Numerical General Relativistic Magnetohydrodynamics

UPPAC

Andrew J. Penner
Department of Physics and Astronomy
University of British Columbia
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Outline

- Motivation
- Plasma
- Astrophysical Phenomenon
- Magnetohydrodynamics
- Solver
- Metric
- Waves
- Instabilities
- Current Project

Motivation

- What is General Relativity?
  - Matter tells space how to curve $\Longleftrightarrow$ Space tells matter how to move
  - General relativity studies how these two interact
- What is General Relativistic Magnetohydrodynamics?
  - The study of compressible, conducting fluids as they exist in curved spacetime.
  - Generally involve the study of three key components
    - Curved Spacetime
    - Hydrodynamic Stress–Energy Tensor
    - Electromagnetism Stress–Energy Tensor
  - Magnetohydrodynamic Stress–Energy Tensor
- Why is magnetohydrodynamics considered important?
  - Magnetic fields are fundamental to many astrophysical phenomenon
  - Currently believed to be a model for the dynamics of an AGN, and Supernovae

Astrophysical Phenomenon

- Accretion Disks
  - Material that is caught in the gravitational field of a massive body
  - May come from a binary star
  - Accretion phenomenon is thought to be much more efficient than nuclear fusion
- Relativistic Jets
  - A steady collimated beam of material emanating from a central body
  - Still not well understood
  - Mechanisms for collimation of the jet is still an open question
- Active Galactic Nuclei
  - This astrophysical phenomenon is said to be at the centre of an active galaxy, understanding this may lead to a better understanding of the origins of our own galaxy
- Angular Momentum Transport
  - There is still ongoing debate as to the mechanism that allows for angular momentum transport within accretion disks.

Cosmology

- Galactic Dynamics
  - Study the evolution of the universe in present day.
  - Can treat galaxies as fluid particles.
  - Many open questions especially if one considers the cosmic scale magnetic fields.
- Dark Fluid
  - A really new idea linking dark matter and dark energy.
  - Needs a lot of work before numerical models can be developed.
- Cosmological Perturbation Theory
  - Evolution from the big bang to present day.
  - models that are closer in time to the big bang itself are modeled using a hydrodynamic description of the matter and energy in the universe.

Magnetohydrodynamic Dimensionless Numbers

- Hydrodynamic dimensionless numbers
  - Knudsen number - $\lambda/L \ll 1$
  - Reynolds number (Re) - VL/ν
  - Mach number (M) - V/c_s
- Where
  - λ - mean free path between particles
  - L - characteristic length scale $\sim R_{EventHorizon}$
  - V - characteristic velocity scale $\sim c$
  - ν - fluid viscosity $\rightarrow 0$
  - c_s - speed of sound $\sim \sqrt{\frac{P}{\rho_o}}$

Magnetohydrodynamic Dimensionless Numbers

- Additional dimensionless numbers when considering magnetic field contributions
  - Magnetic Reynolds number (R_M) - VL/η
  - beta (β) - $P_{thermal}/P_{magnetic}$
  - Alfven number (a) - V/v_A
  - Ratio of energy densities (δ) - E_B/E_{rest}
- where
  - v_A - the Alfvén velocity $= B/\sqrt{\rho_o + B^2}$
  - η - the resistivity $\rightarrow 0$
  - $E_B \sim |B|^2$
  - $E_{rest} \sim \rho_o$
- The free parameters to survey are
  - β, M, a, and δ

General Relativistic Magnetohydrodynamics

- The Conservative Equations
  - General conservative form

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla f(\mathbf{q}) = \mathbf{\Sigma}(\mathbf{q}) \neq \mathbf{\Sigma}(\mathbf{q}, \nabla \mathbf{q})$$

  - This is a system of hyperbolic differential equations
  - Conservative variables $\mathbf{q}$
  - Finite volume techniques have been employed to solve these equations
  - Integral form - allows for shock capturing
  - Typically the conservative variables $(\mathbf{q})$ will be functions of what are known as primitive variables $(\mathbf{p})$, $\mathbf{q} \rightarrow \mathbf{q}(\mathbf{p})$
  - Primitive variable conversion is required to obtain the required physical variables, in GR there is no closed form solution to this inversion
  - Equation of State also requires primitive variables

Conservation Form

- To consider the evolution of a magnetohydrodynamic system we use the conservation equations for baryons, energy–momentum, and the magnetic induction equation, as the fluid flows through some volume of space

- Baryon: $\nabla_\mu(J^\mu) = 0$, $J^\mu = \rho_o u^\mu$
  - u^μ is the fluid's 4-velocity
  - This is an expression for the conservation of matter

- Energy–Momentum: $\nabla_\mu T^{\mu\nu} = 0$
  - Conservation of Mass
  - Conservation of Momentum

