Numerical General Relativistic Magnetohydrodynamics

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Outline

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Motivation

• What is General Relativity?
  • Matter tells space how to curve $\iff$ Space tells matter how to move
  • General relativity studies how these two interact

• What is General Relativistic Magnetohydrodynamics?
  • The study of compressible, conducting fluids as they exist in curved spacetime.
  • Generally involve the study of three key components
    • Curved Spacetime
    • Hydrodynamic Stress–Energy Tensor
    • Electromagnetism Stress–Energy Tensor
    • Magnetohydrodynamic Stress–Energy Tensor

• Why is magnetohydrodynamics considered important?
  • Magnetic fields are fundamental to many astrophysical phenomenon
  • Currently believed to be a model for the dynamics of an AGN, and Supernovae
Astrophysical Phenomenon

● Accretion Disks
  ● Material that is caught in the gravitational field of a massive body
  ● May come from a binary star
  ● Accretion phenomenon is thought to be much more efficient than nuclear fusion

● Relativistic Jets
  ● A steady collimated beam of material emanating from a central body
  ● Still not well understood
  ● Mechanisms for collimation of the jet is still an open question

● Active Galactic Nuclei
  ● This astrophysical phenomenon is said to be at the centre of an active galaxy, understanding this may lead to a better understanding of the origins of our own galaxy

● Angular Momentum Transport
  ● There is still ongoing debate as to the mechanism that allows for angular momentum transport within accretion disks.
Cosmology

- **Galactic Dynamics**
  - Study the evolution of the universe in present day.
  - Can treat galaxies as fluid particles.
  - Many open questions especially if one considers the cosmic scale magnetic fields.

- **Dark Fluid**
  - A really new idea linking dark matter and dark energy.
  - Needs a lot of work before numerical models can be developed.

- **Cosmological Perturbation Theory**
  - Evolution from the big bang to present day.
  - models that are closer in time to the big bang itself are modeled using a hydrodynamic description of the matter and energy in the universe.
Magnetohydrodynamic Dimensionless Numbers

- Hydrodynamic dimensionless numbers
  - Knudsen number - $\frac{\lambda}{L} \ll 1$
  - Reynolds number (Re) - $\frac{VL}{\nu}$
  - Mach number (M) - $\frac{V}{c_s}$

- Where
  - $\lambda$ - mean free path between particles
  - $L$ - characteristic length scale $\sim R_{EventHorizon}$
  - $V$ - characteristic velocity scale $\sim c$
  - $\nu$ - fluid viscosity $\rightarrow 0$
  - $c_s$ - speed of sound $\sim \sqrt{\frac{P}{\rho_o}}$
Magnetohydrodynamic Dimensionless Numbers

- Additional dimensionless numbers when considering magnetic field contributions
  - Magnetic Reynolds number ($R_M$) - $V L / \eta$
  - beta ($\beta$) - $P_{\text{thermal}} / P_{\text{magnetic}}$
  - Alfven number (a) - $V / v_A$
  - Ratio of energy densities ($\delta$) - $E_B / E_{\text{rest}}$

- where
  - $v_A$ - the Alfvén velocity $= B / \sqrt{\rho_o + B^2}$
  - $\eta$ - the resistivity $\to 0$
  - $E_B \sim |B|^2$
  - $E_{\text{rest}} \sim \rho_o$

- The free parameters to survey are
  - $\beta$, $M$, $a$, and $\delta$
The Conservative Equations

- General conservative form

\[ \frac{\partial q}{\partial t} + \nabla f(q) = \Sigma(q) \neq \Sigma(q, \nabla q) \]

- This is a system of hyperbolic differential equations
- Conservative variables \( q \)
- Finite volume techniques have been employed to solve these equations
- Integral form - allows for shock capturing
- Typically the conservative variables \( (q) \) will be functions of what are known as primitive variables \( (p) \), \( q \rightarrow q(p) \)
- Primitive variable conversion is required to obtain the required physical variables, in GR there is no closed form solution to this inversion
- Equation of State also requires primitive variables
To consider the evolution of a magnetohydrodynamic system we use the conservation equations for baryons, energy–momentum, and the magnetic induction equation, as the fluid flows through some volume of space

- **Baryon**: \( \nabla_{\mu}(J^\mu) = 0 \), \( J^\mu = \rho_o u^\mu \)
  - \( u^\mu \) is the fluid’s 4-velocity
  - This is an expression for the conservation of matter

- **Energy–Momentum**: \( \nabla_\mu T^{\mu\nu} = 0 \)
  - Conservation of Mass
  - Conservation of Momentum

- **Induction Equation**: \( \nabla_\mu *F^{\mu\nu} = 0 \)
  - \( *F^{\mu\nu} \) is the dual Faraday tensor
  - Using differential geometry terminology, just to say we use Maxwell’s relations
To close the system of equations, we also need an equation of state.

