# 2+1+1

Ignacio (Iñaki) Olabarrieta

## 1 Derivation of hydrodynamic equations in the 2+1+1 formalism.

In this appendix we give a detail derivation the of the hydrodynamic equation in the 2+1+1 formalism filling the gaps not given in [?]. The stress-energy tensor can be written as:

$$T^{\mu\nu} = \frac{1}{s^4}\tau\xi^\mu\xi^\nu + \frac{1}{s^2}\tau^a\xi^\mu g^\nu{}_a + \frac{1}{s^2}\tau^a\xi^\nu g^\mu{}_a + \tau^{ab}g^\mu{}_a g^\nu{}_b \tag{1}$$

where:

$$\begin{aligned}
\tau &= T_{\mu\nu}\xi^\mu\xi^\nu & (2)\\
\tau^a &= g^{a\mu}\xi^\nu T_{\mu\nu} & (3)\\
\tau_{ab} &= g^\mu_a g^\nu_b T_{\mu\nu} & (4)
\end{aligned}$$

the projection operator is defined as:

$$g_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{s^2}\xi_\mu\xi_\nu \tag{5}$$

$\xi_\mu$ has the following properties:

$$\begin{aligned}
\xi^\mu\xi_\mu &= s^2 & (6)\\
\xi^\mu g^\nu_\mu &= 0 & (7)\\
\xi^\mu{}_{;\mu} &= 0 & (8)
\end{aligned}$$

The fluid equations are derived from:

$$T^\mu{}_{\nu;\mu} = 0 \tag{9}$$

which can be decomposed in the following way:

$$\begin{aligned}
\xi^\nu \left(T^\mu{}_\nu\right)_{;\mu} &= 0 & (10)\\
g_{a\nu}\left(T^{\mu\nu}\right)_{;\mu} &= 0 & (11)
\end{aligned}$$

Let's try to modify first, equation (10):

$$0 = \xi^\nu \left(T^\mu{}_\nu\right)_{;\mu} = \left(\xi^\nu T^\mu{}_\nu\right)_{;\mu} - T^\mu{}_\nu \left(\xi^\nu\right)_{;\mu} \tag{12}$$

the second term on the *rhs* is zero because $\xi^\nu$ satisfices the Killing equations and $T^\mu{}_\nu$ is symmetric. Substituting (1) in (12) and using properties (6) and (7) we get:

$$0 = \left(\frac{1}{s^2}\tau\xi^\mu\right)_{;\mu} + \left(\tau^a g^\mu{}_a\right)_{;\mu} = \left(\frac{\tau}{s^2}\right)_{,\mu}\xi^\mu + \frac{\tau}{s^2}\left(\xi^\mu\right)_{;\mu} + \left(\tau^a\right)_{|a} + \tau^a\left(g^\mu{}_a\right)_{;\mu} \tag{13}$$

where we have used the fact that $g^{\mu}{}_{a}\nabla_{\mu} = D_a$ in the third term of the *rhs*. The first term of (13) is zero because $\pounds_{\xi}(\tau/s^2) = 0$ because $\xi$ is Killing. The second is zero because it is proportional to the trace of the Killing equations (property (8)). The result is:

$$0 = (\tau^a)_{|a} + \tau^a \left(\gamma^{\mu}{}_{a} - \frac{1}{s^2}\xi^{\mu}\xi_a\right)_{;\mu} = (\tau^a)_{|a} - \frac{\tau^a}{s^2}\xi^{\mu}\left(\xi_a\right)_{;\mu} \tag{14}$$

Using the Killing equations, we can express the second term as the derivative of $s^2$:

$$0 = (\tau^a)_{|a} + \frac{\tau^a}{2s^2}\left(s^2\right)_{|a} = \frac{1}{s}\left(s\tau^a\right)_{|a} \tag{15}$$