- Induction Equation: $\nabla_\mu {}^*F^{\mu\nu} = 0$
  - ${}^*F^{\mu\nu}$ is the dual Faraday tensor
  - Using differential geometry terminology, just to say we use Maxwell's relations

Equation of State

- To close the system of equations, we also need an equation of state
- Ideal Gas Equation of state
  - $P = (\Gamma - 1)\rho_o\epsilon$
    - P - Pressure
    - $\rho = \rho_o(1 + \epsilon)$ - total mass energy density
    - Γ - adiabatic constant
    - ϵ - specific energy
    - ρ_o - rest mass density
    - $h = 1 + \epsilon + \frac{P}{\rho}$ - specific enthalpy

Relativistic Magnetohydrodynamics

- The Relativistic MHD Equations in Conservative Form

$$
\begin{aligned}
\partial_t(\sqrt{-g}\rho_o u^t) + \partial_i(\sqrt{-g}\rho_o u^i) &= 0 \\
\partial_t(\sqrt{-g}T^t_\nu) + \partial_i(\sqrt{-g}T^i_\nu) &= \sqrt{-g}T^{\mu\kappa}\partial_\mu g_{\nu\kappa} - \sqrt{-g}T^\kappa_\lambda \Gamma^\lambda_{\nu\kappa} \\
\partial_t(\sqrt{-g}B^i) + \partial_j(\sqrt{-g}\left(u^j b^i - u^i b^j\right)) &= 0 \\
\partial_i(\sqrt{-\gamma}B^i) &= 0
\end{aligned}
$$

where g is the determinant of the metric and

$$
\begin{aligned}
T^{\mu\nu} &= (\rho + P + b^2)u^\mu u^\nu + (P + \frac{1}{2}b^2)g^{\mu\nu} - b^\mu b^\nu \\
b^\mu &\equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}u_\nu F_{\lambda\kappa}
\end{aligned}
$$

Metrics

- Life would be considerably easier if all matter was too small to distort space and time surrounding them
- However this would be boring so we are left with different means to measure the distance between points on the spacetime "grid"
  - Minkowski - flat
  - Schwarzschild - static and stationary
  - Kerr - just stationary
  - Fully dynamic Spacetimes - neither static nor stationary
  - Others do exist but they pertain more to cosmology than surrounding individual massive bodies
- Minkowski is used when the massive body has no real effect on the spacetime itself, the masses are too small, or too far away.
- Schwarzschild, first let us consider a cow, next we assume it is a sphere...
- Kerr, let us take that cow and assume it is elliptical... then make it spin?
- Fully dynamic spacetimes are complicated and would need to be considered if the accretion mass was sufficient to significantly modify the mass of the accretor.

Kerr

- Since more "realistic" systems are expected to at a minimum have angular momentum, we use the Kerr metric
- The Kerr metric requires the following assumptions
  - Accretion rate is insufficient to modify the spacetime considerably
  - The massive bodies are axisymmetric
  - The massive bodies in space are rotating
- Important features of Kerr
  - Maximum angular momentum, J, is equal to the mass M (geometric units)
  - Has an event horizon
  - Has an ergosphere

Symmetries

- When performing calculations one needs to be aware of symmetries in the system

- These will allow a smaller domain to be considered without losing any information

  - Spherical symmetry $\rightarrow$ No angular dependence
    - $u^\theta = u^\phi = 0$ and $\partial_\theta = \partial_\phi = 0$
  - Axisymmetry $\rightarrow$ No azimuthal angular dependence
    - $u^\phi = 0$ and $\partial_\phi = 0$
  - Thin Disk $\rightarrow$ Material may only move in a small region around the equatorial plane
    - $u^\theta = 0$ and $\partial_\theta = 0$

Waves

- Characteristic velocities
  - Magnetohydrodynamics involves three type of characteristic velocities
    - Alfven Velocity
    - Fast Magnetosonic Velocity
    - Slow Magnetosonic Velocity
  - Alfven velocity comes from the magnetic field wave equation
  - The Fast and slow magnetosonic velocities come from the velocity wave equation
  - Fast waves have characteristic velocities greater than the Alfven velocity
  - Slow waves have characteristic velocities less than the Alfven velocity
  - In the relativistic case these must be solved numerically

Real Parameters

- For real accretion disks I will have to attempt to setup the initial conditions as seen from astronomical data

- For these systems we expect the following:
  - $Re > 10^{14}$
  - $R_M \sim 10^{10}$ (Solar Corona)
  - $\rho_o \sim 10^{-5} \rightarrow 10^{32} cm^{-3}$
  - $|B| \sim 0 \rightarrow 10^{15} G$
  - $\beta \sim 10^{-5} \rightarrow 10^{20}$