Ideal Gas Equation of state

- $P = (\Gamma - 1)\rho_o\epsilon$
  - $P$ - Pressure
  - $\rho = \rho_o(1 + \epsilon)$ - total mass energy density
  - $\Gamma$ - adiabatic constant
  - $\epsilon$ - specific energy
  - $\rho_o$ - rest mass density
  - $h = 1 + \epsilon + \frac{P}{\rho}$ - specific enthalpy
The Relativistic MHD Equations in Conservative Form

\begin{align*}
\partial_t (\sqrt{-g} \rho u^t) + \partial_i (\sqrt{-g} \rho u^i) &= 0 \\
\partial_t (\sqrt{-g} T^t \nu) + \partial_i (\sqrt{-g} T^i \nu) &= \sqrt{-g} T^{\mu \kappa} \partial_\mu g_{\nu \kappa} - \sqrt{-g} T^\kappa_\lambda \Gamma^\lambda_{\nu \kappa} \\
\partial_t (\sqrt{-g} B^i) + \partial_j (\sqrt{-g} (u^j b^i - u^i b^j)) &= 0 \\
\partial_i (\sqrt{-\gamma} B^i) &= 0
\end{align*}

where \( g \) is the determinant of the metric and

\begin{align*}
T^{\mu \nu} &= (\rho + P + b^2)u^\mu u^\nu + (P + \frac{1}{2} b^2)g^{\mu \nu} - b^\mu b^\nu \\
b^\mu &\equiv \frac{1}{2} \epsilon^{\mu \nu \kappa \lambda} u_\nu F_{\lambda \kappa}
\end{align*}
• Life would be considerably easier if all matter was too small to distort space and time surrounding them

• However this would be boring so we are left with different means to measure the distance between points on the spacetime “grid”

  • Minkowski - flat
  • Schwarzschild - static and stationary
  • Kerr - just stationary
  • Fully dynamic Spacetimes - neither static nor stationary
  • Others do exist but they pertain more to cosmology than surrounding individual massive bodies

• Minkowski is used when the massive body has no real effect on the spacetime itself, the masses are too small, or too far away.

• Schwarzschild, first let us consider a cow, next we assume it is a sphere...

• Kerr, let us take that cow and assume it is elliptical... then make it spin?

• Fully dynamic spacetimes are complicated and would need to be considered if the accretion mass was sufficient to significantly modify the mass of the accretor.
Kerr

- Since more “realistic” systems are expected to at a minimum have angular momentum, we use the Kerr metric

- The Kerr metric requires the following assumptions
  - Accretion rate is insufficient to modify the spacetime considerably
  - The massive bodies are axisymmetric
  - The massive bodies in space are rotating

- Important features of Kerr
  - Maximum angular momentum, $J$, is equal to the mass $M$ (geometric units)
  - Has an event horizon
  - Has an ergosphere
Symmetries

- When performing calculations one needs to be aware of symmetries in the system

- These will allow a smaller domain to be considered without losing any information

  - Spherical symmetry $\rightarrow$ No angular dependence
    - $u^\theta = u^\phi = 0$ and $\partial_\theta = \partial_\phi = 0$

  - Axisymmetry $\rightarrow$ No azimuthal angular dependence
    - $u^\phi = 0$ and $\partial_\phi = 0$

  - Thin Disk $\rightarrow$ Material may only move in a small region around the equatorial plane
    - $u^\theta = 0$ and $\partial_\theta = 0$
Waves

• Characteristic velocities

  • Magnetohydrodynamics involves three type of characteristic velocities
    • Alfven Velocity
    • Fast Magnetosonic Velocity
    • Slow Magnetosonic Velocity

  • Alfven velocity comes from the magnetic field wave equation
  • The Fast and slow magnetosonic velocities come from the velocity wave equation
  • Fast waves have characteristic velocities greater than the Alfven velocity
  • Slow waves have characteristic velocities less than the Alfven velocity
  • In the relativistic case these must be solved numerically
Real Parameters