Let's modify equation (11):

$$0 = g_{a\nu}\left(T^{\mu\nu}\right)_{;\mu} = \left(g_{a\nu}T^{\mu\nu}\right)_{;\mu} - T^{\mu\nu}\left(g_{a\nu}\right)_{;\mu} \tag{16}$$

using (1) and the definition of the projection operator:

$$0 = \left(\frac{1}{s^2}\tau^b g^{\nu}{}_{b}\, g_{a\nu}\,\xi^{\mu}\right)_{;\mu} + \left(g_{a\nu}\, g^{\mu}{}_{b}\, g^{\nu}{}_{c}\,\tau^{bc}\right)_{;\mu} + T^{\mu\nu}\xi_{\nu}\left(\frac{\xi_a}{s^2}\right)_{;\mu} \tag{17}$$

futher manipulations gives:

$$0 = \frac{1}{s^2}\left(\tau_a\right)_{;\mu}\xi^{\mu} + \left(\tau^b{}_{a}\right)_{|b} + \tau^b{}_{a}\left(g^{\mu}{}_{b}\right)_{;\mu} + T^{\mu\nu}\xi_{\nu}\left(\frac{\xi_a}{s^2}\right)_{;\mu} \tag{18}$$

which can be further reduced to:

$$0 = \frac{1}{s^2}\left(\tau_a\right)_{;\mu}\xi^{\mu} + \frac{1}{s}\left(s\tau^b{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left(\frac{\xi_a}{s^2}\right)_{;\mu} \tag{19}$$

Let's focus for now in the first term:

$$\left(\tau_a\right)_{;\mu}\xi^{\mu} = \left(g^{\lambda}{}_{a}\,\xi^{\sigma}\,T_{\lambda\sigma}\right)_{;\mu}\xi^{\mu} \tag{20}$$

which reduces to:

$$\left(g^{\lambda}{}_{a}\right)_{;\mu}\xi^{\sigma}T_{\lambda\sigma}\xi^{\mu} + g^{\lambda}{}_{a}\xi^{\sigma}{}_{;\mu}T_{\lambda\sigma}\xi^{\mu} + g^{\lambda}{}_{a}\left(T_{\lambda\sigma}\right)_{;\mu}\xi^{\mu}\xi^{\sigma} \tag{21}$$

Now we use the fact that:

$$0 = \pounds_{\xi}T_{\mu\nu} = \xi^{\lambda}\left(T_{\mu\nu}\right)_{;\lambda} + T_{\lambda\mu}\left(\xi^{\lambda}\right)_{;\nu} + T_{\nu\lambda}\left(\xi^{\lambda}\right)_{;\mu} \tag{22}$$

To further reduce the expression:

$$\left(g^{\lambda}{}_{a}\right)_{;\mu}\xi^{\sigma}T_{\lambda\sigma}\xi^{\mu} + g^{\lambda}{}_{a}\left(\xi^{\sigma}\right)_{;\mu}T_{\lambda\sigma}\xi^{\mu} - g^{\lambda}{}_{a}\xi^{\sigma}T_{\mu\lambda}\left(\xi^{\mu}\right)_{;\sigma} - g^{\lambda}{}_{a}\xi^{\sigma}T_{\mu\sigma}\left(\xi^{\mu}\right)_{;\lambda} \tag{23}$$

The second and the third terms are equal with oposite sings (just by renaming the dummy indexes) which gives us:

$$\left(\tau_a\right)_{;\mu}\xi^{\mu} = \left(g^{\lambda}{}_{a}\right)_{;\mu}\xi^{\sigma}T_{\lambda\sigma}\xi^{\mu} - g^{\lambda}{}_{a}\xi^{\sigma}T_{\mu\sigma}\left(\xi^{\mu}\right)_{;\lambda} = \xi^{\sigma}T_{\mu\sigma}\left[\left(g^{\mu}{}_{a}\right)_{;\lambda}\xi^{\lambda} - g^{\lambda}{}_{a}\left(\xi^{\mu}\right)_{;\lambda}\right] \tag{24}$$

Now we can introduce this result in the equation:

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[\frac{1}{s^{2}}\left(g_{\mu_{a}}\right)_{;\lambda}\xi^{\lambda} - \frac{1}{s^{2}}g^{\lambda}{}_{a}\left(\xi_{\mu}\right)_{;\lambda} + \left(\frac{\xi_{a}}{s^{2}}\right)_{;\mu}\right] \tag{25}$$

the expression in brackets can be simplified:

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[-\frac{1}{s^{4}}\left(\xi_{\mu}\xi_{a}\right)_{;\lambda}\xi^{\lambda} - \frac{1}{s^{2}}\gamma^{\lambda}{}_{a}\left(\xi_{\mu}\right)_{;\lambda} + \frac{1}{s^{4}}\xi^{\lambda}\xi_{a}\left(\xi_{\mu}\right)_{;\lambda} + \left(\frac{\xi_{a}}{s^{2}}\right)_{;\lambda}\right] \tag{26}$$

this results in:

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[-\frac{1}{s^4}\left(\xi_{\mu}\right)_{;\lambda}\xi_{a}\xi^{\lambda} - \frac{1}{s^4}\xi_{\mu}\left(\xi_{a}\right)_{;\lambda}\xi^{\lambda} - \frac{1}{s^2}\left(\xi_{\mu}\right)_{;a} + \frac{1}{s^4}\xi^{\lambda}\xi_{a}\left(\xi_{\mu}\right)_{;\lambda} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] \tag{27}$$

the first term inside bracket is equal to the fourth one but with opposite sign.

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[-\frac{1}{s^4}\xi_{\mu}\left(\xi_{a}\right)_{;\lambda}\xi^{\lambda} - \frac{1}{s^2}\left(\xi_{\mu}\right)_{;a} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] \tag{28}$$

applying Killing equations:

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[\frac{1}{s^4}\xi_{\mu}\left(\xi_{\lambda}\right)_{;a}\xi^{\lambda} - \frac{1}{s^2}\left(\xi_{\mu}\right)_{;a} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] \tag{29}$$

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[\frac{1}{2s^4}\xi_{\mu}\left(s^2\right)_{;a} - \frac{1}{s^2}\left(\xi_{\mu}\right)_{;a} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] \tag{30}$$

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[\frac{1}{s^3}\xi_{\mu}\left(s\right)_{;a} - \left(\frac{\xi_{\mu}}{s^2}\right)_{;a} - \frac{2}{s^3}\xi_{\mu}\left(s\right)_{;a} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] \tag{31}$$

$$0 = \frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[-\left(\frac{\xi_{\mu}}{s^2}\right)_{;a} - \frac{1}{s^3}\xi_{\mu}\left(s\right)_{;a} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] \tag{32}$$

using the same decomposition as to derive Valencia equation we get:

$$\frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} + T^{\mu\nu}\xi_{\nu}\left[-\left(\frac{\xi_{\mu}}{s^2}\right)_{;a} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] - \tau\frac{\partial_{a}s}{s^3} = 0 \tag{33}$$

Using the following expression (A12 from Geroch):

$$\nabla_{\mu}\xi_{\nu} = \frac{1}{2s^2}\epsilon_{\mu\nu\alpha\beta}\xi^{\alpha}\omega^{\beta} + \frac{1}{2s^2}\xi_{[\nu}D_{\mu]}s^2 \tag{34}$$

the second term of the equation above:

$$T^{\mu\nu}\xi_{\nu}\left[-\left(\frac{\xi_{\mu}}{s^2}\right)_{;a} + \left(\frac{\xi_{a}}{s^2}\right)_{;\mu}\right] = -\frac{1}{s^4}\tau^{b}g^{\nu}{}_{b}\,\epsilon_{a\nu\gamma\sigma}\xi^{\gamma}\omega^{\sigma} = \frac{1}{s^3}\tau^{b}\epsilon_{abc}\omega^{c} \tag{35}$$

therefore:

$$\frac{1}{s}\left(s\tau^{b}{}_{a}\right)_{|b} - \tau\frac{s_{|a}}{s^3} + \frac{1}{s^3}\tau^{b}\epsilon_{abc}\omega^{c} \tag{36}$$

Now we continue with the (2+1) decomposition. In this case we use the following projection operator:

$$h^a{}_b = g^a{}_b + n^a n_b \tag{37}$$

with the following properties:

$$\begin{aligned}
n^a n_a &= -1 && (38)\\
a_b &= n^c \left(n_b\right)_{|c} = \partial_b \left(\alpha\right)/\alpha && (39)\\
n_a &= \left(-\alpha, 0, 0\right) && (40)\\
n^a &= \left(\frac{1}{\alpha}, -\frac{\beta^A}{\alpha}\right) && (41)\\
\left(n^a\right)_{|a} &= \frac{1}{\alpha\sqrt{h}} \pounds_t \left(\sqrt{h}\right) - \frac{1}{\alpha}\left(\beta^A\right)_{||A} && (42)
\end{aligned}$$