- these ranges include the Newtonian as well as relativistic cases

Typical Setup

- So for my research the typical parameters for an accretion disk are
  - Radius $\rightarrow$ 0.3a
  - Accretion rate $\rightarrow 10^{-4} M_{\odot}/y$
  - Average flow rate $\rightarrow 10^{5}$ m/s
  - Average pressure $\rightarrow 10^{-10} Pa$
  - Average density $\rightarrow 10^{32} cm^{-3}$
  - Magnetic field strength $\rightarrow 0 - 10^{15}$G
  - Magentic field direction $\rightarrow$ axisymmetric
  - Mass of accreting body $\rightarrow 1 M_{\odot}$
  - Tempurature $\rightarrow 10^{4} - 10^{8}$K

Turbulence & Instabilities

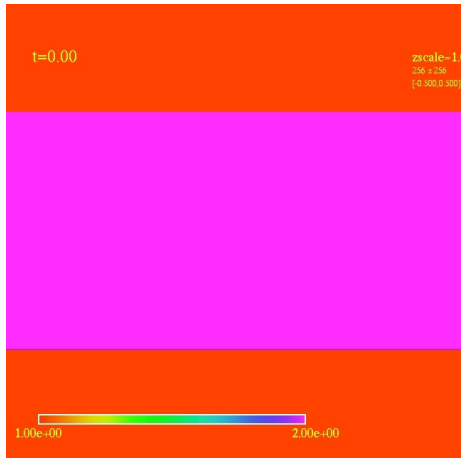
- Instabilities and Turbulence
  - Instabilities arise when the system undergoes a small perturbation and the dispersion relation leads to complex frequencies, thus the amplitudes either expand uncontrollably.
  - Turbulence arises when energy differences in the system undergo a transition from one energy type to another. Typically this energy will cascade from one length scale to the next, until viscous terms become non-negligible
  - There are a two types of turbulence that are of immediate concern in magnetohydrodynamic systems
    - Kelvin Helmholtz Instability
    - Magnetorotational Instability

Preliminary Results - Minkowski spacetime

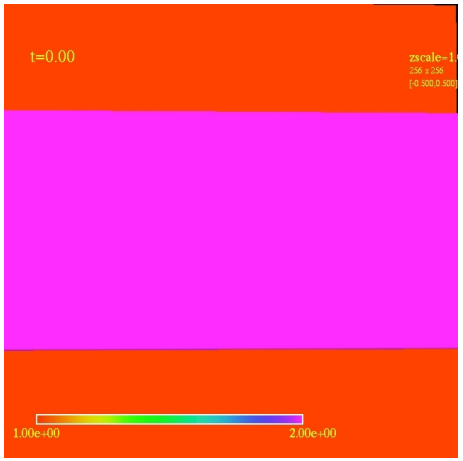
- In the Minkowski spacetime I have implemented the Kelvin-Helmholtz instability

- The Kelvin-Helmholtz instability comes from the over abundance of kinetic energy in a localized region.

- There is slipping between the two layers of the fluid. Relative velocity between the two streams is 0.8c.



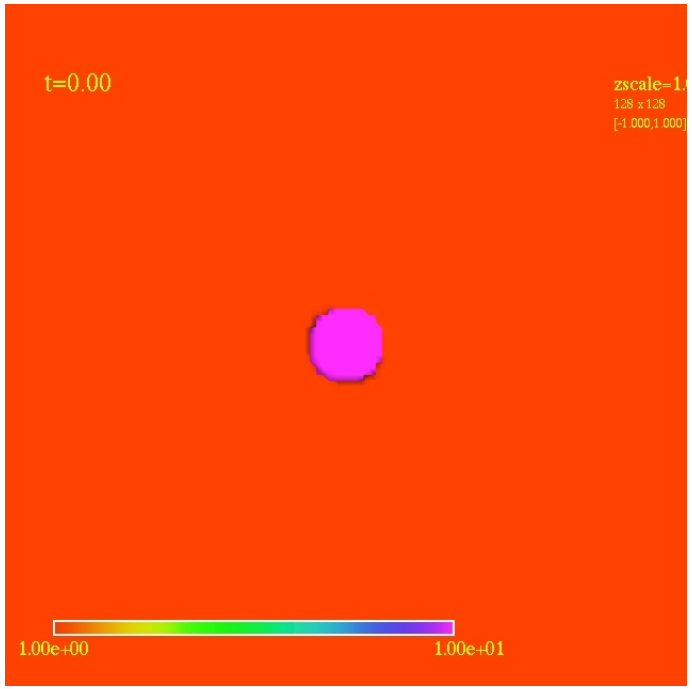
KHI No Magnetic field.



