An operational approach to black hole entropy

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In this paper we calculate the entropy of a thin spherical shell that contracts reversibly from infinity down to its event horizon. We find that, for a broad class of equations of state, the entropy of a non-extremal shell is one-quarter of its area in the black hole limit. The considerations in this paper suggest the following operational definition for the entropy of a black hole: S_{BH} is the equilibrium thermodynamic entropy that would be stored in the material which gathers to form the black hole, if all of this material were compressed into a thin layer near its gravitational radius. Since the entropy for a given mass and area is maximized for thermal equilibrium we expect that this is the maximum entropy that could be stored in the material before it crosses the horizon. In the case of an extremal black hole the shell model does not assign an unambiguous value to the entropy. [S0556-2821(98)01710-X]

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I. INTRODUCTION

As is well known [1] the classical laws of black hole dynamics together with the Hawking temperature

$$T_H=\frac{\kappa}{2\pi} \tag{1}$$

lead to the Bekenstein-Hawking postulate that the entropy of a black hole is given by

$$S_{BH}=\frac{1}{4}A, \tag{2}$$

where κ is the surface gravity of the hole and A is its area (here we work in units with $G=c=\hbar=1$). This expression for the black hole entropy has raised some important questions that still remain unanswered. At what stage in the black hole's evolution is its entropy created? Is it created immediately upon the formation by gravitational collapse or only gradually over the (typically) long course of evaporation? What is the dynamical mechanism that makes S_{BH} a universal function, independent of the hole's past history and detailed internal condition? There are a variety of possible answers. First the subtlest possibility: It is conceivable that no quantum entropy is irreversibly created by the hole. No information is lost and S_{BH} is merely a measure of our own temporary loss of access (during the lifetime of the black hole) to correlations beneath the horizon. When the black hole finally evaporates these correlations will be fully visible to us. A black hole formed by the collapse of matter in a pure state will evaporate into a radiation field whose distribution appears thermal on a coarse-grained level, but will be recognized to be in a pure state once the last particles have left the hole.

Alternatively, it is possible that black hole formation and evaporation are accompanied by an irreversible increase of entropy, and we come back to the questions how, where, and when? The original (pre-1974) motivation for assigning an entropy to a black hole was to keep account of the thermal entropy of objects thrown into a hole (Wheeler's teacup experiment). The generalized second law of black hole thermodynamics, which states that S_{BH} plus the external entropy is non-decreasing, lends support to this view of black hole entropy. Nevertheless, it is not possible (as emphasized by Kundt [2] more than 20 years ago) simply to identify S_{BH} with the thermal entropy of all the matter which collapsed to form the black hole. Since this is an issue of principle, we could, for instance, consider an idealized Oppenheimer-Snyder collapse of cold, pressureless, viscous-free dust: here, no material entropy ever develops.

The view that entropy is somehow created in the process of evaporation also meets with difficulties. Black hole evaporation is very nearly, and can be made exactly, reversible. We simply enclose the hole in a container, so that it comes into equilibrium with its own radiation. We then poke a small hole in the container and let the radiation leak out arbitrarily slowly. Since this process is reversible, no entropy is generated.

Another possible explanation of the entropy of a black hole stems from Frolov and Novikov [3]. This links S_{BH} with modes (produced by vacuum fluctuations) propagating ''outwards'' just inside and alongside the horizon. These modes have positive frequency but their energy is negative as calibrated for an observer at infinity (i.e. including the contribution of gravitational potential energy). Their spectrum is thermal with temperature T_H. Detailed implementation of this picture is so far still plagued with divergences and ambiguities.

Finally we mention that Zurek and Thorne [4] have suggested that S_{BH} should be interpreted as the logarithm of ''the number of quantum-mechanically distinct ways that the black hole could have been made.'' Assuming that the technical difficulties involved in making this statement precise can be overcome, the Zurek-Thorne interpretation uses an ensemble of black holes and thus simply accepts the universality of S_{BH} without offering any dynamical explanation for how it arises in a particular black hole.

