$$
\begin{aligned}
a(t,r,z) &= a_o(t,z) + \frac{a_1(t,z)}{r} + \frac{1}{2}\frac{a_2(t,z)}{r^2} && (1)\\
\alpha(t,r,z) &= \alpha_o(t,z) + \frac{\alpha_1(t,z)}{r} + \frac{1}{2}\frac{\alpha_2(t,z)}{r^2} && (2)\\
c(t,r,z) &= a_o(t,z) + \frac{c_1(t,z)}{r} + \frac{1}{2}\frac{c_2(t,z)}{r^2} && (3)\\
\beta^r(t,r,z) &= \beta^r_o(t,z) + \frac{\beta^r_1(t,z)}{r} + \frac{1}{2}\frac{\beta^r_2(t,z)}{r^2} && (4)\\
\beta^z(t,r,z) &= \beta^z_o(t,z) + \frac{\beta^z_1(t,z)}{r} + \frac{1}{2}\frac{\beta^z_2(t,z)}{r^2} && (5)\\
k_{rr}(t,r,z) &= k_{rro}(t,z) + \frac{\beta^r_1(t,z)}{r} + \frac{1}{2}\frac{k_{rr2}(t,z)}{r^2} && (6)\\
\psi(t,r,z) &= \psi_o(t,z) + \frac{\psi_1(t,z)}{r} + \frac{1}{2}\frac{\psi_2(t,z)}{r^2} && (7)\\
\Pi(t,r,z) &= \Pi_o(t,z) + \frac{\Pi_1(t,z)}{r} + \frac{1}{2}\frac{\Pi_2(t,z)}{r^2} && (8)\\
& && (9)
\end{aligned}
$$

Slicing Condition

$O(0)$**:**

$$\frac{\partial a_o}{\partial z}\frac{\partial \alpha_o}{\partial z}c_o - 2\frac{\partial a_o}{\partial z}\frac{\partial c_o}{\partial z}\alpha_o + 2\frac{\partial^2 a_o}{\partial z^2}\alpha_o c_o - a_o\frac{\partial \alpha_o}{\partial z}\frac{\partial c_o}{\partial z} + a_o\frac{\partial^2 \alpha_o}{\partial z^2}c_o + 8\pi a_o\alpha_o c_o\left(\frac{\partial \psi_o}{\partial z}\right)^2 = 0 \tag{10}$$

Assuming $a_o = \alpha_o = 1$:

$$\boxed{\frac{\partial \psi_o}{\partial z} = 0}$$

$O(1)$**:** Assuming already that $a_o = \alpha_o = 1$:

$$
\begin{aligned}
&16\pi c_o\frac{\partial \psi_o}{\partial z}\frac{\partial \psi_1}{\partial z} - 2\frac{\partial a_1}{\partial z}\frac{\partial c_o}{\partial z} + 8\pi\alpha_1 c_o\left(\frac{\partial \psi_o}{\partial z}\right)^2\\
&-16\pi c_1\left(\frac{\partial \psi_o}{\partial z}\right)^2 + 2\frac{\partial^2 a_1}{\partial z^2}c_o - \frac{\partial \alpha_1}{\partial z}\frac{\partial c_o}{\partial z} + \frac{\partial^2 \alpha_1}{\partial z^2}c_o = 0 && (11)
\end{aligned}
$$

Assuming $\frac{\partial \psi_o}{\partial z} = 0$ then:

$$-2\frac{\partial a_1}{\partial z}\frac{\partial c_o}{\partial z} + 2\frac{\partial^2 a_1}{\partial z^2}c_o - \frac{\partial \alpha_1}{\partial z}\frac{\partial c_o}{\partial z} + \frac{\partial^2 \alpha_1}{\partial z^2} = 0 \tag{12}$$

$O(2)$**:**Assuming asymptotic flatness:

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2 \alpha_2}{\partial z^2} - 4c_1\frac{\partial^2 a_1}{\partial z^2} - 2c_1\frac{\partial^2 \alpha_1}{\partial z^2} + 8\pi\left(\frac{\partial \psi_1}{\partial z}\right)^2 - 2\frac{\partial a_1}{\partial z}\frac{\partial c_1}{\partial z}\\ &+\frac{\partial^2 a_2}{\partial z^2} + \frac{\partial a_1}{\partial z}\frac{\partial \alpha_1}{\partial z} - 2a_1\frac{\partial^2 a_1}{\partial z^2} + 2\frac{\partial^2 a_1}{\partial z^2}\alpha_1 - \frac{\partial \alpha_1}{\partial z}\frac{\partial c_1}{\partial z} = 0\end{aligned} \tag{13}$$

Hamiltonian Constraint

$O(0)$**:**

$$-2\alpha_o^2 c_o^3 {k_{rr}}_o^2 - \frac{1}{2}a_o^6 c_o\left(\frac{\partial \beta_o^r}{\partial z}\right)^2 - 2a_o^3\frac{\partial^2 a_o}{\partial z^2}\alpha_o^2 c_o + 2a_o^3\frac{\partial a_o}{\partial z}\alpha_o^2\frac{\partial c_o}{\partial z} - 8\pi a_o^4 c_o^3\alpha_o^2\Pi_o^2 - 8\pi a_o^4 c_o\alpha_o\frac{\partial \psi_o}{\partial z} = 0 \tag{14}$$

Assuming that $a_o = \alpha_o = 1$ (by asymptotic flatness) results in:

$$-c_o^2 {k_{rr}}_o^2 - \frac{1}{4}\left(\frac{\partial \beta_o^r}{\partial z}\right)^2 - 4\pi c_o^2\Pi_o^2 - 4\pi\left(\frac{\partial \psi_o}{\partial z}\right)^2 = 0 \tag{15}$$

Assuming $\partial\psi_o/\partial z = \Pi_o = 0$:

$$c_o^2 {k_{rr}}_o^2 = -\frac{1}{4}\left(\frac{\partial \beta_o^r}{\partial z}\right)^2 \tag{16}$$

Therefore:

$$\boxed{\frac{\partial \beta_o^r}{\partial z} = 0}$$

$$\boxed{{k_{rr}}_o = 0}$$

$O(1)$**:** Assuming already $a_o = \alpha_o = 1$:

$$16\pi c_1 \left(\frac{\partial \psi_o}{\partial z}\right)^2 - 16\pi c_o \frac{\partial \psi_o}{\partial z}\frac{\partial \psi_1}{\partial z} - 16\pi c_o^3 \Pi_o \Pi_1 \tag{17}$$

$$+\alpha_1 c_o \left(\frac{\partial \beta_o^r}{\partial z}\right)^2 - 4c_o^3 k_{rro} k_{rr1} - 4c_o^3 \beta_o^r k_{rro} + 2\frac{\partial a_1}{\partial z}\frac{\partial c_o}{\partial z} + c1 \left(\frac{\partial \beta_o^r}{\partial z}\right)^2 \tag{18}$$

$$-a_1 c_o l \left(\frac{\partial \beta_o^r}{\partial z}\right)^2 - c_o \frac{\partial \beta_o^r}{\partial z}\frac{\partial \beta_1^r}{\partial z} + 8a_1 c_o^3 {k_{rro}}^2 - 2\frac{\partial^2 a_1}{\partial z^2} c_o = 0 \tag{19}$$

Assuming $\partial\psi_o/\partial z = \Pi_o = \partial\beta_o^r/\partial z = 0$:

$$-4c_o^3 k_{rro} k_{rr1} - 4c_o^3 \beta_o^r k_{rro} + 2\frac{\partial a_1}{\partial z}\frac{\partial c_o}{\partial z} + 8a_1 c_o^3 {k_{rro}}^2 - 2\frac{\partial^2 a_1}{\partial z^2} c_o = 0 \tag{20}$$

Assuming $k_{rro} = 0$:

$$\boxed{\frac{\partial a_1}{\partial z}\frac{\partial c_o}{\partial z} = \frac{\partial^2 a_1}{\partial z^2} c_o}$$