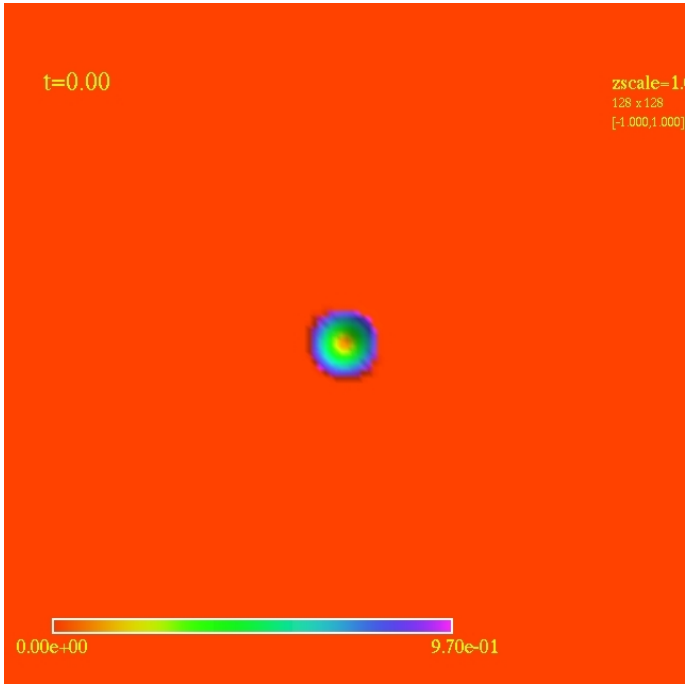
KHI With Magnetic Field, $B_x = 0.5(G\epsilon_o)^{1/2}/c$

Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented a rotor, edge angular velocity $\Omega = 9.75$
- Pressure $= 1.0G/c^4$
- Density $= 10G/c^2$ inside the disk, $1G/c^2$ outside of the disk



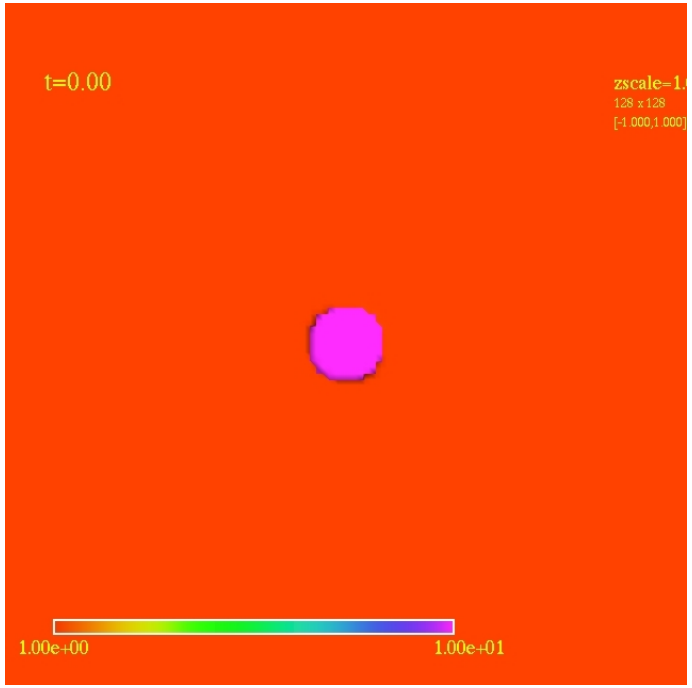
Density



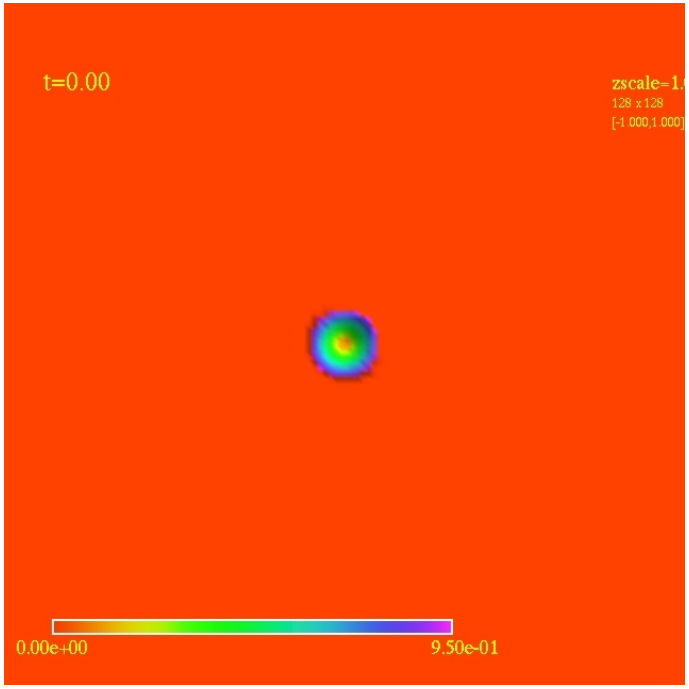
Speed

Preliminary Results - Minkowski spacetime

- Now including the magnetic field, $B_x = 0.5(G\epsilon_o)^{1/2}/c$
- Pressure $= 1.0G/c^4$
- Density $= 10G/c^2$ inside the disk, $1G/c^2$ outside of the disk