No direct insight into the statistical origins of black hole entropy can come from thermodynamics. But if entropy really is a meaningful state function for black hole equilibrium states, then thermodynamics can tell us its value and provide an operational definition of it. To find the entropy of any thermodynamical state one invents a reversible process which arrives at that state from a state of known entropy and computes how the entropy changes in that process using the first law of thermodynamics. In this paper we examine the reversible contraction of a thin spherical shell down to its event horizon. To maintain reversibility, the shell must be in equilibrium with the acceleration radiation seen by observers on the shell. In addition to the classical stress-energy outside the shell there will be a Boulware stress-energy created by the quantum fields that reside in the spacetime. To maintain thermal equilibrium we draw on a source of energy at infinity to ''top up'' the Boulware stress-energy to an appropriate thermal environment. The pressure and surface density of the shell follow from the junction conditions at the shell. Using the first law we find that the entropy of the shell is $A/4$ in the black hole limit for a large class of shell equations of state. These considerations lead us to suggest the following operational definition for the entropy of a black hole: S_{BH} is the equilibrium thermodynamic entropy that would be stored in the material which gathers to form the black hole, if we imagine all of this material compressed into a thin layer near its gravitational radius.

II. ENTROPY OF A CONTRACTING SHELL

In this section we consider compressing a spherical shell reversibly from an infinite radius down to its event horizon. To maintain reversibility at each stage the shell must be in equilibrium with the acceleration radiation that would be measured by an observer on the shell. Thus the temperature of the shell is determined by the local acceleration of static observers at the shell. Our interest is in the end-states of a thermodynamic process: a state of infinite dispersion at infinity and the final black hole state. The shell serves merely as the working substance connecting these two states, and the nature of the material is irrelevant as long as it satisfies the first law of thermodynamics. Its equation of state involves two independent variables: the shell's locally measured mass M and radius R.

For the static spherical geometries inside and outside the shell it will be general enough for our purposes to take the metric to be of the form

$$ds^2 = dr^2/f(r) + r^2 d\Omega^2 - f(r) dt^2. \tag{3}$$

This covers as special cases Minkowski, Schwarzschild, Reissner-Nordström and de Sitter spacetimes. Of course, the classical stress-energy associated (via the Einstein equations) with this metric is not the stress-energy of the ground state for the quantum fields which reside in the spacetime. We know that this is the Boulware state [5], whose stress-energy $(T_{\mu\nu})_B$ depends on the types and number of fields and is unknown. For an ordinary star $(T_{\mu\nu})_B$ is completely negligible, but as the shell approaches its gravitational radius it generally grows without bound, and its backreaction cannot be ignored. We cannot compute this backreaction, but we can compensate for it. By drawing from an energy reservoir at infinity we fill up the shell's surroundings with material whose stress-energy $\Delta T_{\mu\nu}$ tops up $(T_{\mu\nu})_B$ to form a thermal bath which shares the shell's local acceleration temperature $T_{acc}(R)$ at the point of contact. This ''topped-up Boulware (TUB) state'' is constructed in thermal quantum field theory by periodically identifying the coordinate t in the Euclidean sector with period equal to the reciprocal of the shell's red-shifted acceleration temperature $T_\infty = T_{acc}(R) f(R)^{1/2}$. Then the TUB state's local temperature varies in accordance with Tolman's law [6]

$$T(r)(-g_{tt})^{1/2} = T_\infty = \text{const.} \tag{4}$$

The TUB state may be called a generalized Hartle-Hawking (HH) state. Indeed, it becomes the HH state in the limit when the shell approaches its gravitational radius. Its stress-energy

$$(T_{\mu\nu})_{TUB} = (T_{\mu\nu})_B + \Delta T_{\mu\nu} \tag{5}$$

is, like the HH stress-energy, everywhere bounded and small for a large black hole, but non-vanishing at infinity.

To keep effects of backreaction under control, we encase the TUB state in a large spherical container of radius R_{big}. Backreaction is negligible if the total energy of the TUB state is small compared to the shell's mass M, i.e. (in Planck units),

$$T_\infty^4 R_{big}^3 \ll M, \tag{6}$$

or in conventional units,

$$R_{big}/(2GM/c^2) \ll (M/m_{\rm PL})^{2/3} \approx 10^{25} (M/M_\odot)^{2/3}. \tag{7}$$

We assume this condition satisfied, and we shall ignore backreaction and also the entropic contribution of the TUB state.

