Numerical General Relativistic Magnetohydrodynamics

Group Meeting

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Outline

- Motivation
- Magnetohydrodynamics
- Waves
- Turbulence
- Current Project: Relativistic Numerical Magnetohydrodynamics

Motivation

- What is General Relativistic Magnetohydrodynamics?
  - The study of compressible, conducting fluids as they exist in curved spacetime.
  - Generally involve the study of three key components
    - Curved Spacetime
    - Hydrodynamic Stress–Energy Tensor
    - Electromagnetism Stress–Energy Tensor
  - Magnetohydrodynamic Stress–Energy Tensor
- Why study magnetohydrodynamics?
  - Magnetic fields are fundamental to many astrophysical phenomenon
  - Currently believed to be a model for the dynamics of an AGN, and Supernovae

Astrophysical Phenomenon

- Accretion Disks
  - Material that is caught in the gravitational field of a massive body
  - May come from a binary star
  - accretion phenomenon is thought to be much more efficient than nuclear fusion
- Relativistic Jets
  - A steady collimated beam of material emanating from a central body
  - Still not well understood
  - Mechanisms for collimation of the jet is still an open question
- Active Galactic Nuclei
  - This astrophysical phenomenon is said to be at the centre of an active galaxy, understanding this may lead to a better understanding of the origins of our own galaxy

Magnetohydrodynamic Dimensionless Numbers

- Hydrodynamic dimensionless numbers
  - Knudsen number - $\lambda/L \ll 1$
  - Reynolds number (Re) - VL/ν
  - Mach number (M) - V/c_s
- Where
  - λ - mean free path between particles
  - L - characteristic length scale
  - V - characteristic velocity scale
  - ν - fluid viscosity
  - c_s - speed of sound

Magnetohydrodynamic Dimensionless Numbers

- Additional dimensionless numbers when considering magnetic field contributions
  - Magnetic Reynolds number (R_M) - VL/η
  - beta (β) - $P_{thermal}/P_{magnetic}$
  - Alfven number (a) - V/v_A
  - δ - $E_{magnetic}/E_{rest}$
- where
  - v_A - the Alfvén velocity
  - η - the resistivity
- In the ideal MHD case, the viscosity ν, and the resistivity η are both zero
- The free parameters to observe are
  - β, M, a, and δ

General Relativistic Magnetohydrodynamics

- The Conservative Equations
  - General conservative form

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla f(\mathbf{q}) = \mathbf{\Sigma}(\mathbf{q}) \neq \mathbf{\Sigma}(\mathbf{q}, \nabla \mathbf{q})$$

  - This is a system of hyperbolic differential equations
  - Conservative variables $\mathbf{q}$
  - Finite volume techniques have been employed to solve these equations
  - Integral form - allows for shock capturing
  - Typically the conservative variables $(\mathbf{q})$ will be functions of what are known as primitive variables $(\mathbf{p})$, $\mathbf{q} \rightarrow \mathbf{q}(\mathbf{p})$
  - Primitive variable conversion is required to obtain the required physical variables, in GR there is no closed form solution to this inversion
  - Equation of State also requires primitive variables

Conservation Form

- To consider the evolution of a magnetohydrodynamic system we use the conservation equations for baryons, energy–momentum, and the magnetic induction equation, as the fluid flows through some volume of space

- Baryon: $\nabla_\mu (J^\mu) = 0$, $J^\mu = \rho_o u^\mu$
  - u^μ is the fluid's 4-velocity

- Energy–Momentum: $\nabla_\mu T^{\mu\nu} = 0$

- Induction Equation: $\nabla_\mu {}^*F^{\mu\nu} = 0$
  - ${}^*F^{\mu\nu}$ is the dual Faraday tensor

- To close the system of equations, we also need an equation of state

- Ideal Gas Equation of state
  - $P = (\Gamma - 1)\rho_o \epsilon$
    - P - Pressure
    - $\rho = \rho_o(1 + \epsilon)$ - total mass energy density
    - Γ - adiabatic constant
    - ϵ - specific energy
    - ρ_o - rest mass density
    - $h = 1 + \epsilon + \frac{P}{\rho}$ - specific enthalpy

