

# Numerical General Relativistic Magnetohydrodynamics

2nd PhD Committee Meeting

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# Courses

- 2002:
  - 16.742 - Quantum Mechanics -  $> 90$  A+ - 6 crd
  - 16.754 - Statistical Mechanics -  $> 75$  B+ - 3 crd
  - 16.462 - Adv. Classical Mech. -  $> 90$  A+ - 3 crd
- 2003
  - 16.759 - Electromagnetism -  $> 90$  A+ - 3 crd
- 2004F - Term 1:
  - Phys555/410 - Computational Phys - 92 A+ - 3 crd
  - Phys526 - Quantum Electrodynamics - 94 A+ - 3 crd
- 2004W - Term 2:
  - Phys555 - Numerical Relativity - 94 A+ - 3 crd
  - Phys508 - Quantum Field Theory - 88 A - 3 crd

# Courses

- 2005W - Term 2:
  - Phys555 - Meth. in Par. Comp. and Particle Mesh Tech. - 90 A+ - 3 crd
  - Phys530 - Graduate General Relativity - 88 A - 3 crd
- 2006W - Term 2:
  - Phys530 - Quantum Gravity - Aud - 3 crd
  - Phys511 - Quantum Magnetism - Aud - 3 crd

# Outline

- Motivation
- Magnetohydrodynamics
- Waves
- Turbulence
- Current Project: Relativistic Numerical Magnetohydrodynamics

# Motivation

- What is General Relativistic Magnetohydrodynamics?
  - The study of compressible, conducting fluids as they exist in curved spacetime.
  - Generally involve the study of three key components
    - Curved Spacetime
    - Hydrodynamic Stress–Energy Tensor
    - Electromagnetism Stress–Energy Tensor
  - Magnetohydrodynamic Stress–Energy Tensor
- Why study magnetohydrodynamics?
  - Magnetic fields are fundamental to many astrophysical phenomenon
  - Currently believed to be a model for the dynamics of an AGN, and Supernovae

# Astrophysical Phenomenon

- **Accretion Disks**

- Material that is caught in the gravitational field of a massive body
- May come from a binary star
- accretion phenomenon is thought to be much more efficient than nuclear fusion

- **Relativistic Jets**

- A steady collimated beam of material emanating from a central body
- Still not well understood
- Mechanisms for collimation of the jet is still an open question

- **Active Galactic Nuclei**

- This astrophysical phenomenon is said to be at the centre of an active galaxy, understanding this may lead to a better understanding of the origins of our own galaxy

# Magnetohydrodynamic Dimensionless Numbers

- Hydrodynamic dimensionless numbers
  - Knudsen number -  $\lambda/L \ll 1$
  - Reynolds number (Re) -  $VL/\nu$
  - Mach number (M) -  $V/c_s$
- Where
  - $\lambda$  - mean free path between particles
  - $L$  - characteristic length scale
  - $V$  - characteristic velocity scale
  - $\nu$  - fluid viscosity
  - $c_s$  - speed of sound

# Magnetohydrodynamic Dimensionless Numbers

- Additional dimensionless numbers when considering magnetic field contributions
  - Magnetic Reynolds number ( $R_M$ ) -  $VL/\eta$
  - beta ( $\beta$ ) -  $P_{thermal}/P_{magnetic}$
  - Alfvén number ( $a$ ) -  $V/v_A$
  - $\delta$  -  $E_{magnetic}/E_{rest}$
- where
  - $v_A$  - the Alfvén velocity
  - $\eta$  - the resistivity
- In the ideal MHD case, the viscosity  $\nu$ , and the resistivity  $\eta$  are both zero
- The free parameters to observe are
  - $\beta$ ,  $M$ ,  $a$ , and  $\delta$



# General Relativistic Magnetohydrodynamics

- The Conservative Equations

- General conservative form

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla f(\mathbf{q}) = \Sigma(\mathbf{q}) \neq \Sigma(\mathbf{q}, \nabla \mathbf{q})$$

- This is a system of hyperbolic differential equations
- Conservative variables  $\mathbf{q}$
- Finite volume techniques have been employed to solve these equations
- Integral form - allows for shock capturing
- Typically the conservative variables ( $\mathbf{q}$ ) will be functions of what are known as primitive variables ( $\mathbf{p}$ ),  $\mathbf{q} \rightarrow \mathbf{q}(\mathbf{p})$
- Primitive variable conversion is required to obtain the required physical variables, in GR there is no closed form solution to this inversion
- Equation of State also requires primitive variables

