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Quasi-spherical light cones of the Kerr geometry

Frans Pretorius and Werner Israel

Canadian Institute for Advanced Research Cosmology Program, Department of Physics and Astronomy, University of Victoria, PO Box 3055 STN CSC, Victoria BC, Canada V8W 3P6

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Abstract. Quasi-spherical light cones are lightlike hypersurfaces of the Kerr geometry that are asymptotic to Minkowski light cones at infinity. We derive the equations of these surfaces and examine their properties. In particular, we show that they are free of caustics for all positive values of the Kerr radial coordinate r. Useful applications include the propagation of high-frequency waves, the definition of Kruskal-like coordinates for a spinning black hole and the characteristic initial-value problem.

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1. Introduction

The Kerr geometry is a vacuum spacetime with a Weyl tensor of Petrov type D. According to the Goldberg–Sachs theorem [1], it therefore possesses two congruences (ingoing and outgoing) of shear-free lightlike geodesics. Historically, these congruences played an essential role in the discovery of the Kerr solution [2], because the metric takes a simple explicit form in coordinates adapted to them.

In the original [2, 3] (Eddington–Kerr) form of the metric, the ingoing congruence consists of parametric curves of the Eddington–Kerr coordinate v_{EK}, often referred to as an 'advanced time'. This nomenclature can be misleading, because hypersurfaces of constant v_{EK} are actually timelike, not lightlike, if the Kerr rotational parameter a is non-zero. In fact, we have $g^{\alpha\beta}(\partial_\alpha v_{EK})(\partial_\beta v_{EK}) = a^2 \sin^2\theta/(r^2 + a^2\cos^2\theta)$. (The lightlike congruences are 'twisting', i.e. not orthogonal to any hypersurfaces, and hence not tangent to lightlike hypersurfaces if $a \neq 0$.)

Thus, the Eddington–Kerr form of the metric, and the Boyer–Lindquist form [3] derived from it by an elementary coordinate transformation, correspond to a 'threading' [4] (i.e. a one-dimensional foliation) of the manifold by twisting lightlike geodesics. However, a 'slicing' (three-dimensional foliation), in particular a lightlike slicing, is for many purposes more advantageous and often corresponds more closely to the physics.

Light cones of the Kerr geometry have not previously been the subject of systematic study to our knowledge. Perhaps this stems in part from a feeling that such hypersurfaces would quickly develop caustics because of the twist inherent in the metric. Yet as characteristics these surfaces obviously play a key role in the physics of Kerr black holes, e.g. in the propagation of wavefronts and high-frequency waves, and in characteristic initial-value problems.

In this paper we first derive the general solution for axisymmetric lightlike hypersurfaces of the Kerr geometry (section 2). We then focus on a particular foliation (invariant under

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time displacement) by (either ingoing or outgoing) 'quasi-spherical' light cones, defined as degenerate 3-spaces whose spatial sections are asymptotically spherical at infinity. These hypersurfaces become the standard Minkowski light cones when the Kerr mass parameter m vanishes; if $a = 0$, $m \neq 0$, they are the familiar spherical light cones, $t \pm r_* =$ constant, of Schwarzschild spacetime (where r_* is the 'tortoise' coordinate). Quite generally, for arbitrary $m \geqslant 0$ and a, they have the remarkable property of being free of caustics for all positive values of the Kerr radial coordinate r.

These appealing features suggest that coordinates adapted to quasi-spherical surfaces should be especially convenient and useful for the description of the Kerr geometry. The drawback (and it is considerable) is that the metric is no longer expressible in an explicit elementary form, because the new coordinates are elliptic functions of the Boyer–Lindquist coordinates.

We now briefly outline the contents of this paper. We begin in section 2 by solving the eikonal equation for axisymmetric lightlike hypersufaces of the Kerr geometry. The general solution is expressed in terms of integrals of elliptic type, and involves an arbitrary function f of one variable, to be fixed when the 'initial' (e.g. asymptotic) shape of the surface is given. (If $m = 0$ the geometry becomes flat and the integrals can be reduced to elementary form; in section 3 we digress briefly to illustrate this.) Some general properties of axisymmetric lightlike surfaces are developed in section 4. In section 5, guided by the results of section 3 for Minkowski light cones, we make the special choice of f appropriate for asymptotically spherical light cones in Kerr space. The key result, that these are free of caustics for $r > 0$, is demonstrated in section 6. In sections 7 and 8 we show how the 'quasi-spherical' coordinates r_*, θ_* defined by these light cones can be used to construct high-frequency solutions to the wave equation and Kruskal-like coordinates for Kerr black holes. Section 9 summarizes the limiting cases that can provide serviceable approximations for the elliptic functions which appear in the quasi-spherical form of the metric. Finally, in section 10 we present some results of numerical integrations for the inward evolution of the light cone generators.

2. Axisymmetric lightlike hypersurfaces

The equation of an arbitrary axially symmetric lightlike hypersurface of the Kerr geometry is expressible in terms of elliptic integrals as we now proceed to show.