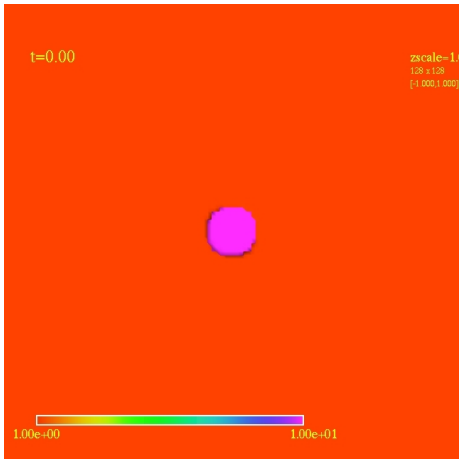
Density



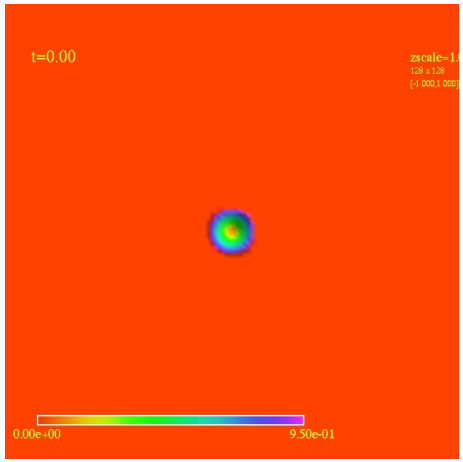
Speed

Preliminary Results - 1D Schwarzschild Spacetime

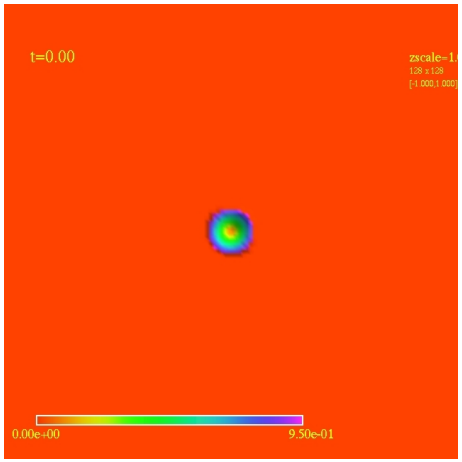
- Now I include a massive spherically symmetric body at the centre of my system $M = 1M_{\odot}$
- System is expected to settle down to a steady state solution after some initial transient behaviour
- Pressure $= 1.0 \times 10^{-12} G/c^4$
- Density $= 1.0 \times 10^{-12} G/c^2$



Density Profile



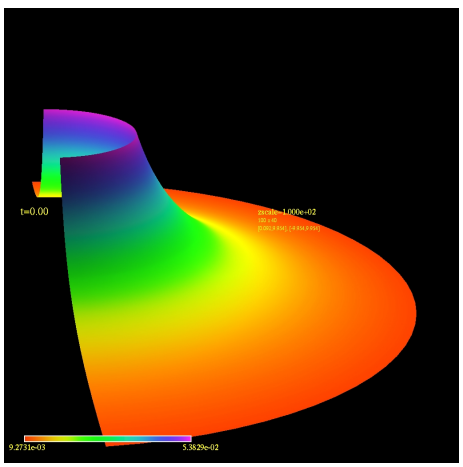
Radial Velocity Profile



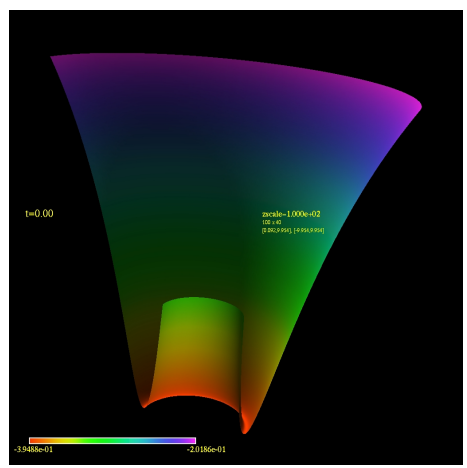
Pressure Profile

Preliminary Results - 2D Schwarzschild Spacetime

- Spherical accretion with $M = 1M_{\odot}$
- System is expected to settle down to a steady state solution after some initial transient behaviour
- Pressure $= 1.0 \times 10^{-12} G/c^4$
- Density $= 1.0 \times 10^{-12} G/c^2$