# Conservation Form

- To consider the evolution of a magnetohydrodynamic system we use the conservation equations for baryons, energy–momentum, and the magnetic induction equation, as the fluid flows through some volume of space
- Baryon:  $\nabla_{\mu}(J^{\mu}) = 0$ ,  $J^{\mu} = \rho_o u^{\mu}$ 
  - $u^{\mu}$  is the fluid's 4-velocity
- Energy–Momentum:  $\nabla_{\mu} T^{\mu\nu} = 0$
- Induction Equation:  $\nabla_{\mu} {}^*F^{\mu\nu} = 0$ 
  - ${}^*F^{\mu\nu}$  is the dual Faraday tensor
- To close the system of equations, we also need an equation of state
- Ideal Gas Equation of state
  - $P = (\Gamma - 1)\rho_o\epsilon$ 
    - $P$  - Pressure
    - $\rho = \rho_o(1 + \epsilon)$  - total mass energy density
    - $\Gamma$  - adiabatic constant
    - $\epsilon$  - specific energy
    - $\rho_o$  - rest mass density
    - $h = 1 + \epsilon + \frac{P}{\rho}$  - specific enthalpy

# Relativistic Magnetohydrodynamics

- The Relativistic MHD Equations in Conservative Form

$$\begin{aligned}\partial_t(\sqrt{-g}\rho_o u^t) + \partial_i(\sqrt{-g}\rho_o u^i) &= 0 \\ \partial_t(\sqrt{-g}T_\nu^t) + \partial_i(\sqrt{-g}T_\nu^i) &= \sqrt{-g}T^{\mu\kappa}\partial_\mu g_{\nu\kappa} - \sqrt{-g}T_\lambda^\kappa\Gamma_{\nu\kappa}^\lambda \\ \partial_t(\sqrt{-g}B^i) + \partial_j(\sqrt{-g}(u^j b^i - u^i b^j)) &= 0 \\ \partial_i(\sqrt{-g}B^i) &= 0\end{aligned}$$

where  $g$  is the determinant of the metric and

$$\begin{aligned}T^{\mu\nu} &= (\rho + P + b^2)u^\mu u^\nu + (P + \frac{1}{2}b^2)g^{\mu\nu} - b^\mu b^\nu \\ b^\mu &\equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}u_\nu F_{\lambda\kappa}\end{aligned}$$

- 3 + 1 formalism: slice spacetime into spacelike hypersurfaces;

# ADM / 3+1 Formalism

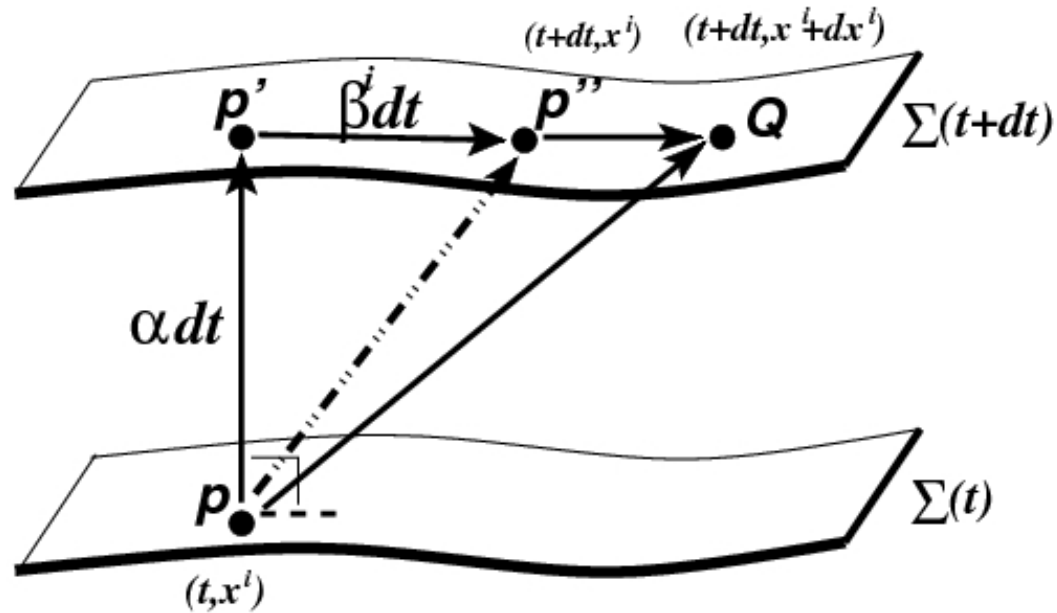


Figure 1: A schematic representation of the ADM space+time decomposition.