We write the Kerr metric in its standard (Boyer–Lindquist) form

$$\mathrm{d}s^2 = \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + R^2\sin^2\theta\,\mathrm{d}\phi^2 - \frac{4mar\sin^2\theta}{\Sigma}\,\mathrm{d}\phi\,\mathrm{d}t - \left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2, \tag{1}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad R^2 = r^2 + a^2 + \frac{2ma^2 r\sin^2\theta}{\Sigma}, \qquad \Delta = r^2 + a^2 - 2mr, \tag{2}$$

and we note the useful identities

$$\Sigma R^2 = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta, \tag{3}$$

$$g_{\phi\phi}g_{tt} - g_{\phi t}^2 = -\Delta\sin^2\theta. \tag{4}$$

The equation

$$v(t, r, \theta) = t + \epsilon r_*(r, \theta) = \text{constant}, \qquad \epsilon = \pm 1 \tag{5}$$

represents an axisymmetric lightlike hypersurface (ingoing for $\epsilon = 1$, outgoing for $\epsilon = -1$) if $g^{\alpha\beta}(\partial_\alpha v)(\partial_\beta v) = 0$, i.e. if

$$\Delta(\partial_r r_*)^2 + (\partial_\theta r_*)^2 = (r^2 + a^2)^2/\Delta - a^2 \sin^2\theta. \tag{6}$$

It is easy to obtain a particular (separable) solution of (6) depending on two arbitrary constants (a 'complete integral') by adding and subtracting an arbitrary separation constant $a^2\lambda$ on the right-hand side. Let us define

$$P^2(\theta, \lambda) = a^2(\lambda - \sin^2\theta), \qquad Q^2(r, \lambda, m) = (r^2 + a^2)^2 - a^2\lambda\Delta. \tag{7}$$

(Note the useful identity

$$Q^2 + \Delta P^2 = \Sigma R^2, \tag{8}$$

which follows from (3).) A complete integral

$$r_* = \rho(r, \theta, \lambda, m), \tag{9}$$

of (6) is then obtained by integrating the exact differential

$$\mathrm{d}r_* = (Q/\Delta)\,\mathrm{d}r + P\,\mathrm{d}\theta \tag{10}$$

at fixed λ. When (10) is integrated, a second, additive integration constant appears, which we shall denote by $a^2 f(\lambda)/2$, where f is an arbitrary function.

We next proceed in the usual way to promote this complete integral, depending upon the arbitrary constants λ and $f(\lambda)$, to a general solution involving an arbitrary function.

In (9), ρ is a function of three independent variables r, θ and λ (not counting m and a), and a more complete expression for its differential is

$$\mathrm{d}\rho = (Q/\Delta)\,\mathrm{d}r + P\,\mathrm{d}\theta + (a^2/2)F\,\mathrm{d}\lambda, \tag{11}$$

where F is the partial derivative

$$(a^2/2)F(r, \theta, \lambda, m) = \partial_\lambda \rho(r, \theta, \lambda, m). \tag{12}$$

Its explicit form may be taken to be

$$F(r, \theta, \lambda, m) = \int_0^\theta \frac{\mathrm{d}\theta'}{P(\theta', \lambda)} + \int_r^\infty \frac{\mathrm{d}r'}{Q(r', \lambda)} + f'(\lambda). \tag{13}$$

Up to now, we have taken λ to be a constant, i.e. $\mathrm{d}\lambda = 0$ in (11). But we achieve the same effect (i.e. (11) still reduces to (10)) even when $\mathrm{d}\lambda \neq 0$ provided we require that $F = 0$. In other words: the function $r_*(r, \theta)$, given by (9) with λ now a *function* $\lambda(r, \theta)$, remains a solution of (6) provided its extra dependence on r, θ through λ does not change the algebraic form of the differential (10). This will indeed be so if we impose the constraint

$$F(r, \theta, \lambda, m) = 0. \tag{14}$$

This condition fixes the dependence of λ on r, θ for any given choice of $f(\lambda)$. Thus we now have a general solution, depending upon an arbitrary *function* $f[\lambda(r, \theta)]$.

The explicit form of the general solution (9) is then

$$\rho(r, \theta, \lambda) = \int \frac{r^2 + a^2}{\Delta(r, m)}\,\mathrm{d}r + \int_r^\infty \frac{r'^2 + a^2 - Q(r', \lambda, m)}{\Delta(r', m)}\,\mathrm{d}r' + \int_0^\theta \mathrm{d}\theta'\,P(\theta', \lambda) + \tfrac{1}{2}a^2 f(\lambda), \tag{15}$$

where the radial dependence has been arranged to ensure convergence of the definite integral at its upper limit. When performing the integrations in (13) and (15), we are allowed to treat

λ as merely a passive constant parameter—it is in fact a function of the limits of integration r,θ, not of the integration variables—with the constraint (14) imposed *a posteriori*. It is possible, though not especially illuminating or useful, to express each of these integrals in terms of standard elliptic integrals.

3. The case $m = 0$: light cones in Minkowski space

As a simple illustrative example (and because we shall need some of the results later), we specialize in this section to the case $m = 0$. The Kerr line-element reduces to the metric of flat spacetime expressed in terms of oblate spheroidal coordinates r, θ—related to Cartesian coordinates by

$$x^2 + y^2 = (r^2 + a^2)\sin^2\theta, \qquad z = r\cos\theta. \tag{16}$$

The second integral in (13) can be reduced to the same general form as the first integral when $m = 0$. Assume that we are in a domain where $\lambda(r, \theta) < 1$. Setting

$$Q_0(r', \lambda) = Q(r', \lambda, m = 0) = \sqrt{r'^2 + a^2}\sqrt{r'^2 + a^2(1-\lambda)}, \tag{17}$$