Relativistic Magnetohydrodynamics

- The Relativistic MHD Equations in Conservative Form

$$
\begin{aligned}
\partial_t(\sqrt{-g}\rho_o u^t) + \partial_i(\sqrt{-g}\rho_o u^i) &= 0 \\
\partial_t(\sqrt{-g}T^t_\nu) + \partial_i(\sqrt{-g}T^i_\nu) &= \sqrt{-g}T^{\mu\kappa}\partial_\mu g_{\nu\kappa} - \sqrt{-g}T^\kappa_\lambda \Gamma^\lambda_{\nu\kappa} \\
\partial_t(\sqrt{-g}B^i) + \partial_j(\sqrt{-g}\left(u^j b^i - u^i b^j\right)) &= 0 \\
\partial_i(\sqrt{-g}B^i) &= 0
\end{aligned}
$$

where g is the determinant of the metric and

$$
\begin{aligned}
T^{\mu\nu} &= (\rho + P + b^2)u^\mu u^\nu + (P + \frac{1}{2}b^2)g^{\mu\nu} - b^\mu b^\nu \\
b^\mu &\equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}u_\nu F_{\lambda\kappa}
\end{aligned}
$$

- 3 + 1 formalism: slice spacetime into spacelike hypersurfaces;

ADM / 3+1 Formalism

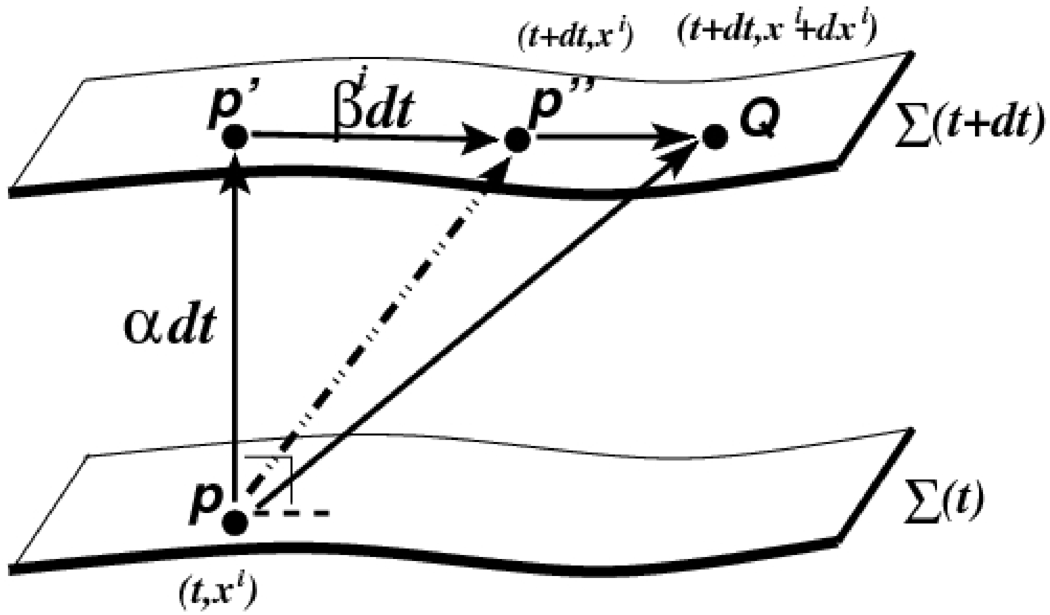


Figure 1: A schematic representation of the ADM space+time decomposition.

- Future directed, time unit normal to the hypersurfaces

$$n_\mu = -\alpha \nabla_\mu t$$

  where α is the lapse function

- Shift vector β^μ defined via

$$t^\mu = \alpha n^\mu + \beta^\mu$$

$$\beta^\mu n_\mu = 0$$

Relativistic Magnetohydrodynamics

- The General Relativistic MHD Equations Written in Conservation Form

Defining the following projections as conservation variables

$$
\begin{aligned}
D &= -n_\mu J^\mu \rightarrow \alpha \rho_o u^t \\
E &= T^{\mu\nu} n_\mu n_\nu \rightarrow \alpha^2 T^{00} \\
S_\lambda &= -T^{\mu\nu} n_\mu \gamma_{\nu\lambda} \rightarrow -\alpha T^{0\nu} \gamma_{\nu\lambda} \\
B_\lambda &= {}^*F^{\nu\mu} n_\nu \gamma_{\mu\lambda} \rightarrow -\alpha {}^*F^{0\mu} \gamma_{\mu\lambda}
\end{aligned}
$$

Where I used: $n_\nu = (-\alpha, 0, 0, 0)$

$\gamma_{\nu\mu}$ is the induced metric

Relativistic Magnetohydrodynamics

- The Conserved Relativistic MHD Equations

$$
\begin{aligned}
\partial_t\left(\sqrt{\gamma} D\right)+\partial_i\left(\sqrt{-g} D V^i\right) &= 0 \\
\partial_t(\sqrt{\gamma} S_j)+\partial_i\left(\sqrt{-g}\left(S_j V^i-b_j B^i / W+P_T \delta_j^i\right)\right) &= \frac{\sqrt{-g}}{2} T^{\mu\nu}\left(\partial_i g_{\mu\nu}-\Gamma_{\mu\nu}^\lambda g_{\lambda i}\right) \\
\partial_t(\sqrt{\gamma} E)+\partial_i\left(\sqrt{-g}\left(E V^i+P_T v^i-\alpha b^t B^i / W\right)\right) &= Q \\
\partial_t(\sqrt{\gamma} B^i)+\partial_j(\sqrt{-g}\left(V^i B^j-V^j B^i\right)) &= 0 \\
\partial_i(\sqrt{\gamma} B^i) &= 0
\end{aligned}
$$