- Future directed, time unit normal to the hypersurfaces

$$n_\mu = -\alpha \nabla_\mu t$$

where  $\alpha$  is the lapse function

- Shift vector  $\beta^\mu$  defined via

$$t^\mu = \alpha n^\mu + \beta^\mu$$

$$\beta^\mu n_\mu = 0$$

# Relativistic Magnetohydrodynamics

- The General Relativistic MHD Equations Written in Conservation Form

Defining the following projections as conservation variables

$$D = -n_\mu J^\mu \rightarrow \alpha \rho_o u^t$$

$$E = T^{\mu\nu} n_\mu n_\nu \rightarrow \alpha^2 T^{00}$$

$$S_\lambda = -T^{\mu\nu} n_\mu \gamma_{\nu\lambda} \rightarrow -\alpha T^{0\nu} \gamma_{\nu\lambda}$$

$$B_\lambda = {}^*F^{\nu\mu} n_\nu \gamma_{\mu\lambda} \rightarrow -\alpha {}^*F^{0\mu} \gamma_{\mu\lambda}$$

Where I used:  $n_\nu = (-\alpha, 0, 0, 0)$

$\gamma_{\nu\mu}$  is the induced metric

# Relativistic Magnetohydrodynamics

- The Conserved Relativistic MHD Equations

$$\partial_t (\sqrt{\gamma} D) + \partial_i (\sqrt{-g} D V^i) = 0$$

$$\partial_t (\sqrt{\gamma} S_j) + \partial_i (\sqrt{-g} (S_j V^i - b_j B^i / W + P_T \delta_j^i)) = \frac{\sqrt{-g}}{2} T^{\mu\nu} (\partial_i g_{\mu\nu} - \Gamma_{\mu\nu}^\lambda g_{\lambda i})$$

$$\partial_t (\sqrt{\gamma} E) + \partial_i (\sqrt{-g} (E V^i + P_T v^i - \alpha b^t B^i / W)) = Q$$

$$\partial_t (\sqrt{\gamma} B^i) + \partial_j (\sqrt{-g} (V^i B^j - V^j B^i)) = 0$$

$$\partial_i (\sqrt{\gamma} B^i) = 0$$

With

$$Q = -\alpha \sqrt{-g} (T^{\mu t} \partial_\mu (\ln \alpha) - T^{\mu\nu} \Gamma_{\mu\nu}^t)$$

$$b^t = W B^i v_i / \alpha$$

$$b^i = \frac{B^i}{W} + \alpha b^t v^i$$

$$V^i = v^i - \beta^i / \alpha$$

$$P_T = P + \frac{b^2}{2}$$

# Waves

- Characteristic velocities
- Luis Anton et al. (arxiv:astro-ph/0506063v1 June 2005)
  - *Numerical 3+1 GRMHD: A Local Characteristic Approach*
  - Found that one can write the EoM in the quasilinear form  $A_B^{\nu A} \nabla_\nu V^B = 0$
  - $V^B = (u^\nu, b^\mu, P, s)$ ,  $A, B = 0 \rightarrow 9$

$$A^\mu = \begin{pmatrix} C u^\mu \delta_\beta^\alpha & -b^\mu \delta_\beta^\alpha + P^{\alpha\mu} b_\beta & l^{\alpha\mu} & 0^{\alpha\mu} \\ b^\mu \delta_\beta^\alpha & -u^\mu \delta_\beta^\alpha & f^{\mu\alpha} & 0^{\alpha\mu} \\ \rho h \delta_\beta^\mu & 0_\beta^\mu & u^\mu / c_s^2 & 0^\mu \\ 0_\beta^\mu & 0_\beta^\mu & 0^\mu & u^\mu \end{pmatrix}$$

$c_s^2$  is the speed of sound

$$\begin{aligned} C &= \rho h + b^2 \\ P^{\alpha\mu} &= g^{\alpha\mu} + 2u^\alpha u^\mu \\ l^{\mu\alpha} &= (\rho h g^{\mu\alpha} + (\rho h - b^2 / c_s^2) u^\mu u^\alpha) / \rho h \\ f^{\mu\alpha} &= (u^\alpha b^\mu / c_s^2 - u^\mu b^\alpha) / \rho h \end{aligned}$$

# Waves Continued

- Projecting  $A^\mu$  onto a  $\phi(x^\mu) = 0$  hypersurface and taking the determinant we have a secular equation

$$\det(A^\mu \phi_\mu) = C a^2 \mathcal{A}^2 \mathcal{N}_4 = 0$$

$$a = u^\mu \phi_\mu$$

$$\mathcal{B} = b^\mu \phi_\mu$$

$$\mathcal{A} = C a^2 - \mathcal{B}$$

$$\mathcal{N}_4 = \rho h \left( \frac{1}{c_s^2 - 1} \right) a^4 - \left( \rho h + \frac{b^2}{c_s^2} \right) a^2 G + \mathcal{B}^2 G$$

$$G = \phi^\mu \phi_\mu$$

- Consider a wave propagating in an arbitrary direction  $\mathbf{x}$  with speed  $\lambda$  normal to the characteristic hypersurface  $\phi(x^\mu) = 0$
- $\phi_\mu = (-\lambda, 1, 0, 0)$
- using this in our secular equation we can look at the individual terms



# Waves Continued

- $a = 0$
- leads to  $\lambda = \alpha v^x - \beta^x$
- $\mathcal{A} = 0$
- leads to the Alfvén waves
- $\lambda = \frac{b^x \pm \sqrt{C} u^x}{b^t \pm \sqrt{C} u^t}$
- $\mathcal{N}_4 = 0$
- leads to the fast and slow magnetosonic waves
- There is no convenient closed form solution for this equation. Must be solved numerically