• For real accretion disks I will have to attempt to setup the initial conditions as seen from astronomical data

• For these systems we expect the following:

  • $Re > 10^{14}$
  • $R_M \sim 10^{10}$ (Solar Corona)
  • $\rho_o \sim 10^{-5} \rightarrow 10^{32} cm^{-3}$
  • $|B| \sim 0 \rightarrow 10^{15} G$
  • $\beta \sim 10^{-5} \rightarrow 10^{20}$

• these ranges include the Newtonian as well as relativistic cases
So for my research the typical parameters for an accretion disk are

- Radius $\rightarrow 0.3a$
- Accretion rate $\rightarrow 10^{-4} M_\odot/y$
- Average flow rate $\rightarrow 10^5$ m/s
- Average pressure $\rightarrow 10^{-10} Pa$
- Average density $\rightarrow 10^{32} cm^{-3}$
- Magnetic field strength $\rightarrow 0 - 10^{15}$ G
- Magnetic field direction $\rightarrow$ axisymmetric
- Mass of accreting body $\rightarrow 1 M_\odot$
- Temperature $\rightarrow 10^4 - 10^8$ K
Instabilities and Turbulence

- Instabilities arise when the system undergoes a small perturbation and the dispersion relation leads to complex frequencies, thus the amplitudes either expand uncontrollably.

- Turbulence arises when energy differences in the system undergo a transition from one energy type to another. Typically this energy will cascade from one length scale to the next, until viscous terms become non-negligible.

- There are two types of turbulence that are of immediate concern in magnetohydrodynamic systems:
  - Kelvin Helmholtz Instability
  - Magnetorotational Instability
In the Minkowski spacetime I have implemented the Kelvin-Helmholtz instability.

The Kelvin-Helmholtz instability comes from the over abundance of kinetic energy in a localized region.

There is slipping between the two layers of the fluid. Relative velocity between the two streams is 0.8c.
KHI No Magnetic field.

KHI With Magnetic Field,

\[ B_x = 0.5(G\epsilon_o)^{1/2}/c \]
Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented a rotor, edge angular velocity $\Omega = 9.75$
- Pressure $= 1.0G/c^4$
- Density $= 10G/c^2$ inside the disk, $1G/c^2$ outside of the disk
Preliminary Results - Minkowski spacetime

- Now including the magnetic field, $B_x = 0.5(G\epsilon_0)^{1/2}/c$
- Pressure = $1.0G/c^4$
- Density = $10G/c^2$ inside the disk, $1G/c^2$ outside of the disk
Preliminary Results - 1D Schwarzschild Spacetime

- Now I include a massive spherically symmetric body at the centre of my system $M = 1M_\odot$

- System is expected to settle down to a steady state solution after some initial transient behaviour

- Pressure $= 1.0 \times 10^{-12} G/c^4$

- Density $= 1.0 \times 10^{-12} G/c^2$
Preliminary Results - 2D Schwarzschild Spacetime

- Spherical accretion with $M = 1M_\odot$
- System is expected to settle down to a steady state solution after some initial transient behaviour
- Pressure = $1.0 \times 10^{-12}G/c^4$
- Density = $1.0 \times 10^{-12}G/c^2$
Main Research Path

- Bondi–Hoyle Accretion
  - Black hole traveling through uniform space
  - Density, and velocities determined by their asymptotic values
    - Tail shock develops over time
  - Mass Accretion Rate
  - Momentum (Drag) Accretion Rate
  - Angular Momentum Rate

- Axisymmetric Evolution
  - Spherically Symmetric Black Hole
  - Axisymmetric Black Hole

- Non-axisymmetric evolution
  - Spherically Symmetric Black Hole
  - Axisymmetric Black Hole
Preliminary Results - 2D Schwarzschild Spacetime

- Bondi–Hoyle axisymmetric accretion onto the black hole
- $M = 1M_\odot$
- Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$
- Pressure = $6.1 \times 10^{-5} G/c^4$
- Density = $11 G/c^2$
- This system is expected to fall into a steady state solution...
Preliminary Results - 2D Kerr Spacetime

- Bondi–Hoyle axisymmetric accretion onto the black hole
- $M = 1M_\odot$, $a = 0.5G/c^3$
- Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$
- Pressure = $6.1 \times 10^{-5} G/c^4$
- Density = $11G/c^2$
- This system is also expected to fall into a steady state solution