and making the substitution

$$r' = a\sqrt{1-\lambda}\sin\chi'/\sqrt{\lambda - \sin^2\chi'}, \tag{18}$$

we find

$$\mathrm{d}r'/Q_0 = \mathrm{d}\chi'/a\sqrt{\lambda - \sin^2\chi'}. \tag{19}$$

Thus the two integrals in (13) can be combined into a single integral, to give

$$aF(r, \theta, \lambda, m = 0) = \int_{\chi(r,\lambda)}^{\theta} \frac{\mathrm{d}\chi'}{\sqrt{\lambda - \sin^2\chi'}} + g'(\lambda), \tag{20}$$

where

$$g(\lambda) = af(\lambda) + 2\int_0^{\theta_*} \sqrt{\lambda - \sin^2\chi'}\,\mathrm{d}\chi', \tag{21}$$

and the form of $\chi(r, \lambda)$ is given by the 'unprimed' version of (18); equivalently,

$$\tan\theta_* = \frac{\sqrt{r^2 + a^2}}{r}\tan\chi, \qquad \lambda \equiv \sin^2\theta_*. \tag{22}$$

The function $g(\lambda)$ is arbitrary. As an example, let us consider the simplest choice, $g(\lambda) = 0$. The functional dependence $\lambda(r, \theta)$ corresponding to this choice is determined by the constraint $F = 0$, which requires that $\chi(r, \lambda) = \theta$ by inspection of (20). Thus (22) gives

$$\tan\theta_* = \frac{\sqrt{r^2 + a^2}}{r}\tan\theta = \frac{\sqrt{x^2 + y^2}}{z} \tag{23}$$

by (16), showing that θ_* is the spherical polar angle. With $\lambda(r, \theta)$ known from (23), it is straightforward to integrate (10) to obtain

$$r_* = \sqrt{r^2 + a^2\sin^2\theta}. \tag{24}$$

Thus r_* is the usual spherical radius, and our solutions $v = t \pm r_* =$ constant of (6) are in this case the Minkowski light cones.

4. Axisymmetric lightlike hypersurfaces: general properties

Returning to the general case, we shall now derive a number of results applicable to all axisymmetric lightlike hypersurfaces of Kerr spacetime.

Since the function $\lambda(r, \theta)$ is determined by the constraint $F = 0$, the partial derivatives of λ can be read off from $\mathrm{d}F = 0$ (13):

$$\mu\,\mathrm{d}\lambda = -\mathrm{d}r/Q + \mathrm{d}\theta/P, \qquad \mu \equiv -\partial F/\partial\lambda. \tag{25}$$

It follows from (10) and (25) that $\nabla r_* \cdot \nabla\lambda = 0$, i.e. that r_* and λ are orthogonal with respect to the intrinsic 2-metric

$$\mathrm{d}\sigma^2 = \Sigma(\mathrm{d}r^2/\Delta + \mathrm{d}\theta^2) \tag{26}$$

of the spatial sections $(\phi, t) =$ constant of the Kerr geometry. Since λ is independent of ϕ and t, it follows further that λ *is constant along the lightlike generators*, i.e.

$$\ell^\alpha \partial_\alpha \lambda = 0, \qquad \ell_\alpha = -\partial_\alpha v = -\partial_\alpha(t + \epsilon r_*). \tag{27}$$

(In fact, λ is just Carter's 'fourth constant of the motion' for Kerr geodesics in the special case where the geodesics are lightlike and have zero angular momentum.)

If r_* and λ are adopted as coordinates in place of r and θ, the 2-metric (26) becomes

$$\mathrm{d}\sigma^2 = R^{-2}(\Delta\,\mathrm{d}{r_*}^2 + L^2\,\mathrm{d}\lambda^2), \qquad L \equiv \mu P Q, \tag{28}$$

where we have made use of (10), (25) and (8). The identity (4) can be re-arranged in the form

$$\left(1 - \frac{2mr}{\Sigma}\right) + \omega_B^2 R^2 \sin^2\theta = \frac{\Delta}{R^2}, \tag{29}$$

where

$$\omega_B = -\frac{g_{\phi t}}{g_{\phi\phi}} = \frac{2mar}{\Sigma R^2} \tag{30}$$

is the Bardeen or ZAMO angular velocity $\mathrm{d}\phi/\mathrm{d}t$ which characterizes orbits having zero angular momentum. This enables us to recast the Kerr metric (1) in the form

$$\mathrm{d}s^2 = \frac{\Delta}{R^2}(\mathrm{d}{r_*}^2 - \mathrm{d}t^2) + \frac{L^2}{R^2}\,\mathrm{d}\lambda^2 + R^2\sin^2\theta(\mathrm{d}\phi - \omega_B\,\mathrm{d}t)^2. \tag{31}$$

Thus

$$g^{r_* r_*} = R^2/\Delta = -g^{tt}, \tag{32}$$

from which the null character of the 3-spaces $t \pm r_* =$ constant is directly evident. The degenerate intrinsic metric of these 3-spaces is

$$(\mathrm{d}s^2)_{LL} = (L/R)^2\,\mathrm{d}\lambda^2 + R^2\sin^2\theta(\mathrm{d}\phi - \omega_B\,\mathrm{d}t)^2, \tag{33}$$

showing that the generators rotate with ZAMO angular velocity relative to stationary observers at infinity. (This is directly obvious from (27), which shows that $\ell_\phi = 0$ for an axisymmetric hypersurface). From (33) we see that caustics will develop when the degenerate volume element tends to zero, i.e. when $L\sin\theta \to 0$ (recalling that λ has a fixed value along each generator.)