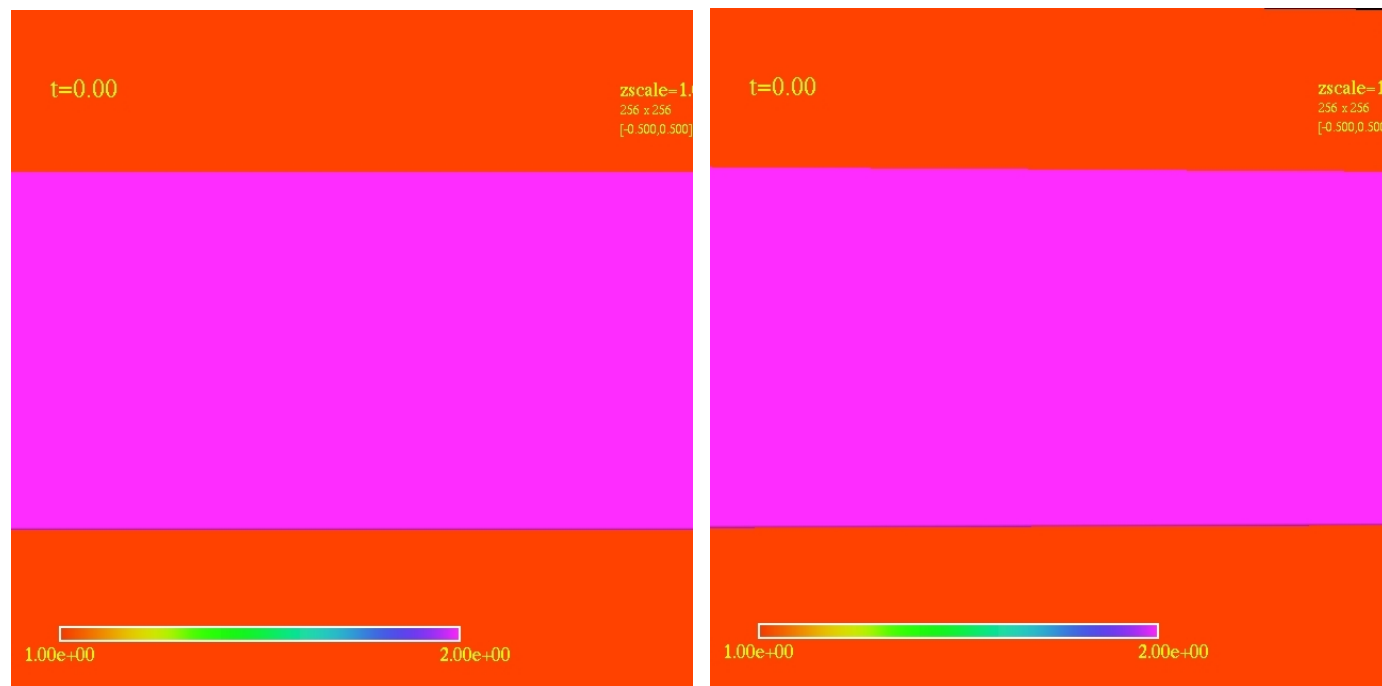
# Turbulence & Instabilities

- Instabilities and Turbulence

- Instabilities arise when the system undergoes a small perturbation and the dispersion relation leads to purely imaginary frequencies, thus the amplitudes either expand uncontrollably or dampen.
- Turbulence arises when energy differences in the system undergo a transition from one energy type to another. Typically this energy will cascade from one length scale to the next, until viscous terms become non-negligible
- There are a few types of turbulence that are of immediate concern in (magneto)hydrodynamic systems
  - Kelvin Helmholtz Instability
  - Magnetorotational Instability

# Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented the Kelvin-Helmholtz instability
- The Kelvin-Helmholtz instability comes from the over abundance of kinetic energy in a localized region. This physically comes from differential rotation which is expected to be found in accretion disks
- There is slipping between the two layers of the fluid. Relative velocity between the two streams is  $0.8c$ .