Let's modify equation (15), the first thing we should do is to decompose $\tau^a$ along $n^a$ and perpendicular to it:

$$\begin{aligned}
\tau^a &= J_\phi n^a + S^A h^a_A && (43)\\
J_\phi &= -n_a \tau^a && (44)\\
S^A &= h^A_b \tau^b && (45)
\end{aligned}$$

Inserting this decomposition into equation (15):

$$0 = \frac{1}{s}\left(sJ_\phi n^a\right)_{|a} + \frac{1}{s}\left(sS^A h^a_A\right)_{|a} \tag{46}$$

$$0 = \frac{1}{s}\left(sJ_\phi\right)_{|a} n^a + J_\phi \left(n^a\right)_{|a} + \frac{1}{s}\left(sS^A\right)_{||A} + S^A \left(h^a_A\right)_{|a} \tag{47}$$

defining $N^a = \alpha\, n^a$

$$0 = \frac{1}{\alpha s}\left(sJ_\phi\right)_{|a} N^a + J_\phi \left[\frac{1}{\alpha\sqrt{h}}\pounds_t\left(\sqrt{h}\right) - \frac{1}{\alpha}\left(\beta^A\right)_{||A}\right] + \frac{1}{s}\left(sS^A\right)_{||A} + S^A \left(n^a n_A\right)_{|a} \tag{48}$$

$$0 = \frac{1}{\alpha s}\pounds_N\left(sJ_\phi\right) + \frac{J_\phi}{\alpha\sqrt{h}}\pounds_t\left(\sqrt{h}\right) - \frac{J_\phi}{\alpha}\left(\beta^A\right)_{||A} + \frac{1}{s}\left(sS^A\right)_{||A} + S^A \left(n^a n_A\right)_{|a} \tag{49}$$

using the fact that $t^a = N^a + \beta^a$ and $n_A = (0,0)$:

$$0 = \frac{1}{\alpha s}\pounds_t\left(sJ_\phi\right) - \frac{1}{\alpha s}\pounds_\beta\left(sJ_\phi\right) + \frac{J_\phi}{\alpha\sqrt{h}}\pounds_t\left(\sqrt{h}\right) - \frac{J_\phi}{\alpha}\left(\beta^A\right)_{||A} + \frac{1}{s}\left(sS^A\right)_{||A} + S^A n^a \left(n_A\right)_{|a} \tag{50}$$

regrouping terms:

$$0 = \frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_\phi\right) - \frac{1}{\alpha s}\left(sJ_\phi\right)_{||A}\beta^A + -\frac{J_\phi}{\alpha}\left(\beta^A\right)_{||A} + \frac{1}{s}\left(sS^A\right)_{||A} + S^A n^a \left(n_A\right)_{|a} \tag{51}$$

$$0 = \frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_\phi\right) - \frac{1}{\alpha s}\left(sJ_\phi\right)_{||A}\beta^A + -\frac{J_\phi}{\alpha}\left(\beta^A\right)_{||A} + \frac{1}{s}\left(sS^A\right)_{||A} + S^A \frac{\alpha_{||A}}{\alpha} \tag{52}$$

$$0 = \frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_\phi\right) - \frac{1}{\alpha s}\left(sJ_\phi\beta^A\right)_{||A} + \frac{1}{\alpha s}\left(\alpha s S^A\right)_{||A} \tag{53}$$

Finally:

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$$0 = \frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_\phi\right) + \frac{1}{\alpha s}\left[\alpha s\left(S^A - J_\phi\frac{\beta^A}{\alpha}\right)\right]_{||A} \tag{54}$$

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Let's decompose now equation (36):

$$0 = \frac{1}{s}\left(s\tau_a^b\right)_{|b} - \tau\frac{s_{|a}}{s^3} + \frac{1}{s^3}\tau^b\epsilon_{abc}\omega^c \tag{55}$$

Let's project this equations along the direction paralell and perpendicular to $n^a$:

$$0 = n^a\frac{1}{s}\left(s\tau^b{}_a\right)_{|b} - \tau n^a\frac{s_{|a}}{s^3} + n^a\frac{1}{s^3}\tau^b\epsilon_{abc}\omega^c \tag{56}$$

$$0 = h^a{}_A\frac{1}{s}\left(s\tau^b{}_a\right)_{|b} - \tau h^a{}_A\frac{s_{|a}}{s^3} + h^a{}_A\frac{1}{s^3}\tau^b\epsilon_{abc}\omega^c \tag{57}$$