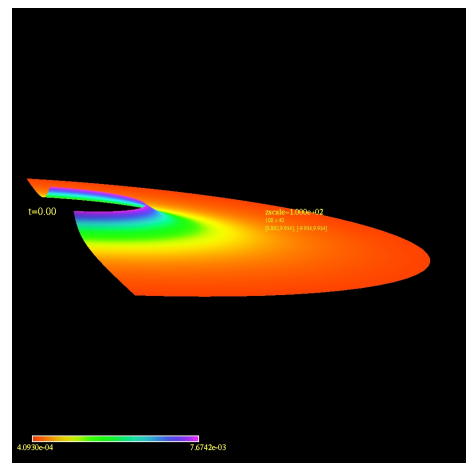


Density Profile



Radial Profile

Velocity



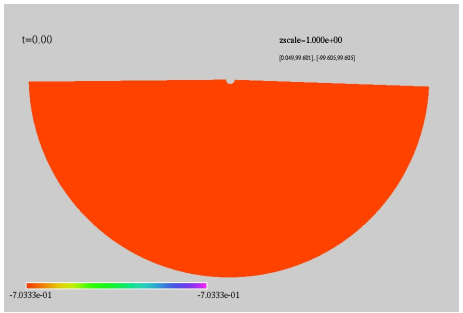
Pressure Profile

Main Research Path

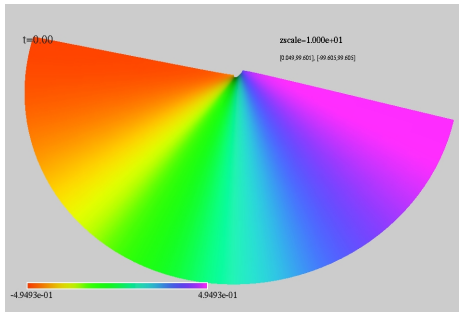
- Bondi–Hoyle Accretion
  - Black hole traveling through uniform space
  - Density, and velocities determined by their asymptotic values
    - Tail shock develops over time
  - Mass Accretion Rate
  - Momentum (Drag) Accretion Rate
  - Angular Momentum Rate
  - Axisymmetric Evolution
    - Spherically Symmetric Black Hole
    - Axisymmetric Black Hole
  - Non-axisymmetric evolution
    - Spherically Symmetric Black Hole
    - Axisymmetric Black Hole

Preliminary Results - 2D Schwarzschild Spacetime

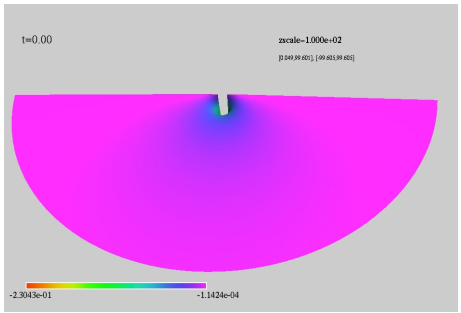
- Bondi–Hoyle axisymmetric accretion onto the black hole
- $M = 1M_{\odot}$
- Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$
- Pressure $= 6.1 \times 10^{-5} G/c^4$
- Density $= 11 G/c^2$
- This system is expected to fall into a steady state solution...