Phenomenology gives us the freedom of a dualistic approach. The thermal equilibrium condition $T_{shell} = T_{TUB}$ corresponds to the viewpoint of a local *stationary* observer. On the other hand, for the stress-energy of the TUB state we adopt the ''objective'' (gravitating) value which appears on the right-hand side of the Einstein equations and corresponds to what is measured by a local *free-falling* observer, and we take this to be negligible for a large black hole. The TUB state would look very different to a stationary observer, for whom the ground state is the Boulware state. In a statistical analysis of the problem, this observer's view is the one we would be forced to adopt, since no technique is currently available for analyzing the statistical thermodynamics of a system in anything other than its stationary rest frame. Such an analysis would lead to values for the TUB state's apparent energy and entropy which are large and divergent in the black hole limit. The extensive literature devoted to this problem resorts to various procedures (e.g. ''brick-wall'' cutoffs [7], renormalization of the gravitational coupling constant [8]) to tame these divergences.

Now consider the surface stress-energy of the shell. The interior and exterior metrics will be of the form (3) with $f(r) = f_1(r)$ and $f = f_2(r)$ respectively. The surface stress-energy is related to the ''jump'' in the extrinsic curvature via [9]

$$8\pi S_{ab} = [K_{ab} - g_{ab} K] \tag{8}$$

where K_{ab} is the extrinsic curvature and $[\cdots]$ denotes the jump in the quantity in brackets (Latin indices, a,b, etc., run from 1 to 3). A simple calculation gives

$$4\pi\sigma=-[\sqrt{f(R)}/R] \tag{9}$$

and

$$16\pi P=[f'/\sqrt{f}+2\sqrt{f}/R] \tag{10}$$

where σ is the proper surface density of the shell and P is the surface pressure. Since the mass of the shell as seen by local free-falling observers is $M=4\pi\sigma R^2$ we have

$$M=-[R\sqrt{f}]. \tag{11}$$

It is instructive to see the explicit form of these expressions for a shell of charge e and gravitational mass m with a flat interior. We therefore set $f_1(r)=1$ and $f_2(r)=1-2m/r+e^2/r^2$ in Eqs. (10) and (11) and find that

$$m=M-\frac{1}{2}\left(\frac{M^2-e^2}{R}\right) \quad \text{and} \quad P=\frac{M^2-e^2}{16\pi R^2(R-M)}. \tag{12}$$

These expressions have obvious Newtonian counterparts and simple intuitive meanings.

As discussed above, if the shell is to be contracted reversibly, it must be in equilibrium with the acceleration radiation that would be seen by observers on the shell. Thus the shell's temperature must be given by [10]

$$T=a/2\pi=f'(R)/4\pi\sqrt{f(R)}. \tag{13}$$

For a Reissner-Nordström space-time $f(r)=1-2m/r+e^2/r^2$ and

$$T=\frac{2MR-M^2-e^2}{4\pi R^2(R-M)}. \tag{14}$$

Since the local gravitational acceleration is discontinuous across the shell, the inner and outer TUB states in which the shell is immersed are at different temperatures. To maintain equilibrium an ''adiabatic'' diaphragm (impermeable to heat) must be interposed between the faces. We can picture the shell as a pair of concentric spherical plates, with inner and outer masses M_1 and M_2, separated by a massless and thermally inert interstitial layer of negligible thickness. How we distribute the total shell mass $M=M_1+M_2$ between the plates is arbitrary. We choose M_1 so that the spacetime is flat between the plates. This generally makes M_1 negative. The two plates thus separate three concentric spherical zones: an inner zone where $f(r)=f_1(r)$, a very thin intermediate zone where $f(r)=1$ and an outer zone where $f(r)=f_2(r)$. Applying Eqs. (10) and (11) to the inner and outer plates gives, for the masses M_i and surface pressures P_i $(i=1,2)$,

$$M_i=R\xi_i(1-V_i(R)) \quad \text{and} \quad 16\pi P_i=\left(\frac{\xi_i f_i'(R)}{V_i(R)}-\frac{2M_i}{R^2}\right), \tag{15}$$

where f_i and $V_i=f_i^{1/2}$ are evaluated at $r=R$, the common radius of the two plates, and $\xi_i=(-1)^i$. The temperature T_i of the plates is given by

$$T_i=\frac{f_i'(R)}{4\pi V_i(R)}. \tag{16}$$

This gives T_i as a function of M_i and R.