Let's start modifiying the 1st term of equation (56):

$$\frac{n^a}{s}\left(s\tau^b{}_a\right)_{|b} = \frac{1}{s}\left(s\,n^a\tau^b{}_a\right)_{|b} - \tau^b{}_a\left(n^a\right)_{|b} \tag{58}$$

using the decomposition:

$$\tau^b{}_a = n_a n^b\rho_H + n_a h^b{}_B J^B + n^b h_{aA}J^A + h^{bB}h_a{}^A S_{AB} \tag{59}$$

where we have made use of the following definitions:

$$\rho_H = n_a n_b\tau^{ab} \tag{60}$$

$$J^B = -n_a h_b^B\tau^{ab} \tag{61}$$

$$S_{AB} = h_{Aa}h_{Bb}\tau^{ab} \tag{62}$$

$$\frac{1}{s}\left[s\left(-n^b\rho_H - h^b{}_B J^B\right)\right]_{|b} - \tau^b{}_a\left(n^a\right)_{|b} \tag{63}$$

$$-\frac{1}{s}\left(s\rho_H\right)_{|b}n^b - \rho_H\left(n^b\right)_{|b} - \frac{h^b{}_B}{s}\left(sJ^B\right)_{|b} - J^B\left(h^b{}_B\right)_{|b} - \tau^b{}_a\left(n^a\right)_{|b} \tag{64}$$

$$-\frac{1}{\alpha s}\pounds_N\left(s\rho_H\right) - \rho_H\left[\frac{1}{\alpha\sqrt{h}}\pounds_t\left(\sqrt{h}\right) - \frac{1}{\alpha}\left(\beta^A\right)_{||A}\right] - \frac{1}{s}\left(s\,J^B\right)_{||B} - J^B\left(n^b n_B\right)_{|b} - \tau^b{}_a\left(n^a\right)_{|b} \tag{65}$$

$$-\frac{1}{\alpha s}\pounds_t\left(s\rho_H\right) + \frac{1}{\alpha s}\pounds_\beta\left(s\rho_H\right) - \frac{\rho_H}{\alpha\sqrt{h}}\pounds_t\left(\sqrt{h}\right) + \frac{\rho_H}{\alpha}\left(\beta^A\right)_{||A} - \frac{1}{s}\left(s\,J^B\right)_{||B} - J^B\frac{\alpha_{||B}}{\alpha} - \tau^b{}_a\left(n^a\right)_{|b} \tag{66}$$

$$-\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}\rho_H\right) + \frac{1}{\alpha s}\left(s\rho_H\right)_{||A}\beta^A + \frac{\rho_H}{\alpha}\left(\beta^A\right)_{||A} - \frac{1}{\alpha s}\left(\alpha s\,J^B\right)_{||B} - \tau^b{}_a\left(n^a\right)_{|b} \tag{67}$$

$$-\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}\rho_H\right) - \frac{1}{\alpha s}\left[\alpha s\left(J^A - \rho_H\frac{\beta^A}{\alpha}\right)\right]_{||A} - \tau^b{}_a\left(n^a\right)_{|b} \tag{68}$$

Using the following property:

$$\left(n_a\right)_{|b} = -K_{ab} - n_b \left(a\right)_{|a} \tag{69}$$

and using the fact that $n^a K_{ab} = 0$ and $n^b a_{|b} = 0$:

$$-\frac{1}{\alpha s \sqrt{h}} \pounds_t \left( s\sqrt{h} \rho_H \right) - \frac{1}{\alpha s} \left[ \alpha s \left( J^A - \rho_H \frac{\beta^A}{\alpha} \right) \right]_{||A} + S_{AB} K^{AB} - J^A \frac{\alpha_{||A}}{\alpha} \tag{70}$$

using now

$$\begin{aligned} S_{AB} &= \left(P + \rho_H\right) v^A v^B + h^{AB} P \qquad (71) \\ J^A &= \left(P + \rho_H\right) v^A \qquad (72) \end{aligned}$$

$$-\frac{1}{\alpha s \sqrt{h}} \pounds_t \left( s\sqrt{h} \rho_H \right) - \frac{1}{\alpha s} \left[ \alpha s \left( J^A - \rho_H \frac{\beta^A}{\alpha} \right) \right]_{||A} + \left(P + \rho_H\right) \left[ v^A v^B K_{AB} - v^A \frac{\alpha_{||A}}{\alpha} \right] + P K^A{}_A \tag{73}$$