The integrability of (25) for the exact differential $\mathrm{d}\lambda$ provides a condition on the integrating factor μ. This takes the form of an evolution equation for μ along the generators. A straightforward calculation, which uses (7), (8), (13) and (25) itself, yields

$$(\partial\mu/\partial r)_\lambda = \frac{a^2}{2}\frac{\Sigma R^2}{P^2 Q^3}, \tag{34}$$

where the subscript λ indicates that the partial derivative is being taken at fixed λ.

Inversion of the differential relations (10) and (25) gives

$$\Sigma R^2\,\mathrm{d}r = \Delta Q(\mathrm{d}r_* - \mu P^2\,\mathrm{d}\lambda), \qquad \Sigma R^2\,\mathrm{d}\theta = P(\Delta\,\mathrm{d}r_* + \mu Q^2\,\mathrm{d}\lambda), \tag{35}$$

which will be of use in subsequent sections.

5. Quasi-spherical light cones

From here on we confine attention to a special class, 'quasi-spherical light cones', defined as lightlike hypersurfaces which asymptotically approach Minkowski spherical light cones at infinity. Define an angle $\theta_*(r,\theta)$ by

$$\lambda = \sin^2\theta_*. \tag{36}$$

In Minkowski space, the generators $\lambda =$ constant of light cones are radial straight lines, suggesting (as we confirmed in section 3) that θ_* is the spherical polar angle. For $r \gg a$, the oblate coordinates (r,θ) become indistinguishable from spherical coordinates, so we have the condition

$$\theta_*(r=\infty,\theta) = \theta \tag{37}$$

for spherical light cones in Minkowski space. This asymptotic condition must therefore also hold for quasi-spherical surfaces in Kerr space. This fixes the arbitrary function $f'(\lambda)$ in (13). We conclude that

$$F(r,\theta,\lambda,m) = \int_r^\infty \frac{\mathrm{d}r'}{Q(r',\lambda,m)} - \int_\theta^{\theta_*} \frac{\mathrm{d}\theta'}{P(\theta',\lambda)} \tag{38}$$

generates the solution for quasi-spherical surfaces. The radial function $\rho(r,\theta,\lambda,m)$ is given by (15) with (compare (21) with $g(\lambda)=0$)

$$af(\lambda) = -2\int_0^{\theta_*} \sqrt{\lambda - \sin^2\theta'}\,\mathrm{d}\theta'. \tag{39}$$

The equation of the hypersurface is then $v = t \pm r_*(r,\theta) =$ constant, with $r_* = \rho(r,\theta,\lambda(r,\theta))$ and the function $\lambda(r,\theta) \equiv \sin^2\theta_*$ determined by the constraint $F = 0$.

6. No caustics for positive r

We now prove the rather remarkable result that quasi-spherical light cones are free of caustics for all positive values of the Kerr radial coordinate r.

This is trivially true if $m = 0$, when these surfaces are simply light cones in Minkowski space with vertices at the spatial origin, represented by $r = 0$, $\theta = 0$ or π in oblate spheroidal coordinates according to (16). We shall prove that it is true *a fortiori* for $m > 0$ by effectively showing that when m is larger, the generators converge *less* rapidly as $r \to 0^+$.

As noted in section 4, formation of a caustic along a generator is signalled by

$$\mu P Q \sin\theta \to 0. \tag{40}$$

We consider in turn the behaviour of each factor in (40) along an ingoing generator, so that r is positive and decreasing, with λ fixed. Because of the equatorial symmetry we need only consider 'northern' generators, i.e. we may assume θ_* (which specifies the initial, asymptotic value of θ when $r = +\infty$) to be acute, and $P(\theta,\lambda)$ positive, at least initially. (Note from (10) that the equatorial symmetry of $r_*(r,\theta)$ implies $P(\lambda,\pi-\theta) = -P(\lambda,\theta)$.)

Since θ decreases with r at fixed λ according to (25), the factor $P = a\sqrt{\lambda - \sin^2\theta}$ must *increase* and remain positive. Any possible caustic in the Kerr positive-r sheet cannot arise from the behaviour of P.

We proceed to consider the other factors in (40). From the definition (7),

$$Q(r, \lambda, m) > Q(r, \lambda, m = 0) > 0 \qquad (r > 0, m > 0). \tag{41}$$

We shall further show that

$$(\partial\theta/\partial m)_{r,\lambda} > 0 \qquad \text{and} \qquad (\partial\mu/\partial m)_{r,\lambda} > 0 \qquad (r > 0, \theta_* < \pi/2). \tag{42}$$

Thus, none of the three factors μ, Q and $\sin\theta$ can reach zero sooner for positive m than they do in flat space, and hence they do not reach zero for any positive value of r.

To establish (42), we note that the function $\theta(r, \lambda, m)$ is determined by the vanishing of $F(r, \theta, \lambda, m)$ as given by (38). Taking the differential at fixed r,λ and using (7) gives

$$(\partial\theta/\partial m)_{r,\lambda} = a^2\lambda P I, \qquad I(r, \lambda, m) \equiv \int_r^\infty r'\,\mathrm{d}r'/Q^3(r', \lambda, m), \tag{43}$$

which is manifestly positive. Turning to μ, this is defined as a function of r, θ, λ and m by $\mu = -\partial F/\partial\lambda$ according to (25). Inverting the order of partial differentiation and using (38),

$$\frac{\partial\mu}{\partial m} = -\frac{\partial}{\partial\lambda}\frac{\partial F}{\partial m} = \frac{\partial}{\partial\lambda}(a^2\lambda I), \tag{44}$$

$$\frac{\partial\mu}{\partial\theta} = -\frac{\partial}{\partial\lambda}\frac{\partial F}{\partial\theta} = -\frac{\partial}{\partial\lambda}\frac{1}{P} = \frac{a^2}{2P^3}. \tag{45}$$

Hence

$$\left(\frac{\partial\mu}{\partial m}\right)_{r,\lambda} = \frac{\partial\mu}{\partial m} + \frac{\partial\mu}{\partial\theta}\left(\frac{\partial\theta}{\partial m}\right)_{r,\lambda} = a^2\left(1 + \frac{1}{2}\frac{a^2\lambda}{P^2}\right)I + \tfrac{3}{2}a^4\lambda J, \tag{46}$$

where

$$J(r, \lambda, m) = \int_r^\infty r'\Delta(r')\,\mathrm{d}r'/Q^5(r', \lambda, m). \tag{47}$$

Each term in (46) is manifestly positive. This completes the proof.