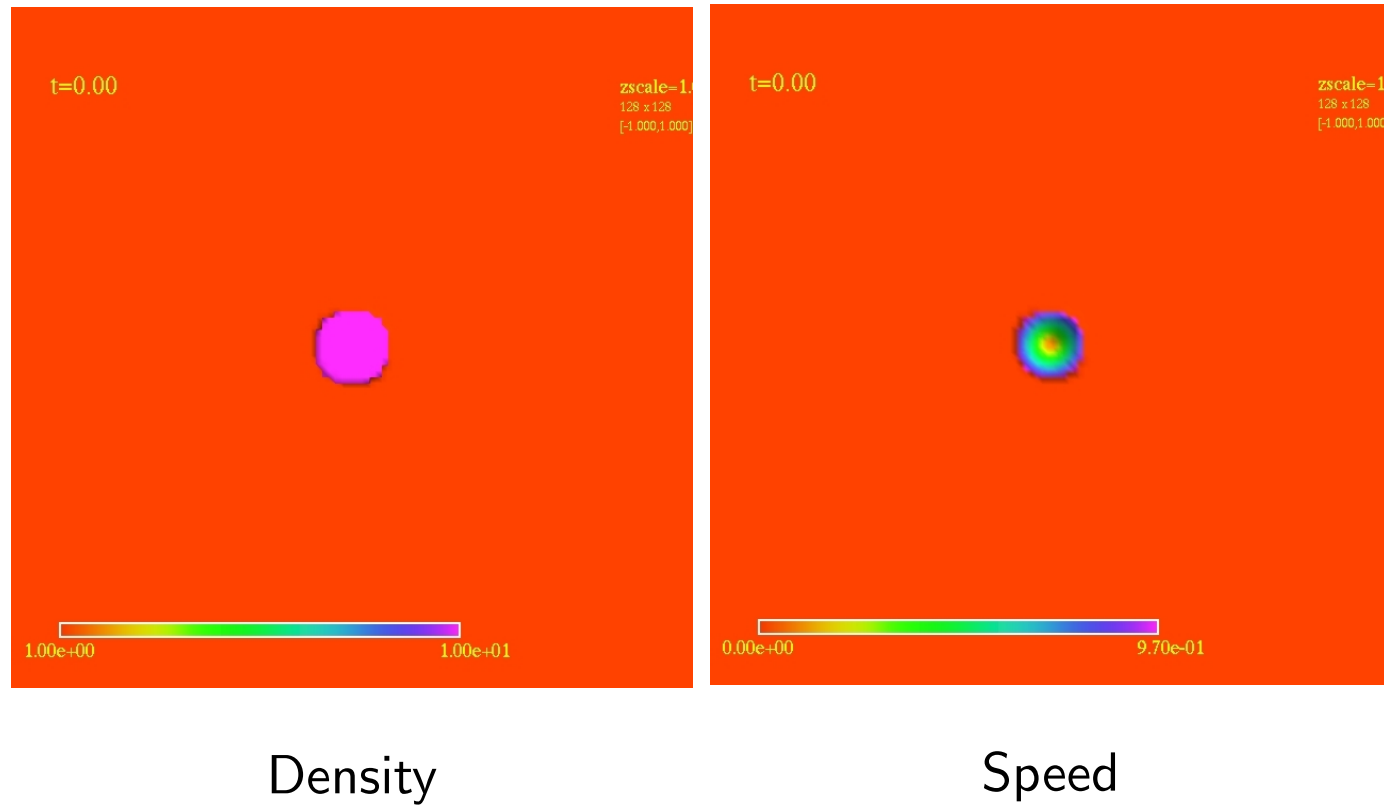


KHI No Magnetic field.

KHI With Magnetic Field,  
 $B_x = 0.5$

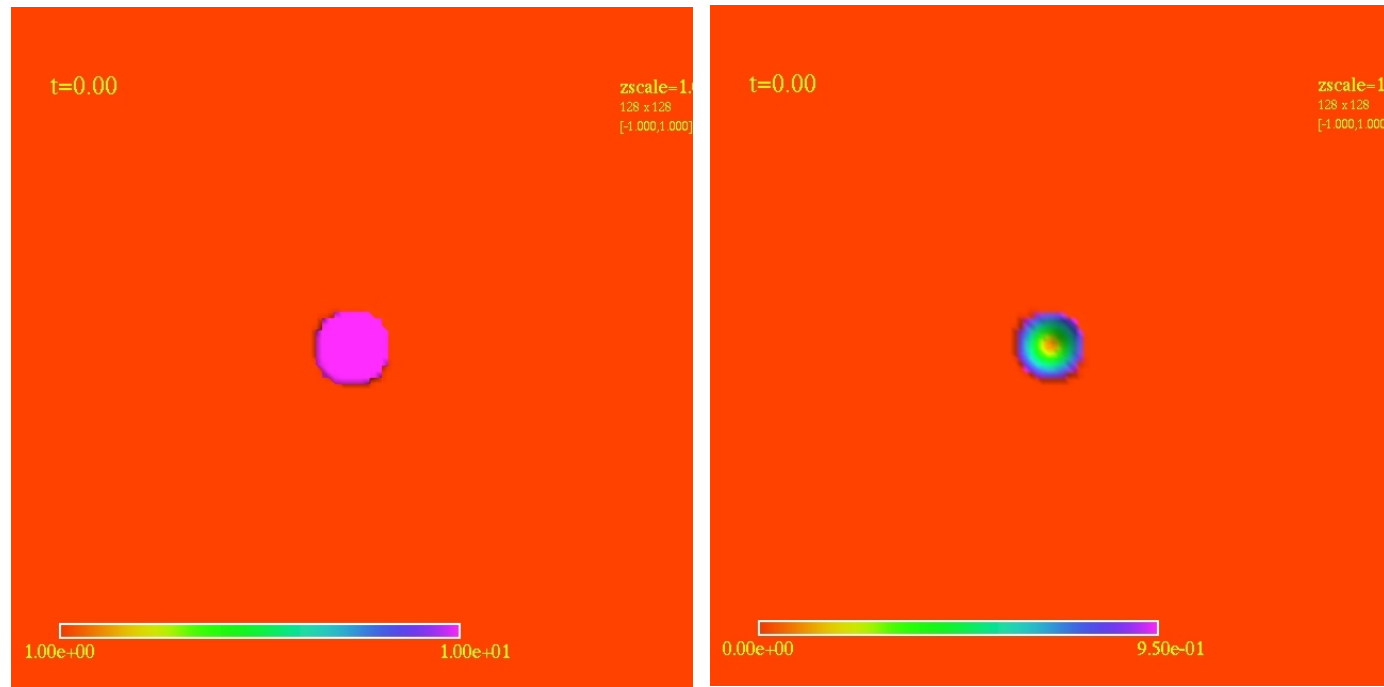
# Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented a rotor, edge angular velocity  $\Omega = 9.75$
- Pressure = 1.0
- Density = 10 inside the disk, 1 outside of the disk