$O(2)$**:** Assuming asymptotic flatness (i.e. the metric goes to minkowski):

$$-2{k_{rr1}}^2 - \frac{\partial^2 a_2}{\partial z^2} - \frac{1}{2}\left(\frac{\partial \beta_1^r}{\partial z}\right)^2 + 2a_1 \frac{\partial^2 a_1}{\partial z^2} + 4c_1 \frac{\partial^2 a_1}{\partial z^2} - 8\pi \frac{\partial \psi_1}{\partial z} + 2\frac{\partial a_1}{\partial z}\frac{\partial c_1}{\partial z} - 8\pi \Pi_1^2 = 0 \tag{21}$$

r-Momentum Constraint

$O(0)$**:**

$$-a_o \frac{\partial \alpha_o}{\partial z} c_o \frac{\partial \beta_o^r}{\partial z} + a_o \alpha_o c_o \frac{\partial^2 \beta_o^r}{\partial z^2} + 3\frac{\partial a_o}{\partial z} \alpha_o c_o \frac{\partial \beta_o^r}{\partial z} - a_o \alpha_o \frac{\partial \beta_o^r}{\partial z} - a_o \alpha_o \frac{\partial c_o}{\partial z}\frac{\partial \beta_o^r}{\partial z} = 0 \tag{22}$$

Assuming $a_o = \alpha_o = 1$, and $\frac{\partial \beta_o^r}{\partial z} = 0$:

$$\boxed{\frac{\partial^2 \beta_o^r}{\partial z^2} = 0}$$

$O(1)$**:**Assuming already $a_o = \alpha_o = 1$, $k_{rro} = 0$, $\frac{\partial \beta_o^r}{\partial z} = \frac{\partial^2 \beta_o^r}{\partial z^2} = 0$:

$$\boxed{\frac{\partial c_o}{\partial z}\frac{\partial \beta_1^r}{\partial z} = c_o\frac{\partial^2 \beta_1^r}{\partial z^2}}$$

$O(2)$**:**Assuming asymptotic flatness:

$$\frac{3}{2}\frac{\partial a_1}{\partial z}\frac{\partial \beta_1^r}{\partial z}-\frac{1}{2}\alpha_1\frac{\partial^2 \beta_1^r}{\partial z^2}+a_1\frac{\partial^2 \beta_1^r}{\partial z^2}-c_1\frac{\partial^2 \beta_1^r}{\partial z^2}-\frac{1}{2}\frac{\partial c_1}{\partial z}\frac{\partial \beta_1^r}{\partial z}-\frac{1}{2}\frac{\partial \alpha_1}{\partial z}\frac{\partial \beta_1^r}{\partial z}+\frac{1}{4}\frac{\partial^2 \beta_2^r}{\partial z^2}+2k_{rr1} = 0 \tag{23}$$

z-Momentum Constraint

$O(0)$**:**

$$\frac{\partial k_{rr\,o}}{\partial z} = 8\pi\Pi_o a_o^2\frac{\partial \psi_o}{\partial z} \tag{24}$$

Assuming $\psi_o = 0$:

$$\boxed{\frac{\partial k_{rr\,o}}{\partial z} = 0}$$

$O(1)$**:** Assuming $\psi_o = \Pi_o = 0$ and $a_o = \alpha_o = 1$ and $\frac{\partial \beta_o^r}{\partial z} = 0$

$$\boxed{\frac{\partial k_{rr1}}{\partial z} = 0}$$

$O(2)$**:**Assuming asymptotic flatness:

$$8\pi\frac{\psi_1}{\partial z}\Pi_1 + 2a_1\frac{\partial k_{rr1}}{\partial z} - \frac{\partial \beta_1^r}{\partial z} - \frac{1}{2}\frac{\partial k_{rr2}}{\partial z} = 0 \tag{25}$$

Evolution equation for a

$O(0)$**:** Assuming $a_o = \alpha_o = 1$:

$$-\alpha_1 k_{rr0} + \frac{\partial a_1}{\partial z}\beta_0^z - k_{rr1} + a_1 k_{rr0} = 0 \tag{26}$$

Therefore, using Assymptotic flatness:

$$\boxed{k_{rr1} = 0}$$

Evolution equation for c

$O(0)$**:** Assuming asymptotic flatness:

$$0 = 0 \tag{27}$$