Now consider the first law of thermodynamics, which would usually relate dS to the quantity $(dM+PdA)/T$ in terms of the variables discussed so far. But since we are using an explicit function $T_i(M_i,R)$, the quantity $(dM+PdA)/T$ will not in general be an exact differential and hence it cannot be a complete representation of the differential dS. Thus we need to introduce another thermodynamic variable $N=N(M,R)$. Since the plates are merely abstract entropy-carrying devices, the physical significance of N is irrelevant. For convenience we interpret N as the number of particles in the shell. The first law now becomes (temporarily dropping the index i)

$$dS=\beta dM+\beta PdA-\alpha dN, \tag{17}$$

where $\beta=1/T$, $\alpha=\mu/T$, and μ is the chemical potential. Using the Gibbs-Duhem relation

$$S=\beta(M+PA)-\alpha N \tag{18}$$

gives

$$n\,d\alpha=\beta dP+(\sigma+P)d\beta, \tag{19}$$

where $n=N/A$. Using the formulas (16) and (15) for T, P and σ in terms of M and R gives

$$n\,d\alpha=\frac{\sigma}{\gamma}d\left(\frac{1}{2\xi\sigma\gamma}\right), \tag{20}$$

where

$$\gamma^2=\frac{f'}{8\pi\xi\sigma V}. \tag{21}$$

The functions n and α can be chosen arbitrarily subject only to the restriction imposed by Eq. (20). The simplest option is to choose plate materials having the ''canonical'' equation of state (denoted by an asterisk, and restoring the index i)

$$n_i^*=\frac{\sigma_i}{\gamma_i} \quad \text{and} \quad \alpha_i^*=(2\xi_i\sigma_i\gamma_i)^{-1}. \tag{22}$$

Now, from Eqs. (16) and (21) we have

$$T_i=2\xi_i\sigma_i\gamma_i^2. \tag{23}$$

Thus the canonical chemical potential $\mu_i^*=T_i\alpha_i^*$ obeys the simple relation

$$\mu_i^* n_i^*=\sigma_i. \tag{24}$$

Substituting the above into Eq. (18) and noting from Eqs. (15) and (21) that the surface pressures can be expressed as

$$P_i = \frac{1}{2}\sigma_i(\gamma_i^2 - 1), \tag{25}$$

we obtain the entropy density $s_i^* = S_i^*/A$ of the plates as

$$s_i^* = \beta_i P_i = \frac{1}{4}\xi_i(1-\gamma_i^{-2}). \tag{26}$$

When the slow contraction of the shell terminates it is hovering just outside the horizon [$r=r_0$, defined by $f(r_0)=0$] of the exterior geometry. Now consider the non-extremal case where the surface gravity $\kappa = f_2'(r_0)/2 \neq 0$ (the extremal case will be examined in Sec. V). Then Eq. (21) shows that γ_2^2 diverges according to

$$\gamma_2^2 \approx \frac{\kappa_2}{M_2/R^2} V_2^{-1} \quad \text{with} \quad V_2^2 = 2\kappa(R-r_0) \tag{27}$$

as $R \to r_0$. Thus, from Eq. (26),

$$\lim_{R\to r_0} s_2^* = \frac{1}{4}. \tag{28}$$

That is, *the entropy of the outer plate is one-quarter of its area in Planck units in the black hole limit*. In the simplest situation the spherical cavity inside the shell is flat and empty. In this case $f_1(r)=1$, $M_1=P_1=s_1=T_1=0$ and the outer plate contributes all of the mass and entropy of the shell.

III. BLACK HOLE ENTROPY

In the previous section we found that the entropy of a shell with a flat interior is one-quarter of its area in the black hole limit. From an observer's perspective at infinity there is nothing to distinguish the shell in its final stages of compression from a black hole. We could even arrange for the shell to leak out energy and entropy in a simulated Hawking evaporation. This suggests an operational definition for the entropy of a black hole, namely the limiting entropy of the associated shell.