With

$$
\begin{aligned}
Q &= -\alpha\sqrt{-g}\left(T^{\mu t}\partial_\mu(\ln\alpha)-T^{\mu\nu}\Gamma_{\mu\nu}^t\right) \\
b^t &= W B^i v_i/\alpha \\
b^i &= \frac{B^i}{W}+\alpha b^t v^i \\
V^i &= v^i-\beta^i/\alpha \\
P_T &= P+\frac{b^2}{2}
\end{aligned}
$$

Waves

- Characteristic velocities
- Luis Anton et al. (arxiv:astro-ph/0506063v1 June 2005)
  - *Numerical 3+1 GRMHD: A Local Characteristic Approach*
  - Found that one can write the EoM in the quasilinear form $A^{\nu A}_{B} \nabla_\nu V^B = 0$
  - $V^B = (u^\nu, b^\mu, P, s)$, $A, B = 0 \to 9$

$$A^\mu \quad = \quad \begin{pmatrix} C u^\mu \delta^\alpha_\beta & -b^\mu \delta^\alpha_\beta + P^{\alpha\mu} b_\beta & l^{\alpha\mu} & 0^{\alpha\mu} \\ b^\mu \delta^\alpha_\beta & -u^\mu \delta^\alpha_\beta & f^{\mu\alpha} & 0^{\alpha\mu} \\ \rho h \delta^\mu_\beta & 0^\mu_\beta & u^\mu / c_s^2 & 0^\mu \\ 0^\mu_\beta & 0^\mu_\beta & 0^\mu & u^\mu \end{pmatrix}$$

c_s^2 is the speed of sound

$$\begin{aligned} C &= \rho h + b^2 \\ P^{\alpha\mu} &= g^{\alpha\mu} + 2u^\alpha u^\mu \\ l^{\mu\alpha} &= \left(\rho h g^{\mu\alpha} + \left(\rho h - b^2/c_s^2\right) u^\mu u^\alpha\right) / \rho h \\ f^{\mu\alpha} &= \left(u^\alpha b^\mu / c_s^2 - u^\mu b^\alpha\right) / \rho h \end{aligned}$$

Waves Continued

- Projecting A^μ onto a $\phi(x^\mu) = 0$ hypersurface and taking the determinant we have a secular equation

$$\det\left(A^\mu \phi_\mu\right) = Ca^2 \mathcal{A}^2 \mathcal{N}_4 \quad = \quad 0$$

$$\begin{aligned}
a &= u^\mu \phi_\mu \\
\mathcal{B} &= b^\mu \phi_\mu \\
\mathcal{A} &= Ca^2 - \mathcal{B} \\
\mathcal{N}_4 &= \rho h \left(\frac{1}{c_s^2 - 1}\right) a^4 - \left(\rho h + \frac{b^2}{c_s^2}\right) a^2 G + \mathcal{B}^2 G \\
G &= \phi^\mu \phi_\mu
\end{aligned}$$

- Consider a wave propagating in an arbitrary direction x with speed λ normal to the characteristic hypersurface $\phi(x^\mu) = 0$

- $\phi_\mu = (-\lambda, 1, 0, 0)$

- using this in our secular equation we can look at the individual terms

Waves Continued

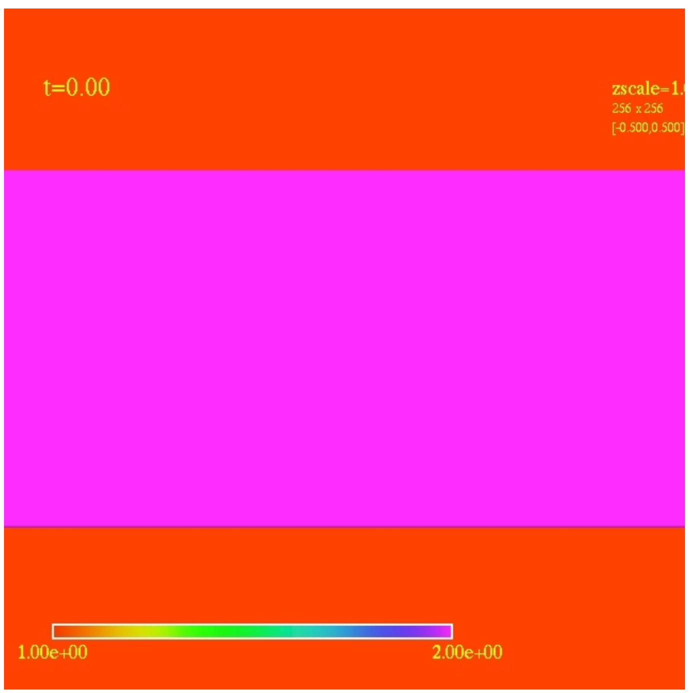
- $a = 0$
- leads to $\lambda = \alpha v^x - \beta^x$
- $\mathcal{A} = 0$
- leads to the Alfven waves
- $\lambda = \frac{b^x \pm \sqrt{C} u^x}{b^t \pm \sqrt{C} u^t}$
- $\mathcal{N}_4 = 0$
- leads to the fast and slow magnetosonic waves
- There is no convenient closed form solution for this equation. Must be solved numerically