# Preliminary Results - Minkowski spacetime

- Now including the magnetic field,  $B_x = 0.5$
- Pressure = 1.0
- Density = 10 inside the disk, 1 outside of the disk



Density

Speed

# Future work

- Magnetorotational Instability
  - Rotational energy is converted to magnetic energy
  - Thought to underlie viscous dissipation when matter falls onto the compact object from a companion
  - Infalling matter has considerable angular momentum
  - This forms the rotating disk around the object, with differential rotation
  - Inner regions lose angular momentum due to viscosity and fall inward, outer regions gain angular momentum and move outward
  - infalling matter perturbs the outer regions preventing the outflow

# Representative Recent Work

- L. Del Zanna, O. Zanotti, N. Bucciantini, P. Londrillo, (2007) ESO
  - *ECHO: and Eulerian Conservative High Order scheme for general relativistic magnetohydrodynamics and magnetodynamics*
  - Developed a GRMHD code to use stationary metrics
  - Created several test suits for GRMHD for Schwarzschild and Kerr spacetimes
- JWS. Blokland, R. Keppens, JP. Goedbloed astro-ph/0703581v1 Astronomy & Astrophysics (2007)
  - *Unstable magnetohydrodynamical continuous spectrum of accretion disks*
  - Study localized MHD instabilities in 2D MHD accretion disks
  - Find localized MHD instabilities provide an ideal linear route to MHD turbulence in strong B fields.

# Current Project - Numerical Magnetohydrodynamics

- Motivation

- Existing theories and codes appear to work well, however they all rely on an ideal equation of state, other equations of state may lead to results that come closer to the observed phenomenon.
- To study the turbulence in accretion disks
- To study "puffy" disks, or accretion shells
- Ultimately, to better understand the complex dynamics of accretion disks and jets around compact objects such as black holes



# Current Project - Numerical Magnetohydrodynamics

- Questions to consider
  - Does the KHI have any impact on the formation of jets?
  - How does a more realistic EoS impact the turbulence in MRI?
  - Will this realistic EoS allow for angular momentum transport in the accretion disk?
  - Does super-Eddington accretion take place in the region between the accretion disk and the black hole event horizon

# What Has Been Done

- Milestones
  - 1D special relativistic hydrodynamics
  - Cartesian 2D Newtonian hydrodynamic codes, along with consistency/convergence tests
  - Cartesian 2D Newtonian MHD, with proper magnetic field treatment
- Since last we met,
  - Completed 2D Cartesian SRMHD with magnetic flux corrections
  - Completed Polar SRMHD with magnetic flux corrections
  - Run several tests, including KHI, the blast waves, rigid rotor
  - Parallelized the code
  - Developed subroutines to perform tensor manipulations numerically

# The Plan - Numerical Magnetohydrodynamics

- Phases of the project
  - The final goal is to run detailed 2D simulations of accretion disks around massive bodies
  - To achieve this several small projects must be completed;
    - Next month
      - Finalize 2D general relativistic magnetohydrodynamics
      - Test Schwarzschild background
      - Test Kerr background
    - Next 4-6 months
      - Using above code, attempt to model real astrophysical data on the accretion disks
      - Perform a parameter survey of the different plasma parameters
      - Observe turbulent behaviour of the accretion disk
  - Afterwards
    - Modify the code for use with FLAMR infrastructure that is being constructed by Martin Snajdr and Frans Pretorius
    - Start investigating 3D relativistic accretion disks with relativistic jets

# The Plan - Numerical Magnetohydrodynamics

- Full 3D Code
- Next 8-12 months
  - Super Eddington Luminosity
    - Open question about goings on between the accretion disk and the black hole horizon
  - Jets
    - There are also open questions about the formation of the relativistic jets
  - Turbulence
    - Attempt to monitor turbulence and instabilities in full 3D evolutions