But is this definition of entropy additive? Suppose that, in the field of a pre-existing black hole with Bekenstein-Hawking entropy S_{old} (or of any object, e.g., a star, having this entropy), we lower a shell of entropy $S_{shell}=S_2+S_1$ to the point where an outer black hole, of area A_{new}, is about to form, so that $S_2=A_{new}/4$ for the outer plate according to Eq. (28). Is the new Bekenstein-Hawking entropy obtained by simple addition as $S_{old}+S_{shell}$? At this point certainly not. The upper plate by itself already accounts for the full Bekenstein-Hawking entropy of the new configuration, and so it would be necessary for the negative entropy of the inner plate to cancel exactly the entropy of whatever was inside the cavity initially; i.e., S_1+S_{old} would need to be 0. This is generally not true. However, we are still free to carry out a further reversible procedure: we can sweep all the material inside the cavity onto the outer shell [if this material includes an inner black hole, this involves inflating the shell representing it until it merges with the lower plate of the new shell—in effect, a (reversible) ''evaporation'' of the inner black hole]. This ''flattens'' the cavity inside the new shell and dematerializes the lower plate. [In a more general (rotating) context, evacuation will not flatten the cavity, but could still reduce the acceleration temperature at the lower plate to zero.] With the shell's entropy thus modified, the total entropy at the end is $S'_{shell}=A_{new}/4$, which is S_{BH} for the final black hole. In this specific sense, S_{BH} may be called ''additive,'' but it is perhaps more correct to say it is ''forgetful'': S_{BH} for the final configuration betrays no clue about the entropy originally contained in the space now occupied by the hole.

These considerations suggest the following operational definition: *S_{BH} is the equilibrium thermodynamic entropy that would be stored in the material which gathers to form the black hole, if all of this material were compressed into a thin layer near its gravitational radius*. Since the entropy for a given mass and area is maximized for thermal equilibrium we expect that this is the maximum entropy that could be stored in the material before it crosses the horizon.

Of course, this imagined process bears no resemblance to any real scenario of black hole formation. But as mentioned it can give a fair schematic description of the evaporation process since Hawking's mechanism of virtual pair creation is a skin effect confined to a thin layer near the horizon. In the real process, the horizon is a port where gravity temporarily detains the evaporating particles on their way out of the hole; the shell model assembles all of them there at one time. Kundt's description of S_{BH} as ''evaporation entropy'' sums up the situation rather well, with the proviso that the evaporation process itself (being virtually reversible) cannot be the *source* of S_{BH}; it only acts as its conduit.

IV. ALTERNATIVE PLATE MATERIAL

The key result (28) was established for a special ''canonical'' form of the plate material. How sensitive are the results to the properties of the material? The most general functions n and α satisfying Eq. (20) are obtained by replacing α_i^* by an arbitrary function of itself $g_i(\alpha_i^*)$, and replacing n_i^* with $n_i^*/g_i'(\alpha_i^*)$. This yields the general formulas

$$\alpha_i = g_i(\alpha_i^*), \quad n_i = n_i^*/g_i'(\alpha_i^*) \tag{29}$$

and

$$\frac{\mu_i n_i}{\sigma_i} = \frac{g_i}{\alpha_i^* g_i'}. \tag{30}$$

The most general expression for the entropy density of the plates is

$$s_i = \frac{1}{4}\xi_i[1+\gamma^{-2}(1-2\mu_i n_i/\sigma_i)]. \tag{31}$$

Thus, Eq. (28) is invariant under arbitrary transformations of the form (29) which leave $\mu_i n_i$ bounded in the high temperature limit. Indeed, we can allow transformations which are singular in this limit, provided that

$$\lim_{T_i\to\infty} \frac{\mu_i n_i}{T_i} = 0, \tag{32}$$

recalling Eq. (23). Expression (28) is not invariant under arbitrary singular transformations. At the root of this problem is the fact that the black hole end-state is a singular state of the plate material (P and T become infinite). In these circumstances there is no *a priori* justification for excluding or constraining asymptotically singular behavior of thermodynamic quantities. However, it is reassuring to note that the loose constraint (32) guarantees that our conclusions are independent of the plate material for a very broad class of equations of state.

The freedom contained in the transformations (29) can be used to ''improve'' the behavior of the plate material at low temperatures. For canonical material the total entropy (not the entropy density) goes as $S^*_{i(\mathrm{tot})} \approx R$ $(R \to \infty)$, since $\gamma^2 \approx 1 - M/R$ for $R \to \infty$. By a suitable change of g_i in Eq. (29) we can arrange that $S^*_{i(\mathrm{tot})}$ is finite in the limit $R \to \infty$ (and $T_i \to 0$).