The second term of equation (56) :

$$-\tau \frac{s_{|a} n^a}{s^3} = \frac{\tau}{s^3} \kappa \tag{74}$$

The third term:

$$\frac{1}{s^3} \tau^b \epsilon_{abc} \omega^c n^a \tag{75}$$

using:

$$w^a = -n^a \Omega_H + h^{aB} \Omega_B \tag{76}$$

where we have defined:

$$\begin{aligned} \Omega_H &= n_a \omega^a \qquad (77) \\ \Omega_A &= h^b{}_A \omega^b \qquad (78) \end{aligned}$$

$$\frac{1}{s^3} \left( J_\phi n^b + S^A h^b{}_A \right) \epsilon_{abc} \left( -n^c \Omega_H + h^{cB} \right) n^a \tag{79}$$

the only term that survives is the one with only one $n^a$ by the antisymmetry of $\epsilon_{abc}$:

$$\frac{1}{s^3} S^A h^b{}_A h^{cB} \Omega_B \epsilon_{abc} n^a \tag{80}$$

which results in the following:

$$\frac{1}{s^3} \tau^b \epsilon_{abc} \omega^c n^a = \frac{1}{s^3} S_A \Omega_B \epsilon^{AB} = \frac{2}{s^2} E^A S_A \tag{81}$$

where we have used the definitions:

$$\begin{aligned} \epsilon_{ab} &= n^c \epsilon_{cab} \qquad (82) \\ E^A &= \frac{1}{2s} \epsilon^{AB} \Omega_B \qquad (83) \end{aligned}$$

$$\frac{1}{s^2} \tau^b \epsilon_{abc} \omega^c n^a = \frac{1}{s^3} S_A \Omega_B \epsilon^{AB} = J_\phi \frac{2}{s^2} E_A v^A \tag{84}$$

collecting all the terms together:

$$-\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}\rho_H\right)-\frac{1}{\alpha s}\left[\alpha s\left(J^A-\rho_H\frac{\beta^A}{\alpha}\right)\right]_{||A} \tag{85}$$
$$+\left(P+\rho_H\right)\left[v^Av^BK_{AB}-v^A\frac{\alpha_{||A}}{\alpha}\right]+PK^A{}_A+\tau\frac{\kappa}{s^3}+J_\phi\frac{2}{s^2}E_Av^A=0$$

Note that $J^A=\left(\rho_H+P\right)v^A$, $\tau=J_\phi v^\phi s^2+Ps^2$, $E_\phi=\kappa s$. The final equation for the conservation of energy is:

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$$-\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}\rho_H\right)-\frac{1}{\alpha s}\left[\alpha s\left(J^A-\rho_H\frac{\beta^A}{\alpha}\right)\right]_{||A} \tag{86}$$
$$+\left(P+\rho_H\right)\left[v^Av^BK_{AB}-v^A\frac{\alpha_{||A}}{\alpha}\right]+P\left(K^A{}_A+\frac{1}{s}\kappa\right)+\frac{1}{s^2}J_\phi\left(\frac{1}{s^2}E_\phi v_\phi+2E_Av^A\right)=0$$

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Now we take care of the equation (57), the first term of that equation gives us:

$$0=h^a{}_A\frac{1}{s}\left(s\tau^b{}_a\right)_{|b}=\frac{1}{s}\left(sh^a{}_A\tau^b{}_a\right)_{|b}-\tau^b{}_a\left(h^a{}_A\right)_{|b} \tag{87}$$

Applying the decomposition

$$\tau^b{}_a=n_an^b\rho_H+n_ah^b{}_BJ^B+n^bh_{aA}J^A+h^{bB}h_a{}^CS_{BC} \tag{88}$$

the equation results in:

$$\frac{1}{s}\left[sh^a{}_An^bh_{aC}J^C+h^{bB}h^a{}_AS_{BC}h^C{}_a\right]_{|b}-\tau^b{}_a\left(h^a{}_A\right)_{|b} \tag{89}$$

$$\frac{1}{s}\left(sJ_An^b+sS_{AB}h^{bB}\right)_{|b}-\tau^b{}_a\left(h^a{}_A\right)_{|b} \tag{90}$$