Turbulence & Instabilities

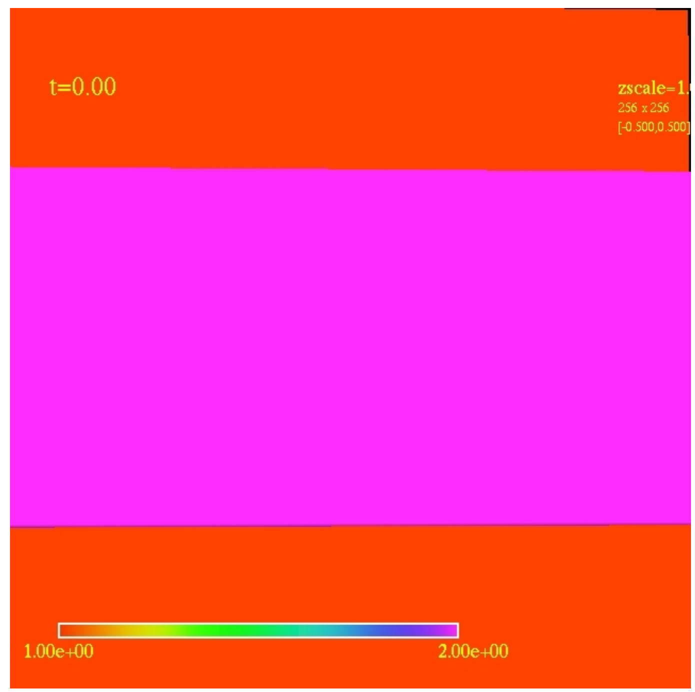
- Instabilities and Turbulence
  - Instabilities arise when the system undergoes a small perturbation and the dispersion relation leads to purely imaginary frequencies, thus the amplitudes either expand uncontrollably or dampen.
  - Turbulence arises when energy differences in the system undergo a transition from one energy type to another. Typically this energy will cascade from one length scale to the next, until viscous terms become non-negligible
  - There are a few types of turbulence that are of immediate concern in (magneto)hydrodynamic systems
    - Kelvin Helmholtz Instability
    - Magnetorotational Instability

Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented the Kelvin-Helmholtz instability
- The Kelvin-Helmholtz instability comes from the over abundance of kinetic energy in a localized region. This physically comes from differential rotation which is expected to be found in accretion disks
- There is slipping between the two layers of the fluid. Relative velocity between the two streams is 0.8c.



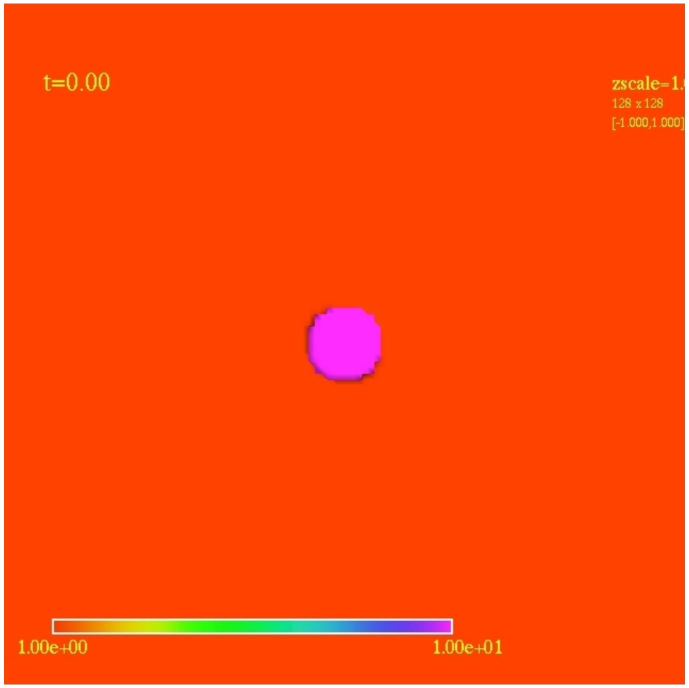
KHI No Magnetic field.