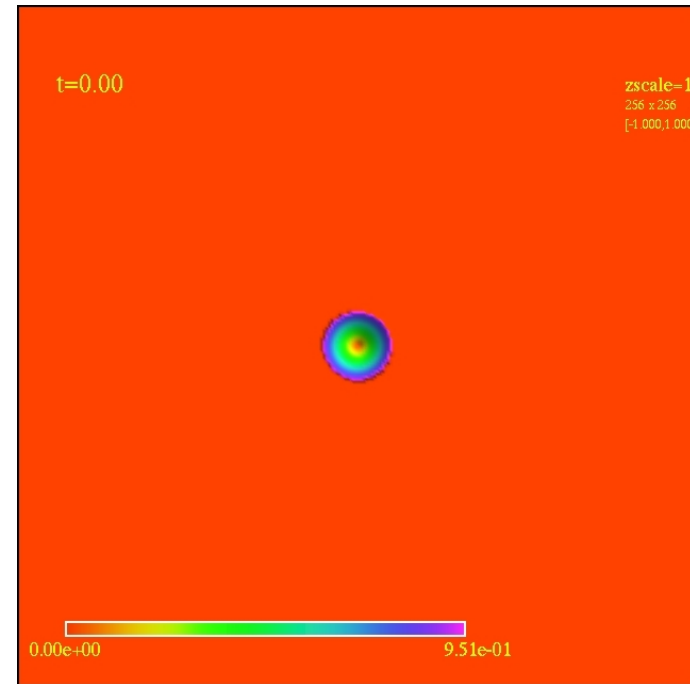
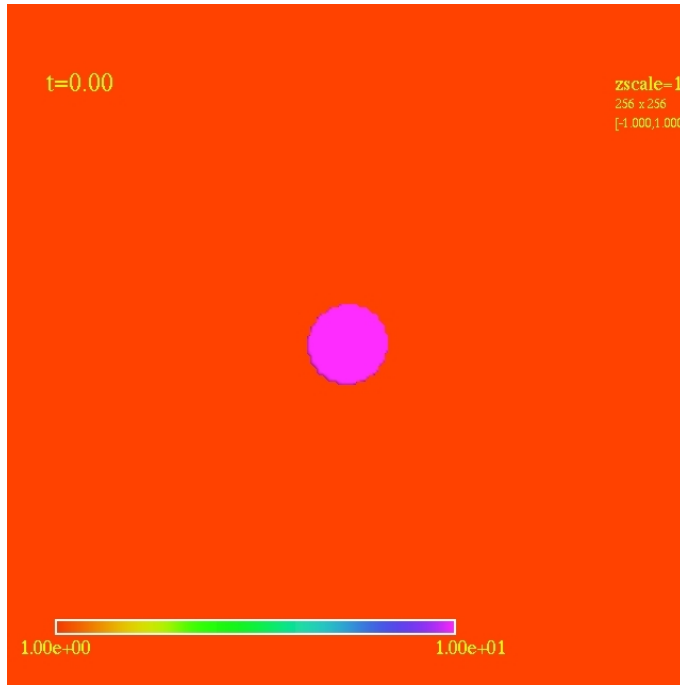
# Future considerations

- For real accretion disks I will have to attempt to setup the initial conditions as seen from astronomical data
- For these systems we expect the following:
  - $Re > 10^4$
  - $R_M \sim 10^{10}$  (Solar Corona)
  - $\rho_o \sim 10^{-5} - 10^{32} cm^{-3}$
  - $T \sim 10 - 10^9 K$
  - $|B| \sim 10^{-12} - 10^{20} G$
  - $\beta \sim 10^{-5} - 10^{20}$

