Metric

$$ds^2 = \left(-\alpha^2 + a^2\beta^{r\,2} + c^2\beta^{z\,2}\right) dt^2$$

$$+ \quad 2a^2\beta^r\, dt\, dr + 2c^2\beta^z\, dt\, dz + a^2\, dr^2 + c^2\, dz^2 + r^2\, d\Omega^2$$

Evolution of the 4-metric

$$\dot{a} = -\frac{k_{rr}\alpha}{a} + \beta^r a_{,r} + \beta^z a_{,z} + a\beta^r{}_{,r}$$

$$\dot{c} = \frac{c k_{rr}\alpha}{(a)^2} + 2\,\frac{c\beta^r}{r} + \beta^r c_{,r} + \beta^z c_{,z} + c\beta^z{}_{,z}$$

Evolution of the Extrinsic Curvature

$$\dot{k}_{rr} = -\; 2\,\frac{(k_{rr})^2\,\alpha}{(a)^2} + 2\,k_{rr}\beta^r{}_{,r} + \beta^r(k_{rr})_{,r} + \beta^z(k_{rr})_{,z}$$

$$+ \; 1/2\,\frac{(c)^2\,(\beta^z{}_{,r})^2}{\alpha} - 1/2\,\frac{(a)^4\,(\beta^r{}_{,z})^2}{\alpha\,(c)^2} - \alpha_{,rr} + \frac{a_{,r}\alpha_{,r}}{a}$$

$$- \; \frac{a a_{,z}\alpha_{,z}}{(c)^2} + \frac{\alpha a a_{,z} c_{,z}}{(c)^3} - \frac{\alpha a a_{,zz}}{(c)^2} - \frac{\alpha c_{,rr}}{c} + \frac{\alpha c_{,r} a_{,r}}{ac} + 2\,\frac{\alpha a_{,r}}{ra} - 8\pi\,\alpha\left(\frac{\partial}{\partial r}\psi\right)^2$$

Constraint Equations

Hamiltonian Constraint:

$$2 \quad \frac{a_{,z}c_{,z}}{(c)^3\,a} - 2\,\frac{a_{,zz}}{a\,(c)^2} - 2\,\frac{c_{,rr}}{c\,(a)^2} + 2\,\frac{c_{,r}a_{,r}}{(a)^3\,c} + 4\,\frac{a_{,r}}{(a)^3\,r}$$

$$- \; 4 \quad \frac{c_{,r}}{(a)^2\,rc} + 2\,r^{-2} - 2\,\frac{1}{r^2\,(a)^2} - 1/2\,\frac{(a)^2\,(\beta^r{}_{,z})^2}{(c)^2\,(\alpha)^2} - \frac{\beta^r{}_{,z}\beta^z{}_{,r}}{(\alpha)^2}$$

$$- \quad 1/2 \quad \frac{(c)^2 (\beta^z{,_r})^2}{(a)^2 (\alpha)^2} - 6 \frac{(\beta^r)^2}{r^2 (\alpha)^2} - 2 \frac{(k_{rr})^2}{(a)^4} - 4 \frac{k_{rr}\beta^r}{(a)^2 r\alpha}$$

$$= \quad 8\,\pi\, (\Pi)^2 + 8\pi \frac{\left(\frac{\partial}{\partial r}\psi\right)^2}{(a)^2} + 8\pi \frac{\left(\frac{\partial}{\partial z}\psi\right)^2}{(c)^2}$$

Momemtum Constraint
(r-component)

$$3/2 \quad \frac{a a{,_z} \beta^r{,_z}}{\alpha (c)^2} + 1/2 \frac{a{,_z}\beta^z{,_r}}{\alpha a} + 2 \frac{c{,_r} k_{rr}}{c (a)^2} + 2 \frac{c{,_r}\beta^r}{r\alpha c} + 2 \frac{k_{rr}}{(a)^2 r}$$

$$- \quad 1/2 \quad \frac{(a)^2 c{,_z}\beta^r{,_z}}{\alpha (c)^3} + 1/2 \frac{c{,_z}\beta^z{,_r}}{\alpha c} + 1/2 \frac{(a)^2 \beta^r{,_{zz}}}{\alpha (c)^2} + 1/2 \frac{\beta^z{,_{rz}}}{\alpha}$$

$$- \quad 1/2 \quad \frac{(a)^2 \alpha{,_z}\beta^r{,_z}}{(c)^2 (\alpha)^2} - 1/2 \frac{\alpha{,_z}\beta^z{,_r}}{(\alpha)^2} - 2 \frac{\beta^r}{r^2\alpha} + \frac{(k_{rr}){,_r}}{(a)^2} - 2 \frac{k_{rr} a{,_r}}{(a)^3} = -8\pi \left(\frac{\partial}{\partial r}\psi\right)\Pi$$

