dmpeli.math.mcmaster.ca/Matlab/CLLsoftware/NumMethods/Lecture2-2.html