The fact that adding mass to the Kerr field should *reduce* the focusing of ingoing radial light rays is at first sight paradoxical, but can be explained by the circumstance that the material source is not at the origin, but located on the singular equatorial ring $x^2 + y^2 = a^2$, $z = 0$ (see (16)). The source has a peculiar, 'demi-pole' structure [5], made possible by the double-sheeted structure of the Kerr manifold: a ring of positive mass in the sheet $r > 0$ is bonded to a ring of negative mass in the sheet $r < 0$. Light rays heading inwards toward the origin in the sheet $r > 0$ are deflected outward by the ring and defocused. Repulsive effects have become dominant by the time they pass through the disk $r = 0$: the rays are then refocused and form a caustic in the sheet $r < 0$. This description is more than just hand-waving: the Keres Newtonian analogue model of the Kerr field [6] has a source structure and equations of motion closely resembling Kerr (apart from frame dragging), and displays precisely this behaviour.

Figure 1 (a result of numerical integrations described in section 10) shows the ingoing generators mapped onto the flat background of the Kerr–Schild decomposition, using the rectangular coordinates defined in (16) (effects of frame dragging are not shown). According to the Keres–Kerr model, the gravitational 'force' should become repulsive for $r \approx a$, and, indeed, the generators have inflection points near this radius.

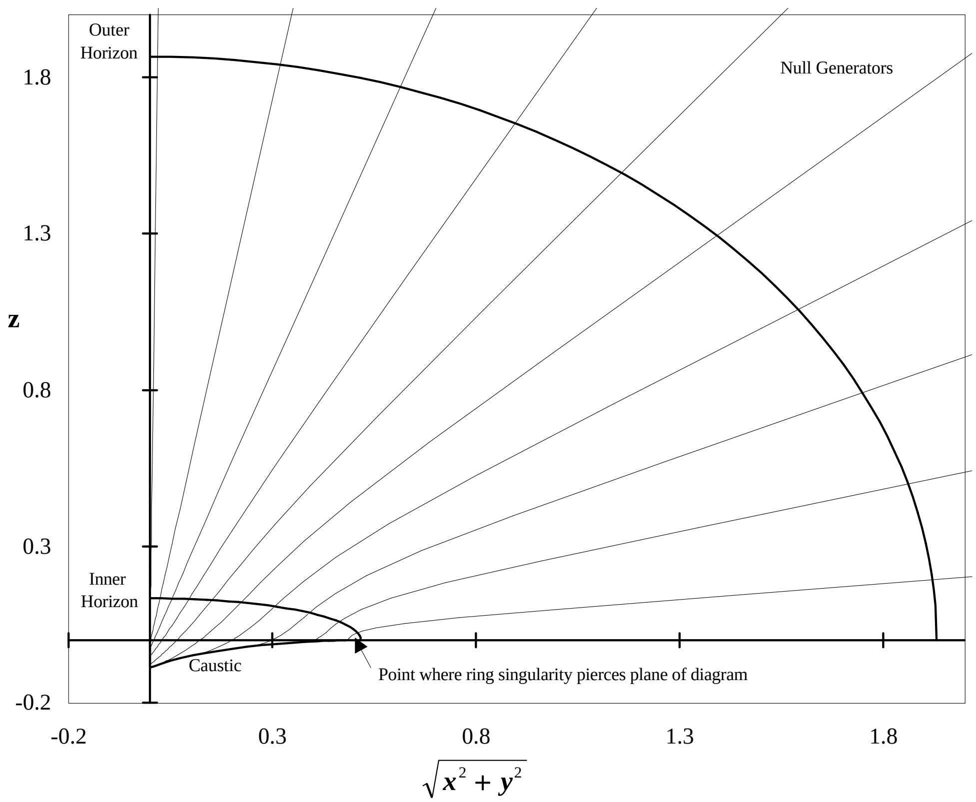

Figure 1. Null generators of the quasi-spherical light cones projected onto the Cartesian plane for $m = 1$, $a = \frac{1}{2}$. The azimuthal twist (not shown in this two-dimensional representation) depends on the choice of azimuthal coordinate; it is zero for the coordinate ϕ_ϵ introduced in section 8. In order to follow the generators through the equatorial disk $r = 0$ and into the negative-r sheet, the 'southern half' ($\theta > \pi/2$) of the positive-r sheet has been cut away, and replaced by the 'northern' half ($\theta < \pi/2$) of the negative-r sheet. Note that only well within the outer horizon does the mass of the spacetime significantly affect the r, θ behaviour of the generators (which would continue as straight lines intersecting the origin if m were zero). One branch of the caustic emerges from the singular ring $r = 0$, $\theta = \pi/2$ (if $m > 0$) and creeps inwards to the axis and 'downwards' toward increasingly negative values of r. There it meets a second branch, which runs down the negative-r segment of the axis $\theta = 0$, beginning from $r = 0$.