(z-component)

$$1/2 \quad \frac{a{,_r}\beta^r{,_z}}{\alpha a} - 1/2 \frac{(c)^2 a{,_r}\beta^z{,_r}}{\alpha (a)^3} + 1/2 \frac{\beta^r{,_{rz}}}{\alpha} + 3/2 \frac{c\beta^z{,_r} c{,_r}}{\alpha (a)^2}$$

$$+ \quad 1/2 \quad \frac{(c)^2 \beta^z{,_{rr}}}{\alpha (a)^2} - 1/2 \frac{\beta^r{,_z}\alpha{,_r}}{(\alpha)^2} - 1/2 \frac{(c)^2 \beta^z{,_r}\alpha{,_r}}{(\alpha)^2 (a)^2} - 2 \frac{a{,_z}\beta^r}{\alpha r a}$$

$$+ \quad 1/2 \quad \frac{c{,_r}\beta^r{,_z}}{c\alpha} - \frac{\beta^r{,_z}}{\alpha r} + \frac{(c)^2 \beta^z{,_r}}{\alpha r (a)^2} - \frac{(k_{rr}){,_z}}{(a)^2} + 2 \frac{\beta^r \alpha{,_z}}{(\alpha)^2 r} = -8\pi \left(\frac{\partial}{\partial z}\psi\right)\Pi$$

Slicing Condition:

$$2 \quad \frac{\alpha}{r^2 (a)^2} - 2 \frac{\alpha}{r^2} + 8 \frac{\alpha\pi \left(\frac{\partial}{\partial r}\psi\right)^2}{(a)^2} + 2 \frac{\alpha c{,_{rr}}}{c (a)^2} + \frac{c{,_r}\alpha{,_r}}{c (a)^2} + \frac{\alpha{,_{rr}}}{(a)^2} + \frac{\alpha{,_{zz}}}{(c)^2} - \frac{a{,_r}\alpha{,_r}}{(a)^3}$$

$$+ \quad 2 \quad \frac{\alpha{,_r}}{r (a)^2} - \frac{c{,_z}\alpha{,_z}}{(c)^3} + 4 \frac{\alpha c{,_r}}{c (a)^2 r} + 8 \frac{\alpha\pi \left(\frac{\partial}{\partial z}\psi\right)^2}{(c)^2} - 4 \frac{\alpha a{,_r}}{(a)^3 r}$$

$$- \quad 2 \quad \frac{\alpha a{,_z} c{,_z}}{(c)^3 a} - 2 \frac{\alpha c{,_r} a{,_r}}{(a)^3 c} + 2 \frac{\alpha a{,_{zz}}}{a (c)^2} + \frac{a{,_z}\alpha{,_z}}{a (c)^2} = 0$$

Re-ordering:

$$\frac{1}{acr^2}\left(\frac{cr^2}{a}\alpha_{,r}\right)_{,r} + \frac{1}{ac}\left(\frac{a}{c}\alpha_{,z}\right)_{,z}$$
$$= -2\alpha\left\{\frac{1}{r^2}\left(\frac{1}{a^2}-1\right) + 4\pi\left(\frac{(\psi_{,r})^2}{a^2} + \frac{(\psi_{,z})^2}{c^2}\right) - \frac{2a_{,r}}{a^3 r} - \frac{a_{,r}\,c_{,r}}{a^3 c} + \frac{2c_{,r}}{a^2 cr} + \frac{c_{,rr}}{a^2 c} + \frac{a_{,zz}}{ac^2} - \frac{a_{,z}\,c_{,z}}{ac^3}\right\}$$

Equations for the Matter

$$\dot{\psi} = \alpha\Pi + \beta^z\frac{\partial}{\partial z}\psi + \beta^r\frac{\partial}{\partial r}\psi$$

$$\dot{\Pi} = \frac{\alpha c_{,r}\frac{\partial}{\partial r}\psi}{c\,(a)^2} + \beta^z\frac{\partial}{\partial z}\Pi + \frac{a_{,z}\alpha\frac{\partial}{\partial z}\psi}{(c)^2\,a} + \frac{\alpha_{,z}\frac{\partial}{\partial z}\psi}{(c)^2} + \beta^r\frac{\partial}{\partial r}\Pi + \frac{\alpha_{,r}\frac{\partial}{\partial r}\psi}{(a)^2}$$
$$+ \frac{\alpha\frac{\partial^2}{\partial r^2}\psi}{(a)^2} - \frac{\alpha a_{,r}\frac{\partial}{\partial r}\psi}{(a)^3} + \frac{\alpha\frac{\partial^2}{\partial z^2}\psi}{(c)^2} - \frac{c_{,z}\alpha\frac{\partial}{\partial z}\psi}{(c)^3} + 2\,\frac{\left(\frac{\partial}{\partial r}\psi\right)\alpha}{r\,(a)^2}$$