$$J_A\left(n^b\right)_{|b}+\frac{1}{s}\left(sJ_A\right)_{|b}n^b+\frac{1}{s}\left(sS_{AB}\right)_{|b}h^{bB}+S_{AB}\left(h^{bB}\right)_{|b}-\tau^b{}_a\left(n^an_A\right)_{|b} \tag{91}$$

using property (42):

$$J_A\left[\frac{1}{\alpha\sqrt{h}}\pounds_t\left(\sqrt{h}\right)-\frac{1}{\alpha}\left(\beta^B\right)_{||B}\right]+\frac{1}{\alpha s}\pounds_N\left(sJ_A\right)+\frac{1}{s}\left(sS^B{}_A\right)_{||B}+$$
$$S^B{}_A\frac{\alpha_{||B}}{\alpha}+\rho_Hn^b\left(n_A\right)_{|b}+h^b{}_BJ^B\left(n_A\right)_{|b} \tag{92}$$

using the fact that $t^a=N^a+\beta^a$

$$\frac{J_A}{\alpha\sqrt{h}}\pounds_t\left(\sqrt{h}\right)-\frac{J_A}{\alpha}\left(\beta^B\right)_{||B}+\frac{1}{\alpha s}\pounds_t\left(sJ_A\right)-\frac{1}{\alpha s}\left(sJ_A\right)_{||B}\beta^B-$$
$$\frac{J_B}{\alpha}\left(\beta^B\right)_{||A}+\frac{1}{\alpha s}\left(\alpha sS^B{}_A\right)_{||B}+\rho_Hn^b\left(n_A\right)_{|b}+J^B\left(n_A\right)_{||B} \tag{93}$$

collecting terms:

$$\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_A\right)+\frac{1}{\alpha s}\left[\alpha s\left(S^B{}_A-\frac{\beta^B}{\alpha}J_A\right)\right]_{||B}-$$
$$\frac{J_B}{\alpha}\left(\beta^B\right)_{||A}+\rho_H n^b\left(n_A\right)_{|b}+J^B\left(n_A\right)_{||B} \tag{94}$$

Using identity (4.7.6) from Weinberg:

$$\left(Y^B{}_A\right)_{||B}=\frac{1}{\sqrt{h}}\left(\sqrt{h}Y^B{}_A\right)_{,B}-Y^{BC}\left(h_{CA}\right)_{,B}+h_{CA}\Gamma^C{}_{BD}Y^{BD} \tag{95}$$

it results in the following:

$$\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_A\right)+\frac{1}{\alpha s\sqrt{h}}\left[\alpha s\sqrt{h}\left(S^B{}_A-\frac{\beta^B}{\alpha}J_A\right)\right]_{,B}-$$
$$\left(S^{BC}-\frac{\beta^B}{\alpha}J^C\right)\left[\left(h_{CA}\right)_{,B}-{}^{(2)}\Gamma_{ABC}\right]-$$
$$\frac{J_B}{\alpha}\left(\beta^B\right)_{||A}+\rho_H n^b\left(n_A\right)_{|b}+J^B\left(n_A\right)_{||B} \tag{96}$$

expanding the Christoffel symbols:

$$\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_A\right)+\frac{1}{\alpha s\sqrt{h}}\left[\alpha s\sqrt{h}\left(S^B{}_A-\frac{\beta^B}{\alpha}J_A\right)\right]_{,B}-$$
$$\left(S^{BC}-\frac{\beta^B}{\alpha}J^C\right)\left[\frac{1}{2}\left(h_{AC}\right)_{,B}-\frac{1}{2}\left(h_{AB}\right)_{,C}+\frac{1}{2}\left(h_{BC}\right)_{,A}\right]-$$
$$\frac{J_B}{\alpha}\left\{\left(\beta^B\right)_{,A}+\frac{1}{2}h^{BD}\beta^C\left[\left(h_{DC}\right)_{,A}+\left(h_{DA}\right)_{,C}-\left(h_{AC}\right)_{,D}\right]\right\}+$$
$$\rho_H n^b\left(n_A\right)_{|b}+J^B\left(n_A\right)_{||B} \tag{97}$$