7. Eikonal solution of the wave equation

Since the quasi-spherical light cones $t \pm r_* =$ constant are characteristics of the wave operator, the coordinates r_*, λ are well adapted for representing asymptotically spherical high-frequency modes.

The wave equation for $\Psi(r_*, \lambda, \phi, t)$ on the Kerr background (31) is

$$\frac{\Delta}{R^2}\Box\Psi = (\partial_{r_*}^2 - \partial_t^2)\Psi + (\partial_{r_*}\ln\gamma)(\partial_{r_*}\Psi) + \gamma^{-1}\partial_\lambda(\gamma^{-1}\Delta\sin^2\theta\,\partial_\lambda\Psi) + \left(1 - \frac{2mr}{\Sigma}\right)\frac{1}{R^2\sin^2\theta}\partial_\phi^2\Psi - 2\omega_B\partial_\phi\partial_t\Psi = 0 \qquad (48)$$

where $\gamma = \mu P Q \sin\theta$ gives the area element (degenerate volume element) on the light cone (33).

Introducing the ansatz

$$\Psi = \Phi(r_*, \lambda)\mathrm{e}^{\mathrm{i}\omega(t+\epsilon r_*)}\mathrm{e}^{\mathrm{i}n\phi} \qquad (\epsilon = \pm 1) \tag{49}$$

into (48), we find

$$2\epsilon\mathrm{i}\omega\left[\partial_{r_*}\Phi + \Phi\left(\tfrac{1}{2}\partial_{r_*}\ln\gamma - \epsilon\mathrm{i}n\omega_B\right)\right] + B = 0, \tag{50}$$

where

$$B = \partial^2_{r_*}\Phi + (\partial_{r_*}\ln\gamma)(\partial_{r_*}\Phi) + \gamma^{-1}\partial_\lambda(\gamma^{-1}\Delta\sin^2\theta\,\partial_\lambda\Phi) - \frac{n^2}{R^2\sin^2\theta}\left(1 - \frac{2mr}{\Sigma}\right)\Phi \tag{51}$$

has no explicit ω-dependence. We can re-express ω_B in (50) as a partial derivative with respect to r_*. Defining an angular function

$$\alpha(r, \lambda) = \int^r \frac{2mar'}{\Delta(r')Q(r', \lambda)}\,\mathrm{d}r' \tag{52}$$

and recalling (35), we find

$$\left(\frac{\partial\alpha}{\partial r_*}\right)_\lambda = \frac{2mar}{\Delta Q}\left(\frac{\partial r}{\partial r_*}\right)_\lambda = \frac{2mar}{\Sigma R^2} = \omega_B. \tag{53}$$

Therefore (50) can be rewritten as

$$2\epsilon\mathrm{i}\omega\partial_{r_*}(\sqrt{\gamma}\mathrm{e}^{-\epsilon\mathrm{i}n\alpha}\Phi) = -\sqrt{\gamma}\mathrm{e}^{-\epsilon\mathrm{i}n\alpha}B. \tag{54}$$

If ω is large, we can apply an iterative procedure to (54) and (51) to develop the solution for Φ in inverse powers of ω. In the lowest approximation, we simply equate the coefficient of ω in (54) to zero. This yields the 'eikonal approximation'

$$\Psi \approx \gamma^{-1/2}f(\lambda)\mathrm{e}^{\mathrm{i}\omega(t+\epsilon r_*)}\mathrm{e}^{\mathrm{i}n(\phi+\epsilon\alpha)} \tag{55}$$

with an arbitrary function $f(\lambda)$. By linear superposition we can build from this an arbitrary high-frequency solution

$$\Psi \approx \gamma^{-1/2}F(\lambda, \phi + \epsilon\alpha, t + \epsilon r_*), \tag{56}$$

where F is an arbitrary function of its three arguments varying rapidly with time.

From (55) and (53) we see that crests of high-frequency azimuthal (ϕ-dependent) waves twist about the axis with angular frequency ω_B, the ZAMO angular velocity. Because ω_B and α depend on λ, propagation of azimuthal waves produces latitude-dependent phase shifts. The latitude dependence is, however, always bounded, even when $\alpha \to \pm\infty$ at horizons.

In particular, wave-tails propagating inwards from the event to the Cauchy horizon of a spinning black hole experience a blueshift which is modulated by an oscillatory, latitude-dependent factor when $n \neq 0$—an effect first noted by Ori [7]. However, since $|n|$ is no larger than the multipole order ℓ, this effect is more than offset by the natural power-law decay of the higher multipole wavemodes with advanced time v, $\Psi \sim v^{-(2\ell+2)}$. It does not affect the uniformity of the leading-order divergence $\mathrm{e}^{|\kappa_-|v}$ of blueshift at the Cauchy horizon, where κ_- (the inner-horizon surface gravity) is a constant. These features, which are expected to extend at least qualitatively to waves having (initially) lower frequencies, are important when considering the back-reaction of blueshifted wave-tails on the geometry near the Cauchy horizon [8].