# C'est Fin

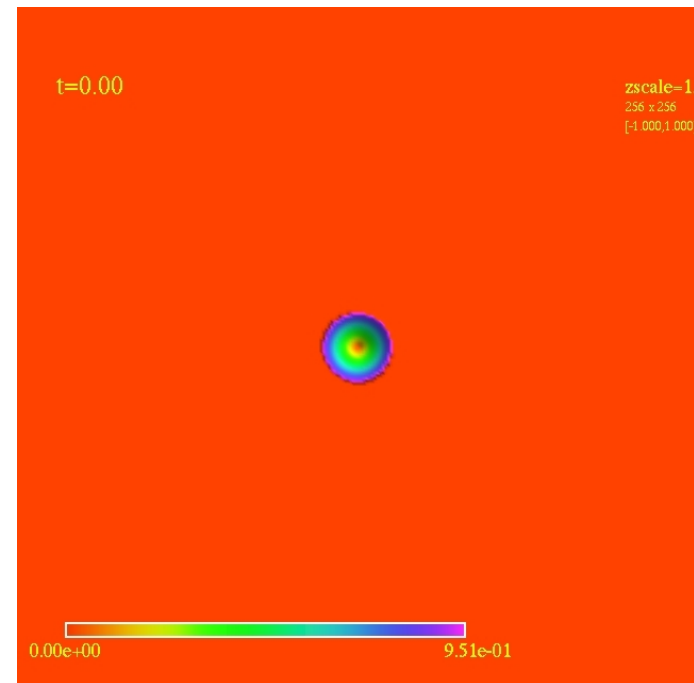
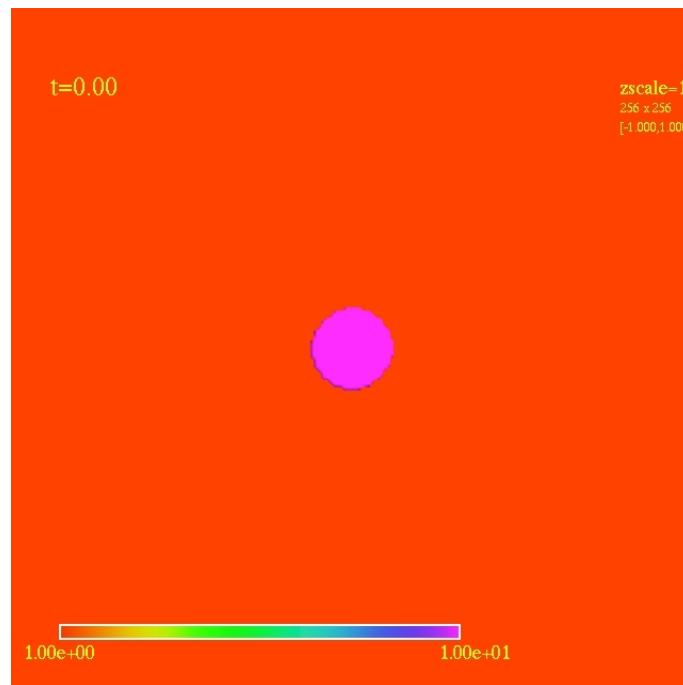
- Questions?

# More Preliminary Results - Minkowski spacetime



# Preliminary Results - Minkowski spacetime

- Now including the magnetic field





# Plasma

- Plasma Physics
  - A plasma is a highly ionized gas. When the plasma constituents are sufficiently collisional we use single fluid magnetohydrodynamics to describe the system.
  - If the conductance is sufficiently large we can neglect the resistive terms
  - If the system has isotropic pressure, we assume heat flow is small and can neglect the viscous terms.
- dimensionless plasma parameters
  - $\Lambda$  - the number of particles in the Debye sphere
  - $\beta$  - the ratio of the thermal pressure to the magnetic pressure
  - $\delta$  - the ratio of the rest mass energy to the magnetic energy

# The Solver

- The Roe Solver

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = \Sigma(q)$$

Cell average:

$$Q_i \approx \frac{1}{\Delta V} \int_{\Delta V} q dV$$

Integrate conservative equation over a small volume, and divide by that same volume

$$\frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial t} q dx + \frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial x} f(q) dV = \frac{1}{\Delta V} \int_{\Delta V} \Sigma(q) dV$$

# The Solver

which can be re-arranged, using Stokes' theorem

$$\frac{1}{\Delta V} \frac{\partial}{\partial t} \int_{\Delta V} q dx + \frac{1}{\Delta V} \int_{\partial V} f(q) dS = \frac{1}{\Delta V} \int_{\Delta V} \Sigma(q) dV$$

So now using our cell average, and the mean value theorem on the source term we get

$$\frac{\partial}{\partial t}(Q_i) + \frac{1}{\Delta V} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) = \Sigma(Q_i)$$

finite difference the time derivative

$$(Q_i)^{n+1} = (Q_i)^n - \frac{\Delta t}{\Delta V} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) + \Delta t \Sigma_i$$

# Flux Approximations

- Shock Capturing

We think of the discretization  $Q_i^n$  as being a piecewise constant reconstruction of the solution  $q(x)$ . Then at every cell boundary we have a Riemann problem (the discontinuity). To estimate the flux in the above equation we write the flux as  $f(q^*)$  where  $q^*$  is the solution at the cell boundary to the problem given by:

$$\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = 0$$

with the Jacobian  $A = \partial F / \partial q$  is constant.

For  $x = 0$ : If  $A > 0$  the flow is to the right, so  $q^* = q_L$ , this corresponds to  $f(q^*) = Aq_L$ . If  $A < 0$  the flow is to the left, so  $q^* = q_R$ , which corresponds to  $f(q^*) = Aq_R$ . Rather than using an if/then approach we use a general form:

$$f_{i+1/2} = f(q_{i+1/2}^*) = \frac{1}{2} (Aq_L + Aq_R - |A|(q_R - q_L))$$

So if  $v > 0$  then  $f = \frac{1}{2}(Aq_L + Aq_L) = Aq_L$