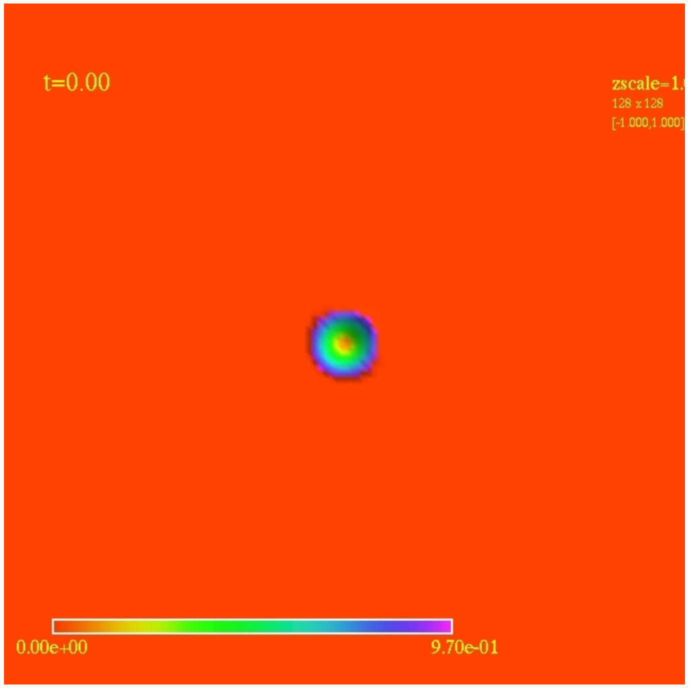
KHI With Magnetic Field, $B_x = 0.5$

Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented a rotor, edge angular velocity $\Omega = 9.75$
- Pressure $=$ 1.0
- Density $=$ 10 inside the disk, 1 outside of the disk



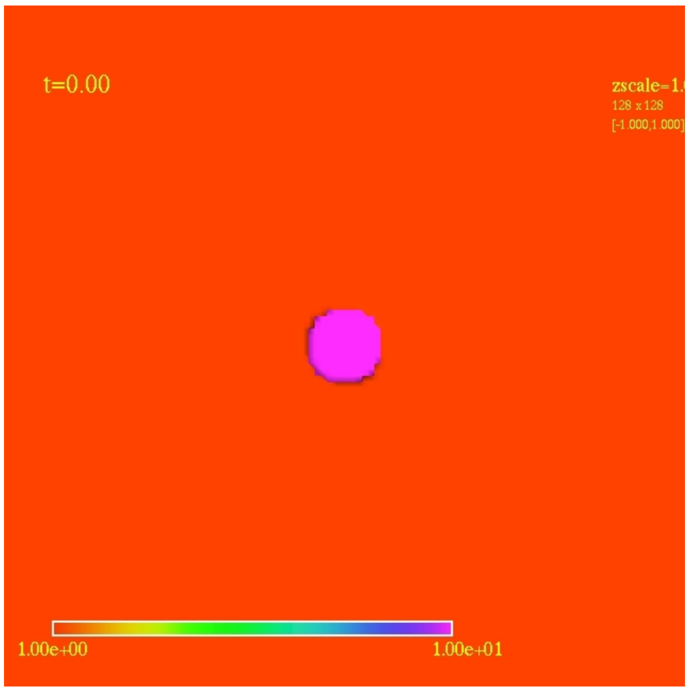
Density



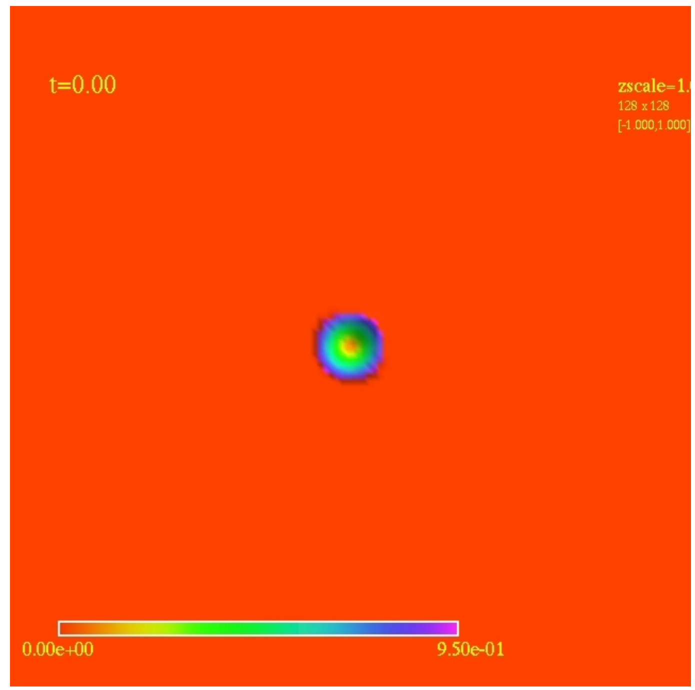
Speed

Preliminary Results - Minkowski spacetime

- Now including the magnetic field, $B_x = 0.5$
- Pressure $= 1.0$
- Density $= 10$ inside the disk, 1 outside of the disk