8. Kruskal coordinates

It is straightforward to transform the Kerr metric (31) into a Kruskal-like form. We introduce retarded and advanced times u and v, and associated Kruskal coordinates U and V, by the definitions

$$-\frac{\mathrm{d}U}{\kappa U} = \mathrm{d}u = \mathrm{d}(t - r_*), \qquad \frac{\mathrm{d}V}{\kappa V} = \mathrm{d}v = \mathrm{d}(t + r_*). \tag{57}$$

Here, κ is the surface gravity of the horizon under consideration, defined for the outer ($r = r_+$) and inner ($r = r_-$) horizons by

$$\kappa_\pm = \pm\sqrt{m^2 - a^2}/(r_\pm^2 + a^2). \tag{58}$$

Then (31) becomes

$$\mathrm{d}s^2 = \frac{1}{\kappa^2 R^2}\frac{\Delta}{UV}\,\mathrm{d}U\,\mathrm{d}V + \frac{L^2}{R^2}\,\mathrm{d}\lambda^2 + R^2\sin^2\theta(\mathrm{d}\phi - \omega_B\,\mathrm{d}t)^2 \tag{59}$$

with

$$-UV = \mathrm{e}^{2\kappa r_*} \qquad \text{and} \qquad -V/U = \mathrm{e}^{2\kappa t}. \tag{60}$$

The first ($\mathrm{d}U\,\mathrm{d}V$) term is manifestly regular at the horizon sheets $U = 0$ and $V = 0$. The last term, involving the Boyer–Lindquist coordinates ϕ and t, is not. We therefore define the advanced and retarded angular coordinates ϕ_+, ϕ_- by

$$\phi_\epsilon = \phi + \epsilon\alpha(r, \lambda) \qquad (\epsilon = \pm 1) \tag{61}$$

where α was defined in (52). It is straightforward to show that

$$\mathrm{d}\alpha = \omega_B\,\mathrm{d}r_* - N\,\mathrm{d}\lambda, \qquad N = \mu P^2\omega_B + ma^3 I \tag{62}$$

with I defined as in (43). Thus the last term in (59) can be expressed in a manifestly regular way as

$$\mathrm{d}\phi - \omega_B\,\mathrm{d}t = \mathrm{d}\phi_\epsilon - \omega_B\,\mathrm{d}u_\epsilon + \epsilon N\,\mathrm{d}\lambda, \tag{63}$$

with $u_+ = v$, $u_- = u$. In (63) we can choose either sign for ϵ, depending on which sheet of the horizon is of interest. For example, ϕ_+ and u_+ are regular on the 'upward' (future) sheets of both outer and inner horizons, with the nice bonus feature that u_+ is constant over each ingoing light cone and ϕ_+ is constant along each ingoing generator.

Is there a single azimuthal coordinate which regularizes the metric simultaneously on both past and future sheets of (say) the outer horizon, including the bifurcation surface? Since ω_B takes a constant value $\omega_H = a/2mr_+$ over this horizon, a possible choice for the desired coordinate is $\Phi = \phi - \omega_H t$, and Φ in fact agrees with ϕ_+ on the future sheet and ϕ_- on the past sheet. But Φ develops the undesirable features characteristic of rigidly rotating axes in the outer regions of the space.

We briefly mention that following standard methods one could construct a Penrose diagram of the Kerr spacetime using the Kruskal-like coordinates (60), valid up to formation of the caustic surface. Such a diagram would not look any different from the usual textbook examples, see for example [9], except that a *single* two-dimensional diagram is all that would be needed to illustrate the causal structure of the spacetime. This is because the ingoing (V = constant) and outgoing (U = constant) lightlike congruences intersect at surfaces of constant r_*, represented by a single point on a diagram where the compactified coordinate system is derived from (60). The causal future of observers on the surface of intersection is entirely contained within the future-directed wedge of U = constant, V = constant (as is evident from (31) by noting that when $|\mathrm{d}r_*| > |\mathrm{d}t|$ the line-element is spacelike).

9. Limiting cases and approximations

The quasi-spherical coordinates r_* and $\lambda = \sin^2\theta_*$, and the coefficients in the quasi-spherical forms (31) and (59) of the Kerr metric are complicated elliptic functions of the Kerr–Boyer–Lindquist coordinates r,θ. It may therefore be useful to record here the simple forms these functions reduce to in the two opposite limiting cases, $m/a \to 0$ and $m/a \to \infty$.
(i) $m = 0$, $a \neq 0$: this was studied in section 3, and we simply list the results

$$r_* = \sqrt{r^2 + a^2\sin^2\theta}, \qquad \tan\theta_* = \frac{\sqrt{r^2+a^2}}{r}\tan\theta, \tag{64}$$

$$Q = \frac{r(r^2+a^2)}{\sqrt{r^2+a^2\sin^2\theta}}, \qquad P = \frac{a^2\sin\theta\cos\theta}{\sqrt{r^2+a^2\sin^2\theta}}, \qquad \mu P = \frac{1}{2}\frac{\sin\theta\cos\theta}{(\sin\theta_*\cos\theta_*)^2}, \tag{65}$$

$$\gamma = \frac{1}{2}\frac{r^2+a^2\sin^2\theta}{\cos\theta_*}. \tag{66}$$

(ii) $m \neq 0$, $a = 0$: This is just Schwarzschild spacetime, and one readily finds

$$r_* = \int \frac{\mathrm{d}r}{1-2m/r}, \qquad \theta_* = \theta, \qquad Q = r^2, \qquad \mu P = \frac{1}{\sin 2\theta}, \tag{67}$$

$$\gamma = \frac{1}{2}\frac{r^2}{\cos\theta}. \tag{68}$$

Comparison with numerical results described in the next section suggests that the $m = 0$, $a \neq 0$ case functions can provide serviceable analytic approximations outside the outer horizon for broad ranges of m and a. For example, with $m = 1$ and $a = \frac{1}{2}$, all of the quantities except r_* in (64)–(66) differ by no more than roughly 1–3% from the true (numerically integrated) values, outside of r_+. (r_* converges only logarithmically to the Minkowski value in the limit $r \to \infty$, and of course diverges at the horizon.)