Density Profile, $\Gamma = 2$



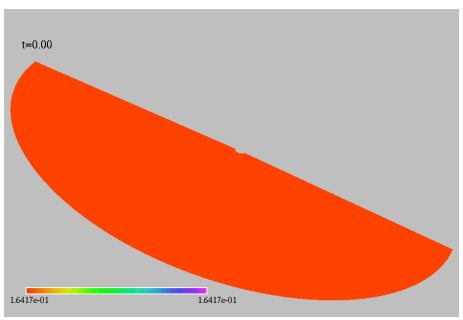
Radial Velocity Profile



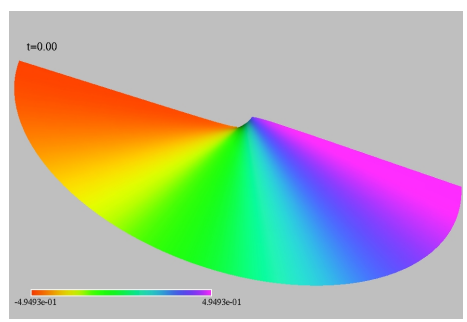
Angular Velocity Profile

Preliminary Results - 2D Kerr Spacetime

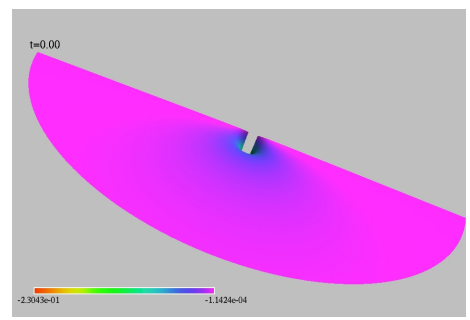
- Bondi–Hoyle axisymmetric accretion onto the black hole
- $M = 1M_{\odot}$, $a = 0.5G/c^3$
- Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$
- Pressure $= 6.1 \times 10^{-5} G/c^4$
- Density $= 11G/c^2$
- This system is also expected to fall into a steady state solution



Density Profile, $\Gamma = 5/3$



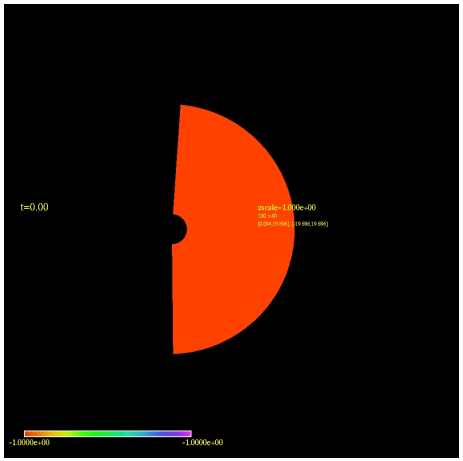
Radial Velocity Profile



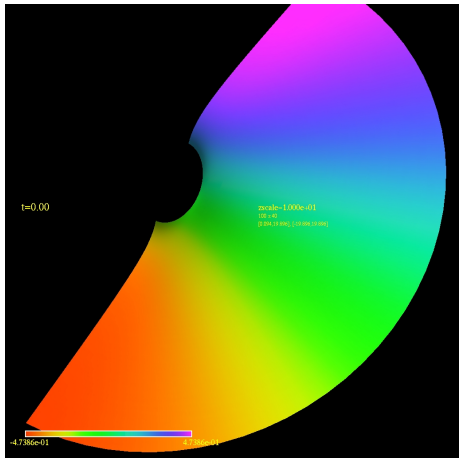
Angular Velocity Profile

Preliminary Results - 2D Schwarzschild Spacetime

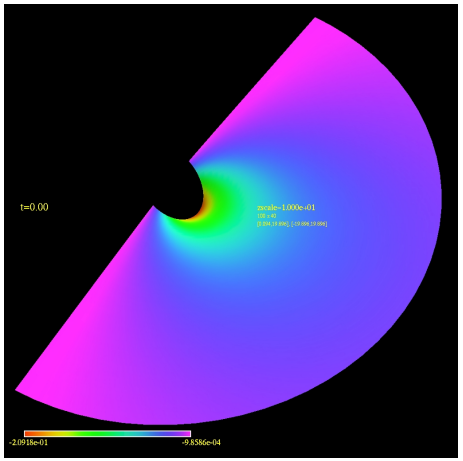
- Bondi–Hoyle non-axisymmetric accretion onto the black hole
- $M = 1M_\odot$
- Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$
- Pressure $= 6.1 \times 10^{-5} G/c^4$
- Density $= 11 G/c^2$
- This system is expected to fall into a steady state solution...