Density



Speed

Future work

- Magnetorotational Instability
  - Rotational energy is converted to magnetic energy
  - Thought to underlie viscous dissipation when matter falls onto the compact object from a companion
  - Infalling matter has considerable angular momentum
  - This forms the rotating disk around the object, with differential rotation
  - Inner regions lose angular momentum due to viscosity and fall inward, outer regions gain angular momentum and and move outward
  - infalling matter perturbs the outer regions preventing the outflow

Representative Recent Work

- L.Del Zanna, O. Zanotti, N. Bucciantini, P Londrillo,(2007) ESO
  - *ECHO: and Eulerian Conservative High Order scheme for general relativistic magnetohydrodynamics and magnetodynamics*
  - Developed a GRMHD code to use stationary metrics
  - Created several test suits for GRMHD for Schwarzschild and Kerr spacetimes
- JWS. Blokland, R. Keppens, JP. Goedbloed astro-ph/0703581v1 Astronomy & Astrophysics (2007)
  - *Unstable magnetohydrodynamical continuous spectrum of accretion disks*
  - Study localized MHD instabilities in 2D MHD accretion disks
  - Find localized MHD instabilities provide an ideal linear route to MHD turbulence in strong B fields.

Current Project - Numerical Magnetohydrodynamics

- Motivation
  - Existing theories and codes appear to work well, however they all rely on a ideal equation of state, other equations of state may lead to results that come closer to the observed phenomenon.
  - To study the turbulence in accretion disks
  - To study "puffy" disks, or accretion shells
  - Ultimately, to better understand the complex dynamics of accretion disks and jets around compact object such as black holes

Current Project - Numerical Magnetohydrodynamics

- Questions to consider
  - Does the KHI have any impact on the formation of jets?
  - How does a more realistic EoS impact the turbulence in MRI?
  - Will this realistic EoS allow for angular momentum transport in the accretion disk?
  - Does super-Eddington accretion take place in the region between the accretion disk and the black hole event horizon

What Has Been Done

- Milestones
  - 1D special relativistic hydrodynamics
  - Cartesian 2D Newtonian hydrodynamic codes, along with consistency/convergence tests
  - Cartesian 2D Newtonian MHD, with proper magnetic field treatment
  - Completed 2D Cartesian SRMHD with magnetic flux corrections
  - Completed Polar SRMHD with magnetic flux corrections
  - Completed Schwarzschild and Kerr metric implementations (generalized)
  - Run several tests for the Newtonian code and the SRMHD cases, including KHI, the blast waves, rigid rotor
  - Parallelized the code
  - Developed subroutines to perform tensor manipulations numerically
  - Found and implemented methods to calulcate relativistic vorticity
  - Implemented an interpolation method to allow for convergence tests

The Plan - Numerical Magnetohydrodynamics

- Phases of the project
  - The final goal is to run detailed 2D simulations of accretion disks around massive bodies
  - To achieve this several small projects must be completed;
    - Next 4-6 months?(Reality be damned)?
      - Using above code, attempt to model real astrophysical data on the accretion disks
      - Perform a parameter survey of the different plasma parameters
      - Observe turbulent behaviour of the accretion disk
  - Afterwards
    - Modify the code for use with FLAMR infrastructure that is being constructed by Martin Snajdr and Frans Pretorius
    - Start investigating 3D relativistic accretion disks with relativistic jets

The Plan - Numerical Magnetohydrodynamics

- Full 3D Code

- Next 8-12 months

  - Super Eddington Luminosity
    - Open question about goings on between the accretion disk and the black hole horizon
  - Jets
    - There are also open questions about the formation of the relativistic jets
  - Turbulence
    - Attempt to monitor turbulence and instabilities in full 3D evolutions
    - Implement a dynamic sapcetime background code