Spherical Symmetry

(i.e. $\partial/\partial z = \partial^2/\partial z^2 = 0$, $\partial\psi/\partial r = \Phi$)

Evolution Equations

$$
\begin{aligned}
\dot{a} &= -\frac{k_{rr}\alpha}{a} + \beta^r a_{,r} + a{\beta^r}_{,r} \\
\dot{c} &= \frac{ck_{rr}\alpha}{(a)^2} + 2\,\frac{c\beta^r}{r} + \beta^r c_{,r} \\
\dot{k}_{rr} &= -2\,\frac{\alpha\,(k_{rr})^2}{(a)^2} + 2\,k_{rr}{\beta^r}_{,r} + \beta^r (k_{rr})_{,r} \\
+\quad 1/2 &\quad \frac{(c)^2\,({\beta^z}_{,r})^2}{\alpha} - \alpha_{,rr} + \frac{a_{,r}\alpha_{,r}}{a} - \frac{\alpha c_{,rr}}{c} + \frac{\alpha c_{,r} a_{,r}}{ca} + 2\,\frac{\alpha a_{,r}}{ra} \\
-\quad 8 &\quad \pi\,\alpha\,\Phi^2
\end{aligned}
$$

Constraint Equations:

Hamiltonian C.:

$$
\begin{aligned}
-\quad 2 &\quad \frac{c_{,rr}}{c\,(a)^2} + 2\,\frac{c_{,r}a_{,r}}{c\,(a)^3} + 4\,\frac{a_{,r}}{(a)^3\,r} - 4\,\frac{c_{,r}}{c\,(a)^2\,r} + 2\,r^{-2} \\
-\quad 2 &\quad \frac{1}{(a)^2\,r^2} - 1/2\,\frac{(c)^2\,({\beta^z}_{,r})^2}{(a)^2\,(\alpha)^2} - 6\,\frac{(\beta^r)^2}{r^2\,(\alpha)^2} - 2\,\frac{(k_{rr})^2}{(a)^4} \\
-\quad 4 &\quad \frac{k_{rr}\beta^r}{(a)^2\,r\alpha} = 8\,\pi\,(\Pi)^2 + 8\,\pi\frac{\Phi^2}{(a)^2}
\end{aligned}
$$

Hamiltonian C. II

$$\left(2\,\frac{c_{,r}}{c}+4\,r^{-1}\right)a_{,r}\,(a)^{-1}+\left(-6\,\frac{(\beta^r)^2}{r^2\,(\alpha)^2}+2\,r^{-2}\right)(a)^2$$

$$-\quad 4\quad\frac{c_{,r}}{cr}-2\,\frac{c_{,rr}}{c}-2\,r^{-2}-1/2\,\frac{(c)^2\,(\beta^z{}_{,r})^2}{(\alpha)^2}$$

$$-\quad 4\quad\frac{k_{rr}\beta^r}{r\alpha}-2\,\frac{(k_{rr})^2}{(a)^2}=8\,\pi\,\Pi^2a^2+8\,\pi\,\Phi^2$$

Momentum C.:
(r-component)

$$2\,\frac{c_{,r}k_{rr}}{c\,(a)^2}+2\,\frac{c_{,r}\beta^r}{c\alpha r}+2\,\frac{k_{rr}}{r\,(a)^2}-2\,\frac{\beta^r}{\alpha r^2}+\frac{(k_{rr})_{,r}}{(a)^2}-2\,\frac{a_{,r}k_{rr}}{(a)^3}=-8\pi\,\left(\frac{\partial}{\partial r}\psi\right)\Pi$$

Momentum C.2:
(r-component)

$$\beta^r=-\frac{r^2\alpha c_{,r}k_{rr}}{(a)^2\,(c_{,r}r-c)}-\frac{crk_{rr}\alpha}{(a)^2\,(c_{,r}r-c)}-1/2\,\frac{r^2\alpha c(k_{rr})_{,r}}{(a)^2\,(c_{,r}r-c)}+\frac{r^2\alpha ca_{,r}k_{rr}}{(a)^3\,(c_{,r}r-c)}-4\,\frac{r^2\alpha\Phi\Pi\pi\,c}{c_{,r}r-c}$$