Density Profile, $\Gamma = 5/3$



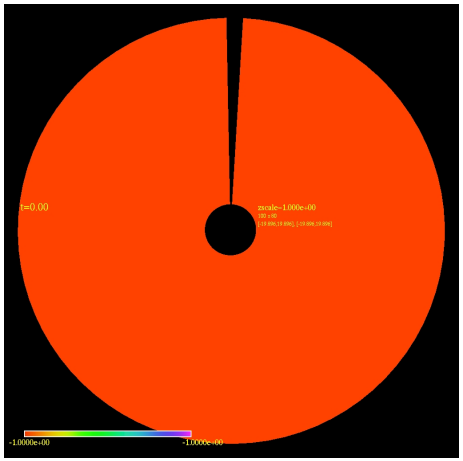
Radial Velocity Profile



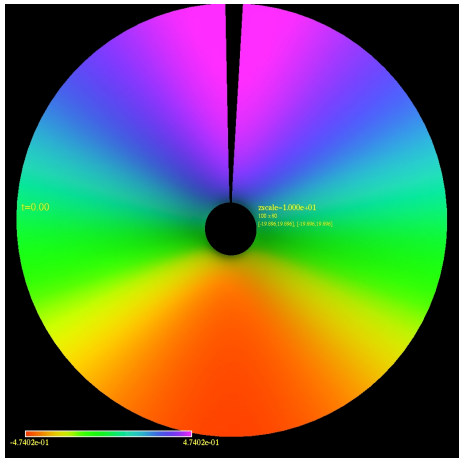
Angular Velocity Profile

Preliminary Results - 2D Kerr Spacetime

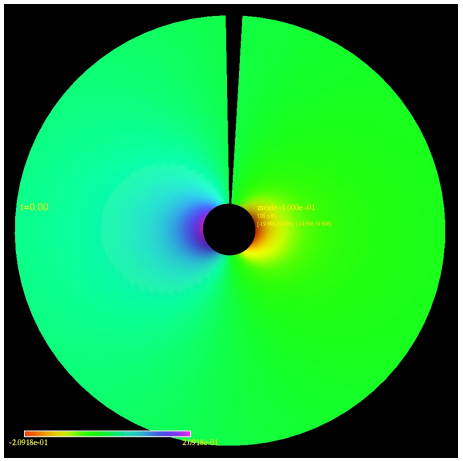
- Still let $M = 1M_{\odot}$, $a = 0.5G/c^3$
- Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$
- Pressure $= 0.061 \times 10^{-5} G/c^4$
- Density $= 11G/c^2$
- This system is expected to fall into a steady state solution...



Density Profile



Radial Velocity Profile



Angular Velocity Profile

What next?

- Raditation Hydrodynamics
  - In systems treated so far radiation is assumed to be caught by the accretion disk and not emmitted
  - This approximation is not valid for optically thin disks
  - these systems must allow for at least black body radiation
- Viscous Hydrodynamics
  - Non-adiabatic evolution
  - Newtonian viscous dissipation is parabolic $\longrightarrow$ infinite propagation speeds
  - Relativity cannot allow this, we must express hyberbolic viscous terms $\longrightarrow$ finite propagation speed

Numerical Magnetohydrodynamics Open Questions (well at least a few)

- Questions to consider
  - How well can we resolve large magnetic fields expected in extreme MRI?
  - How much magnetic flux enters the event horizon before a backlog occurs?
  - Can a steady state solution be achieved when one involves a magnetic field?
  - Does turbulence aid in the angular momentum transport by dissipating the energy to smaller scales?
  - What happens if matter infalls as lumps (high density regions) rather than as a steady stream of thin matter?
  - What are the effects of frame dragging for infalling matter to an observer at infinity (earth).

C'est Fin

- Questions?

More Preliminary Results - Minkowski spacetime

Angular Velocity Profile

Preliminary Results - Minkowski spacetime

- Now including the magnetic field

The Solver

- The Roe Solver

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = \Sigma(q)$$

Cell average:

$$Q_i \approx \frac{1}{\Delta V} \int_{\Delta V} q dV$$

Integrate conservative equation over a small volume, and divide by that same volume

$$\frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial t} q dx + \frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial x} f(q) dV = \frac{1}{\Delta V} \int_{\Delta V} \Sigma(q) dV$$

The Solver

which can be re-arranged, using Stokes' theorem

$$\frac{1}{\Delta V}\frac{\partial}{\partial t}\int_{\Delta V} q dx + \frac{1}{\Delta V}\int_{\partial V} f(q) dS = \frac{1}{\Delta V}\int_{\Delta V} \Sigma(q) dV$$

So now using our cell average, and the mean value theorem on the source term we get

$$\frac{\partial}{\partial t}(Q_i) + \frac{1}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) = \Sigma(Q_i)$$

finite difference the time derivative

$$(Q_i)^{n+1} = (Q_i)^n - \frac{\Delta t}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) + \Delta t \Sigma_i$$

Flux Approximations

- Shock Capturing

We think of the discretization Q_i^n as being a piecewise constant reconstruction of the solution q(x). Then at every cell boundary we have a Riemann problem (the discontinuity). To estimate the flux in the above equation we write the flux as $f(q^*)$ where q^* is the solution at the cell boundary to the problem given by:

$$\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} = 0$$

with the Jacobian $A = \partial F/\partial q$ is constant.

For $x = 0$: If $A > 0$ the flow is to the right, so $q^* = q_L$, this corresponds to $f(q^*) = Aq_L$. If $A < 0$ the flow is to the right, so $q^* = q_R$, which corresponds to $f(q^*) = Aq_R$. Rather than using an if/then approach we use a general form:

$$f_{i+1/2} = f(q^*_{i+1/2}) = \frac{1}{2}\left(Aq_L + Aq_R - |A|(q_R - q_L)\right)$$

So if $A > 0$ then $f = \frac{1}{2}(Aq_L + Aq_L) = Aq_L$