Future considerations

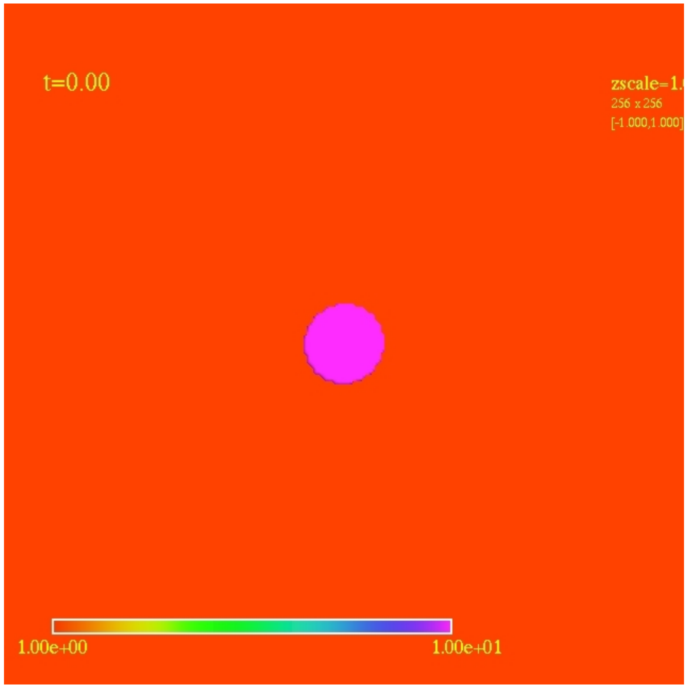
- For real accretion disks I will have to attempt to setup the initial conditions as seen from astronomical data

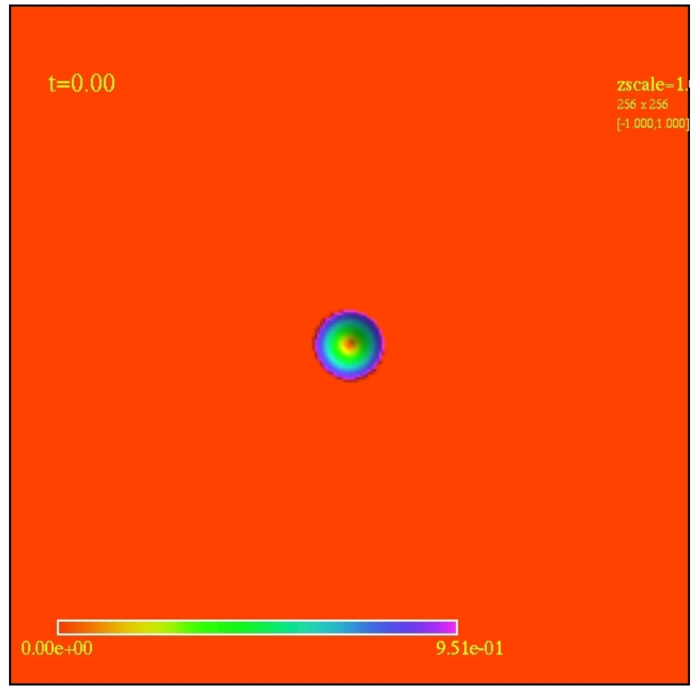
- For these systems we expect the following:
  - $Re > 10^4$
  - $R_M \sim 10^{10}$ (Solar Corona)
  - $\rho_o \sim 10^{-5} - 10^{32} cm^{-3}$
  - $T \sim 10 - 10^9 K$
  - $|B| \sim 10^{-12} - 10^{20} G$
  - $\beta \sim 10^{-5} - 10^{20}$

C'est Tout

- Pergunta?
- Bonpac?
- Fragen?
- Domande?
- Vragen?
- Questions?

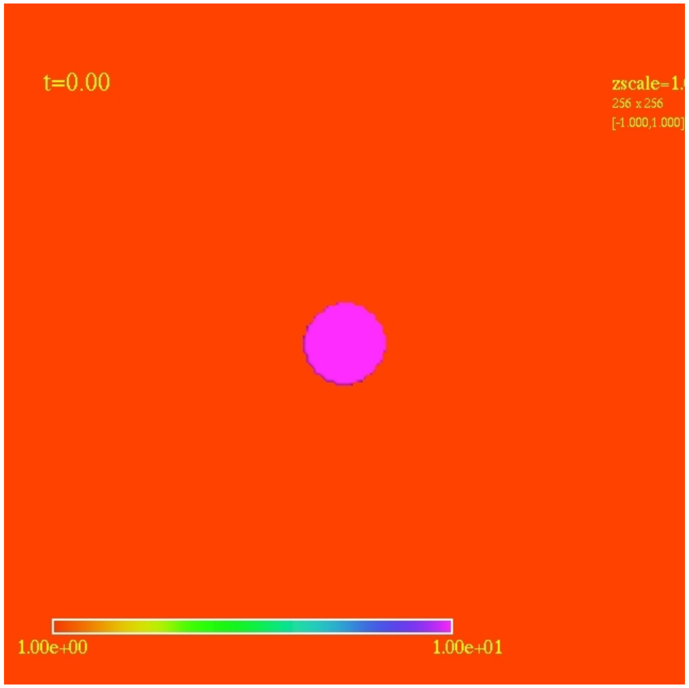
More Preliminary Results - Minkowski spacetime