10. Numerical evolution of the generators

In figure 1 we illustrated the behaviour of the generators and the nature of the caustic that forms in the negative-r sheet of the Kerr manifold. The $\lambda =$ constant curves were obtained by numerically integrating the evolution equations of the null generators of the hypersurface:

$$\ell^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau = -g^{\alpha\beta}\partial_\beta v, \tag{69}$$

with affine parameter τ. In particular

$$\dot{r} \equiv \mathrm{d}r/\mathrm{d}\tau = -\epsilon Q/\Sigma, \qquad \dot{\theta} \equiv \mathrm{d}\theta/\mathrm{d}\tau = -\epsilon P/\Sigma. \tag{70}$$

To detect the formation of a caustic (see 40) we track the evolution of μ, and to reconstruct surfaces of constant r_* in the r,θ plane we calculate the change of r_* along the generators. If we treat r as the independent variable, from (70), (15), (25) and (34) we find

$$(\partial\theta/\partial r)_\lambda = \frac{P}{Q}, \tag{71}$$

$$(\partial r_*/\partial r)_\lambda = \Sigma R^2/\Delta Q, \tag{72}$$

and

$$(\partial\mu/\partial r)_\lambda = \frac{a^2}{2}\frac{\Sigma R^2}{P^2 Q^3}. \tag{73}$$

At the horizons r_* and its derivative with respect to r diverge. These divergences are coordinate singularities which can be avoided numerically by subtracting off the infinities that occur there. Thus define $\tilde{r}$ as

$$\tilde{r} = r_* - \frac{\ln|r - r_+|}{2\kappa_+} - \frac{\ln|r - r_-|}{2\kappa_-}, \tag{74}$$

where κ_+ (κ_-) is the surface gravity at the outer (r_+) (inner (r_-)) horizon as defined in (58). So we actually integrate $\tilde{r}$ along the generators and retrieve r_* from the result using (74).

For asymptotic intitial conditions for μ and θ, we use the relations listed in (64) and (65) for a sufficiently large initial r. Using (43) and (46) one can show that $\mu(r, \lambda, m) - \mu(r, \lambda, m = 0) \approx (a^2\lambda m/8 \sin\theta \cos\theta)(1/r^2)$ and $\theta(r, \lambda, m) - \theta(r, \lambda, m = 0) \approx (a^4\lambda m \sin\theta \cos\theta/4)(1/r^5)$ for large r along a given generator. The initial condition for r_* is arbitrary ($r_* +$ constant is still a solution to (6)).

Figure 2 is a close-up within the inner horizon of the case illustrated in figure 1, showing the projections of surfaces of constant r_* onto the $\sqrt{x^2 + y^2}$, z plane. This is the only region where $r_* =$ constant surfaces start showing significant deviations from the spheres of the massless scenario. Note that the caustic surface does not coincide with a hypersurface slice $r_* =$ constant: following the generators inward a caustic first develops where the hypersurface meets the ring singularity, after which it quickly 'unravels'. We only present plots for $m = 1$ and $a = \frac{1}{2}$; except for variations in scale there are no qualitative differences in the shapes of the curves and surfaces for arbitrary non-zero a and positive m.

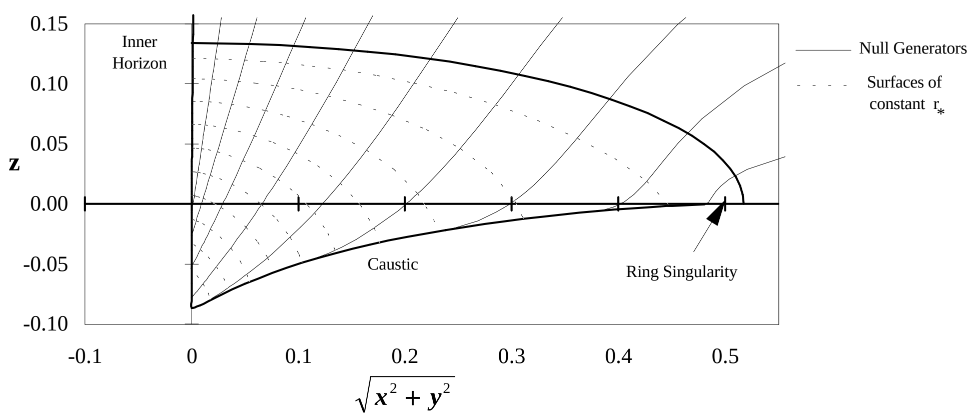

Figure 2. Null generators and surfaces of constant r_* projected onto the Cartesian plane ($m = 1, a = \frac{1}{2}$). This is a close-up of figure 1 focusing on the region between the inner horizon and caustic. As in figure 1, the northern half of the negative-r sheet replaces the southern half of the positive-r sheet to illustrate the evolution of the generators through the equatorial disk.

11. Concluding remarks

Despite the complexity of the coordinate transformations linking them to the familiar Boyer–Lindquist coordinates, the quasi-spherical coordinates r_*, θ_* introduced in this paper and the light cones associated with them provide new and useful insights into the structure of the Kerr geometry and wave propagation in Kerr spacetime. We anticipate that they will find increasing use as their special advantages become apparent, in particular for numerical schemes to evolve characteristic initial-value data in problems involving spinning black holes [10].

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