V. EXTREMAL BLACK HOLES AND THE THIRD LAW

There are essentially two distinct versions of the third law of thermodynamics. The first version, proposed by Nernst in 1906, states that isothermal processes become isentropic in the zero temperature limit. An essentially equivalent form states that the temperature of a system cannot be reduced to zero in a finite number of operations. The second version, proposed by Planck in 1911, states that the entropy of any system tends, as $T \to 0$, to an absolute constant, which may be taken as zero.

In their 1973 paper on ''the four laws of black hole dynamics,'' Bardeen, Carter and Hawking [11] proposed a form of the third law patterned after Nernst's unattainability principle: ''It is impossible by any process, no matter how idealized, to reduce the surface gravity to zero in a finite sequence of operations.'' A more specific form [12], which makes precise the meaning of ''a finite sequence of operations,'' states that ''a non-extremal black hole cannot become extremal at finite advanced time in any continuous process in which the stress-energy of accreted matter stays bounded and satisfies the weak energy condition.'' From this formulation it is clear that quantum processes like evaporation, which typically involve the absorption of negative energy, can violate Nernst's form of the third law.

For a long time it was believed that there is no black hole analogue to Planck's version of the third law. Recently, however, this has become a matter of controversy. Arguments based on black hole instanton topology and pair creation [13] suggest that the entropy of extremal black holes is zero. On the other hand, the remarkable indirect calculations of black hole entropy by counting states of strings on D-branes gives the value $S_{BH} = A/4$ for extremal black holes [14].

We can examine this question by considering the quasistatic contraction of an extremally charged spherical shell with a flat interior. Setting $|e| = m$ in Eqs. (11) and (12) we find that $M = |e|$ and $P = 0$. If the shell is made of canonical material, Eq. (26) gives $s^* = 0$ at all stages of the contraction, leading to

$$S^*_{extBH} = 0. \tag{33}$$

This result is, however, quite sensitive to the equation of state of the shell material. For arbitrary material, Eq. (31) gives the shell's entropy density as

$$s_{extBH} = \frac{1}{2}\left(1 - \frac{\mu n}{\sigma}\right) \tag{34}$$

whose value can be made arbitrary by a choice of the function $g(\alpha^*)$ in Eq. (29). Thus no universal quantity can be assigned to the entropy of an extremal shell in any stage of its compression. This suggests that the entropy of extremal black holes may depend on their prior history.

VI. CONCLUSION

In this paper we calculated the entropy of a quasistatically contracting spherical shell and discussed its relationship to the entropy of the black hole that it forms.

Outside the shell the classical stress-energy tensor will be modified by the quantum fields that reside in the spacetime. The Boulware stress-energy produced by these fields was ''topped up'' to provide a thermal environment and reduce the backreaction to negligible levels for large black holes. Since the shell contracts reversibly, its temperature must be equal to the acceleration temperature seen by observers on the shell. The surface pressure and density follow from the junction conditions. We reformulated the Gibbs relation in the form (20), involving $n = N/A$ and $\alpha = \mu/T$ where N is the number of particles, A the area, μ the chemical potential and T the temperature of the shell. The entropy of the shell can easily be calculated once n and α are found. The solution to the equation for n and α is not unique, but we found a simple solution (the ''canonical'' solution). In the non-extremal case the entropy of a shell made of canonical material approaches $A/4$ as the shell approaches its event horizon. This result does not hold for all solutions for α and n. However, it does hold for all equations of state which satisfy $\mu n/T \to 0$ as the shell approaches its event horizon (note that $T \to \infty$ in this limit).

The considerations in this paper led us to suggest the following operational definition for the entropy of a black hole: S_{BH} is the equilibrium thermodynamic entropy that would be stored in the material which gathers to form the black hole, if all of this material were compressed into a thin layer near its gravitational radius. Since the entropy for a given mass and area is maximized for thermal equilibrium, we expect that this is the maximum entropy that could be stored in the material before it crosses the horizon.

For the special case of an extremal shell (charge equals mass in relativistic units) our approach gives ambiguous results; the limiting entropy of the shell depends on the equation of state of the material.

It should be noted that these conclusions go significantly beyond the verifications of the generalized second law [4,15] which show that one-quarter of the *change* in area (when a black hole slowly ingests material) is equal to the entropy absorbed. Here, we have derived the entropy-area relation in integral form, eliminating the possibility of an additive constant.

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