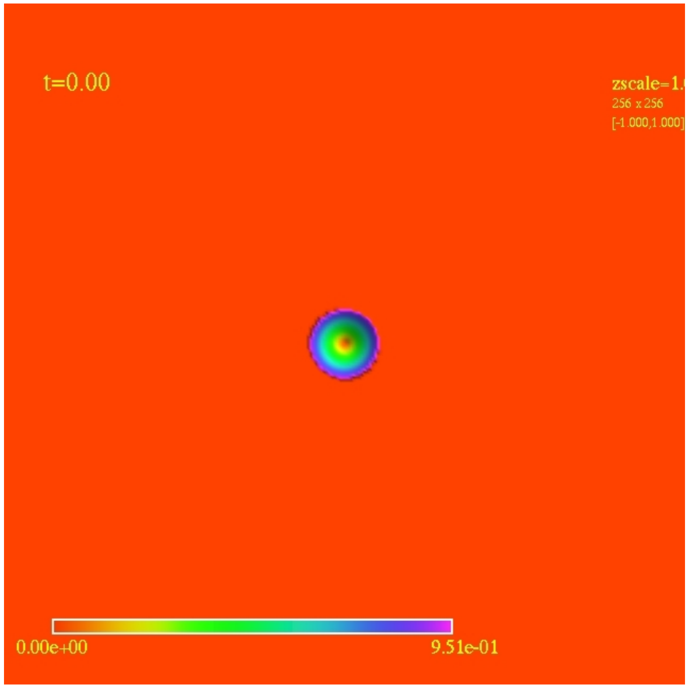




Preliminary Results - Minkowski spacetime

- Now including the magnetic field





Plasma

- Plasma Physics
  - A plasma is a highly ionized gas. When the plasma constituents are sufficiently collisional we use single fluid magnetohydrodynamics to describe the system.
  - If the conductance is sufficiently large we can neglect the resistive terms
  - If the system has isotropic pressure, we assume heat flow is small and can neglect the viscous terms.

- dimensionless plasma parameters
  - Λ - the number of particles in the Debye sphere
  - β - the ratio of the thermal pressure to the magnetic pressure
  - δ - the ratio of the rest mass energy to the magnetic energy

The Solver

- The Roe Solver

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = \Sigma(q)$$

Cell average:

$$Q_i \approx \frac{1}{\Delta V}\int_{\Delta V} q dV$$

Integrate conservative equation over a small volume, and divide by that same volume

$$\frac{1}{\Delta V}\int_{\Delta V}\frac{\partial}{\partial t} q dx + \frac{1}{\Delta V}\int_{\Delta V}\frac{\partial}{\partial x} f(q) dV = \frac{1}{\Delta V}\int_{\Delta V}\Sigma(q) dV$$

The Solver

which can be re-arranged, using Stokes' theorem

$$\frac{1}{\Delta V}\frac{\partial}{\partial t}\int_{\Delta V} q dx + \frac{1}{\Delta V}\int_{\partial V} f(q) dS = \frac{1}{\Delta V}\int_{\Delta V}\Sigma(q) dV$$

So now using our cell average, and the mean value theorem on the source term we get

$$\frac{\partial}{\partial t}(Q_i) + \frac{1}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) = \Sigma(Q_i)$$

finite difference the time derivative

$$(Q_i)^{n+1} = (Q_i)^n - \frac{\Delta t}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) + \Delta t \Sigma_i$$

Flux Approximations

- Shock Capturing

We think of the discretization Q_i^n as being a piecewise constant reconstruction of the solution q(x). Then at every cell boundary we have a Riemann problem (the discontinuity). To estimate the flux in the above equation we write the flux as $f(q^*)$ where q^* is the solution at the cell boundary to the problem given by:

$$\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} = 0$$

with the Jacobian $A = \partial F/\partial q$ is constant.

For $x = 0$: If $A > 0$ the flow is to the right, so $q^* = q_L$, this corresponds to $f(q^*) = Aq_L$. If $A < 0$ the flow is to the right, so $q^* = q_R$, which corresponds to $f(q^*) = Aq_R$. Rather than using an if/then approach we use a general form:

$$f_{i+1/2} = f(q^*_{i+1/2}) = \frac{1}{2}\left(Aq_L + Aq_R - |A|(q_R - q_L)\right)$$

So if $v > 0$ then $f = \frac{1}{2}(Aq_L + Aq_L) = Aq_L$