(z-component)

$$-1/2\,\frac{(c)^2\,a_{,r}\beta^z{}_{,r}}{\alpha\,(a)^3}+3/2\,\frac{c\beta^z{}_{,r}c_{,r}}{\alpha\,(a)^2}+1/2\,\frac{(c)^2\,\beta^z{}_{,rr}}{\alpha\,(a)^2}-1/2\,\frac{(c)^2\,\beta^z{}_{,r}\alpha_{,r}}{(\alpha)^2\,(a)^2}+\frac{(c)^2\,\beta^z{}_{,r}}{r\alpha\,(a)^2}=0$$

Slicing Condition:

$$-\qquad\frac{a_{,r}\alpha_{,r}}{(a)^3}+2\,\frac{\alpha}{(a)^2\,r^2}+2\,\frac{\alpha c_{,rr}}{c\,(a)^2}-2\,\frac{\alpha c_{,r}a_{,r}}{c\,(a)^3}+8\,\frac{\alpha\pi\,\left(\frac{\partial}{\partial r}\psi\right)^2}{(a)^2}$$

$$+\qquad\frac{\alpha_{,rr}}{(a)^2}+2\,\frac{\alpha_{,r}}{(a)^2\,r}+4\,\frac{\alpha c_{,r}}{c\,(a)^2\,r}+\frac{c_{,r}\alpha_{,r}}{c\,(a)^2}-4\,\frac{\alpha a_{,r}}{(a)^3\,r}-2\,\frac{\alpha}{r^2}=0$$

Slicing Equation 2:

$$\left(-4\,\frac{a_{,r}}{ar}-2\,\frac{c_{,r}a_{,r}}{ac}+8\,\pi\,\Phi^2-2\,\frac{(a)^2}{r^2}+2\,r^{-2}+2\,\frac{c_{,rr}}{c}+4\,\frac{c_{,r}}{cr}\right)\alpha$$

$$+\quad\left(-\frac{a_{,r}}{a}+2\,r^{-1}+\frac{c_{,r}}{c}\right)\alpha_{,r}+\alpha_{,rr}=0$$

Equations for the Matter

$$
\begin{aligned}
\dot{\psi} &= \alpha\Pi + \beta^r\Phi \\
\dot{\Pi} &= \frac{\alpha}{a^2}\Phi,_r + \beta^r\Pi,_r + \left(\frac{c,_r\alpha}{ca^2} + \frac{\alpha,_r}{a^2} - \frac{a,_r\alpha}{a^3} + 2\,\frac{\alpha}{ra^2}\right)\Phi \\
&= \frac{1}{a}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\alpha}{a}\Phi\right) + \frac{c,_r\alpha}{ca^2}\Phi + \beta^r\Pi,_r \\
\dot{\Phi} &= \alpha\Pi,_r + \alpha,_r\Pi + \beta^r{}_{,r}\Phi + \beta^r\Phi,_r = \left(\alpha\Pi + \beta^r\Phi\right),_r
\end{aligned}
$$

with the next boundary conditions:

$$
\begin{aligned}
\Pi,_r(0) = 0 &\qquad \dot{\Pi}(\infty) = -\left(\Pi,_r + \frac{1}{r}\Pi\right) \\
\Phi(0) = 0 &\qquad \dot{\Phi}(\infty) = -\left(\Phi,_r + \frac{1}{r}\Phi\right) \\
\psi,_r(0) = 0 &\qquad \dot{\psi}(\infty) = -\left(\psi,_r + \frac{1}{r}\psi\right)
\end{aligned}
$$

To find the boundary conditions at $r = \infty$, we assume that the spacetime is flat.

$$
\begin{aligned}
\dot{\Pi} &= \Phi,_r + \frac{2}{r}\Phi && (1) \\
\dot{\Phi} &= \Pi,_r && (2)
\end{aligned}
$$

This results in the next second order equation for ψ:

$$\ddot{\psi} = \psi,_{rr} + \frac{2}{r}\psi,_r. \qquad (3)$$

This equation can be rewritten as:

$$(r\ddot{\psi}) = (r\psi)\,,_{rr} \qquad (4)$$

Therefore $r\psi = f(r-t)$ is an approximation of outgoing b.c. The equation that ψ satisfices is then:

$$\psi + r\psi,_r + r\dot{\psi} = 0 \qquad (5)$$

Taking the spatial derivative of this equation, we get:

$$2\Phi + r\Phi_{,r} + \Pi + r\Pi_{,r} = 0, \tag{6}$$

which using the eom can be written as:

$$\dot{\Pi} + \Pi_{,r} + \frac{1}{r}\Pi \tag{7}$$

If in (6) we use the approximation at large r of (5) ($\Pi = -\Phi$) we get the other b.c.:

$$\Phi + r\Phi_{,r} + r\dot{\Phi} = 0 \tag{8}$$