Renaming the dummy indexes:

$$\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_A\right)+\frac{1}{\alpha s\sqrt{h}}\left[\alpha s\sqrt{h}\left(S^B{}_A-\frac{\beta^B}{\alpha}J_A\right)\right]_{,B}-S^{BC}\frac{1}{2}\left(h_{BC}\right)_{,A}+$$
$$\rho_H\frac{\alpha_{||A}}{\alpha}+J^B\left(n_A\right)_{||B}-J_B\frac{\beta^B_{,A}}{\alpha} \tag{98}$$

Now we deal with the next term of the equation:

$$\frac{1}{s^3}\tau^b \epsilon_{abc}\omega^c h^a{}_A \tag{99}$$

using the decomposition:

$$\tau^b = n^b J_\phi + h^b{}_C S^C \tag{100}$$

and

$$w^c = -\Omega_H n^c + \Omega_B h^{Bc} \tag{101}$$

we get:

$$-\frac{1}{s^3} J_\phi \epsilon_{AB}\Omega^B - \frac{1}{s^3}\epsilon_{AC}\Omega_H \tag{102}$$

where we have use the facts that $h^b{}_C h^a{}_A h^{cB}\epsilon_{abc} = 0$ and the definition of $\epsilon_{AB}$. Now using $B_\phi = 1/2\Omega_H$ and $E_A = 1/(2s)\epsilon_{AB}\Omega^B$

$$-2\frac{1}{s^2} J_\phi E_A - \frac{2}{s^3} B_\phi S^C \epsilon_{AC} \tag{103}$$

using $S^A = J_\phi v^A$:

$$-\frac{1}{s^2} J_\phi \left(2E_A + \frac{2}{s} B_\phi v^C \epsilon_{AC}\right) \tag{104}$$

collecting all the terms together:

$$\begin{aligned}&\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_A\right) + \frac{1}{\alpha s\sqrt{h}}\left[\alpha s\sqrt{h}\left(S^B{}_A - \frac{\beta^B}{\alpha}J_A\right)\right]_{,B} - S^{BC}\frac{1}{2}\left(h_{BC}\right)_{,A} + \\ &\rho_H\frac{\alpha_{||A}}{\alpha} + J^B\left(n_A\right)_{||B} - J_B\frac{\beta^B_{,A}}{\alpha} - \frac{1}{s^2}J_\phi\left(2E_A + \frac{2}{s}B_\phi v^C\epsilon_{AC}\right) - \tau\frac{s_{||A}}{s^3}\end{aligned} \tag{105}$$

using the fact that $\tau = J_\phi v_\phi + Ps^2$:

$$\begin{aligned}&\frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_A\right) + \frac{1}{\alpha s\sqrt{h}}\left[\alpha s\sqrt{h}\left(S^B{}_A - \frac{\beta^B}{\alpha}J_A\right)\right]_{,B} - S^{BC}\frac{1}{2}\left(h_{BC}\right)_{,A} - J_B\frac{\beta^B_{,A}}{\alpha} \\ &+\rho_H\frac{\alpha_{||A}}{\alpha} + J^B\left(n_A\right)_{||B} - \frac{1}{s^2}J_\phi\left(2E_A + \frac{2}{s}B_\phi v^C\epsilon_{AC}\right) - J_\phi v_\phi\frac{s_{||A}}{s^3} - P\frac{s_{||A}}{s}\end{aligned} \tag{106}$$

since $\epsilon_{AC}\epsilon^{CB} = -\delta^B{}_A$ and $B^C = \epsilon^{AB}\partial_B s$ and $J^B\left(n_A\right)_{||B} = 0$:

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$$\begin{aligned}0 &= \frac{1}{\alpha s\sqrt{h}}\pounds_t\left(s\sqrt{h}J_A\right) + \frac{1}{\alpha s\sqrt{h}}\left[\alpha s\sqrt{h}\left(S^B{}_A - \frac{\beta^B}{\alpha}J_A\right)\right]_{,B} \\ &- \left(\rho_H + P\right)\left[v^B v^C\frac{1}{2}\left(h_{BC}\right)_{,A} + v_B\frac{\beta^B_{,A}}{\alpha}\right] \\ &+\rho_H\frac{\alpha_{||A}}{\alpha} - \frac{1}{s^2}J_\phi\left[2E_A + \epsilon_{AC}\left(\frac{2}{s}B_\phi v^C - \frac{1}{s}B^C v_\phi\right)\right] - P\frac{s_{||A}}{s}\end{aligned} \tag{107}$$

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