![Density Profile](image1.png)

$\Gamma = 5/3$

![Angular Profile](image2.png)

![Radial Profile](image3.png)

![Velocity Profile](image4.png)
• Bondi–Hoyle non-axisymmetric accretion onto the black hole

• $M = 1M_\odot$

• Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$

• Pressure $= 6.1 \times 10^{-5} G/c^4$

• Density $= 11G/c^2$

• This system is expected to fall into a steady state solution...
Density Profile, \( \Gamma = \frac{5}{3} \)

Angular Velocity Profile

Radial Velocity Profile
• Still let $M = 1M_\odot$, $a = 0.5G/c^3$

• Boost the black hole through spacetime at some decent relativistic velocity $v = 0.5c$

• Pressure = $0.061 \times 10^{-5} G/c^4$

• Density = $11G/c^2$

• This system is expected to fall into a steady state solution...
Density Profile

Radial Velocity Profile

Angular Velocity Profile

Velocity
What next?

- **Radiation Hydrodynamics**
  - In systems treated so far radiation is assumed to be caught by the accretion disk and not emitted
  - This approximation is not valid for optically thin disks
  - these systems must allow for at least black body radiation

- **Viscous Hydrodynamics**
  - Non-adiabatic evolution
  - Newtonian viscous dissipation is parabolic \( \rightarrow \) infinite propagation speeds
  - Relativity cannot allow this, we must express hyperbolic viscous terms \( \rightarrow \) finite propagation speed
Numerical Magnetohydrodynamics Open Questions (well at least a few)

- Questions to consider

  - How well can we resolve large magnetic fields expected in extreme MRI?
  - How much magnetic flux enters the event horizon before a backlog occurs?
  - Can a steady state solution be achieved when one involves a magnetic field?
  - Does turbulence aid in the angular momentum transport by dissipating the energy to smaller scales?
  - What happens if matter infalls as lumps (high density regions) rather than as a steady stream of thin matter?
  - What are the effects of frame dragging for infalling matter to an observer at infinity (earth).
Questions?
Angular Velocity Profile
Preliminary Results - Minkowski spacetime

- Now including the magnetic field
The Solver

• The Roe Solver

\[
\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = \Sigma(q)
\]

Cell average:

\[
Q_i \approx \frac{1}{\Delta V} \int_{\Delta V} q dV
\]

Integrate conservative equation over a small volume, and divide by that same volume

\[
\frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial t} q dx + \frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial x} f(q) dV = \frac{1}{\Delta V} \int_{\Delta V} \Sigma(q) dV
\]
which can be re-arranged, using Stokes’ theorem

\[
\frac{1}{\Delta V} \frac{\partial}{\partial t} \int_{\Delta V} q dx + \frac{1}{\Delta V} \int_{\partial V} f(q) dS = \frac{1}{\Delta V} \int_{\Delta V} \Sigma(q) dV
\]

So now using our cell average, and the mean value theorem on the source term we get

\[
\frac{\partial}{\partial t} (Q_i) + \frac{1}{\Delta V} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) = \Sigma(Q_i)
\]

finite difference the time derivative

\[
(Q_i)^{n+1} = (Q_i)^n - \frac{\Delta t}{\Delta V} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) + \Delta t \Sigma_i
\]
Flux Approximations

- Shock Capturing

We think of the discretization $Q^n_i$ as being a piecewise constant reconstruction of the solution $q(x)$. Then at every cell boundary we have a Riemann problem (the discontinuity). To estimate the flux in the above equation we write the flux as $f(q^*)$ where $q^*$ is the solution at the cell boundary to the problem given by:

$$\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = 0$$

with the Jacobian $A = \partial F/\partial q$ is constant.

For $x = 0$: If $A > 0$ the flow is to the right, so $q^* = q_L$, this corresponds to $f(q^*) = Aq_L$. If $A < 0$ the flow is to the right, so $q^* = q_R$, which corresponds to $f(q^*) = Aq_R$. Rather than using an if/then approach we use a general form:

$$f_{i+1/2} = f(q^*_{i+1/2}) = \frac{1}{2} (Aq_L + Aq_R - |A|(q_R - q_L))$$

So if $A > 0$ then $f = \frac{1}{2} (Aq_L + Aq_L) = Aq_L$