Numerical General Relativistic Magnetohydrodynamics

1st PhD Committee Meeting

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December 6, 2005

Courses

- 2002:
  - 16.742 - Quantum Mechanics - > 90 A+ - 6 crd
  - 16.754 - Statistical Mechanics - > 75 B+ - 3 crd
  - 16.462 - Adv. Classical Mech. - > 90 A+ - 3 crd
- 2003
  - 16.759 - Electromagnetism - > 90 A+ - 3 crd
- 2004F - Term 1:
  - Phys555/410 - Computational Phys - 92 A+ - 3 crd
  - Phys526 - Quantum Electrodynamics - 94 A+ - 3 crd
- 2004W - Term 2:
  - Phys555 - Numerical Relativity - 94 A+ - 3 crd
  - Phys508 - Quantum Field Theory - 88 A - 3 crd
- 2005W - Term 2:
  - Phys555 - Meth. in Par. Comp. and Particle Mesh Techniques - 3 crd
  - Phys530 - Graduate General Relativity - 3 crd

Outline

- Motivation
- Magnetohydrodynamics
- Curved spacetime background
- Hydrodynamics
- Electromagnetism
- ADM / 3+1 Formalism
- Previous work: a short overview
- Current Project: Numerical Magnetohydrodynamics

Motivation

- Why study magnetohydrodynamics?
  - Plasma simulation is computationally too expensive for today's computers
  - MHD is a single fluid approximation to plasma theories
  - Magnetic fields are measurable, and been observed in several astrophysical phenomenon
  - Newtonian MHD can be used to model our sun
  - Currently believed to be a model for the dynamics of an AGN, and Supernovae and more...

Motivation

- Why Study Active Galactic Nuclei?
  - This astrophysical phenomenon is said to be at the centre of an active galaxy, understanding this may lead to a better understanding of the origins of our own galaxy
  - accretion phenomenon is thought to be much more efficient than nuclear fusion
  - The accretion disk becomes plasma by frictional forces
- Why Study Supernovae?
  - Thought to be the source of higher mass elements that allowed for life on earth
  - This phenomenon is still poorly understood due to computational difficulties involving shocks

Both of the are observable objects in space

Magnetohydrodynamics

- This theory is an approximation to full plasma physics
- Considers a perfectly conducting fluid
- Ideal MHD assumes the magnetic field is frozen in to the fluid $E_\mu = 0$
- Can be broken into three parts
  - Curved spacetime background
  - Hydrodynamics
  - Electromagnetism

Curved Spacetime Background

- For accretion disks, I will assume a static spacetime background
- First I will study Minkowski spacetime
- Ultimately I will study MHD in Kerr spacetime

Hydrodynamics

- The structure of the hydrodynamic equations is well understood
- The basic idea behind them is the conservation of baryons, energy, and momentum as it flows through some volume of space
- The stress–energy tensor for an ideal fluid
  - $T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}$
- Conservation of stress–energy $\nabla_\mu T^{\mu\nu} = 0$
- Also must consider the baryonic conservation equation
  - $\nabla_\mu(\rho_o u^\mu) = 0$
- Equation of state
  - $P = (\Gamma - 1)\rho_o \epsilon$

Electromagnetism

- The Faraday tensor, and Maxwell's equations are well known

- The Faraday tensor can be utilized to produce the Electromagnetic stress–energy tensor
  - $T^{\mu\nu} = F^{\mu\alpha}F_{\alpha}^{\nu} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$

- Conservation of stress–energy $\nabla_{\mu}T^{\mu\nu} = 0$

- We must also consider the Maxwell equations directly $\nabla_{\mu}^{*}F^{\mu\nu} = 0$

- This leads us to the induction equation, when considering a perfect conductor, as well as the no-monopole constraint
  - $\partial_t(\sqrt{-g}B^j) + \partial_i(\sqrt{-g}(u^i b^j - b^i u^j)) = 0$
  - $\partial_i(\sqrt{-g}(B^i)) = 0$
  - $b^{\mu} \equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}u_{\nu}F_{\lambda\kappa}$

Relativistic Magnetohydrodynamics

- The Relativistic MHD Equations

$$
\begin{aligned}
\partial_t(\sqrt{-g}\rho u^t) + \partial_i(\sqrt{-g}\rho u^i) &= 0 \\
\partial_t(\sqrt{-g}T^t_\nu) + \partial_i(\sqrt{-g}T^i_\nu) - \sqrt{-g}T^\kappa_\lambda \Gamma^\lambda_{\nu\kappa} &= 0 \\
\partial_t(\sqrt{-g}B^i) + \partial_j(\sqrt{-g}\left(u^j b^i - u^i b^j\right)) &= 0 \\
\partial_i(\sqrt{-g}B^i) &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
T^{\mu\nu} &= (\rho + P + b^2)u^\mu u^\nu + (P + \frac{1}{2}b^2)g^{\mu\nu} - b^\mu b^\nu \\
b^\mu &\equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}u_\nu F_{\lambda\kappa}
\end{aligned}
$$

- Its tensorial nature gives rise to several different formalisms
- 3 + 1 formalism: slice spacetime in spacelike hypersurfaces; use Einstein equations to evolve in time the 3-geometry of an initial hypersurface in order to construct the spacetime (i.e. the 4-dimensional metric, $g_{\mu\nu}$)

ADM / 3+1 Formalism

- ADM formalism is a way of writing Einstein equations as an initial value problem

- Manifold with metric $(M, g_{\mu\nu})$ foliated by spacelike hypersurfaces Σ_t

- Coordinates $x^\mu = (t, x^i)$

- Future directed, time unit normal to the hypersurfaces

$$n_\mu = -\alpha \nabla_\mu t$$

  where α is the lapse function

- Shift vector β^μ defined via

$$t^\mu = \alpha n^\mu + \beta^\mu$$
$$\beta^\mu n_\mu = 0$$

ADM / 3+1 Formalism

- Define the projection operator, projects 4-space to hypersurface

$$\perp^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu} n_{\nu}$$

- Hypersurface metric $\gamma_{\mu\nu}$ induced by $g_{\mu\nu}$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_{\mu} n_{\nu}$$

ADM / 3+1 Formalism

- 3+1 line element

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

- Extrinsic curvature

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$$

- 3+1 form of Einstein's equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ derived by considering various projections of Einstein/Ricci and stress–energy tensors

- Projections of $T^{\mu\nu}$

$$\begin{aligned} E &\equiv n_\mu n_\nu T^{\mu\nu} \\ S_\mu &\equiv \gamma_{\alpha\mu} n_\beta T^{\alpha\beta} \\ Q_{\mu\nu} &\equiv \gamma_{\alpha\mu}\gamma_{\beta\nu} T^{\alpha\beta} \end{aligned}$$

Relativistic Magnetohydrodynamics

- The Conserved Relativistic MHD Equations

  Using the following as conserved quantities

$$\begin{aligned} E &= T^{\mu\nu} n_\mu n_\nu \\ S_\lambda &= T^{\mu\nu} n_\mu \gamma_{\nu\lambda} \end{aligned}$$

  and the projected stress–energy conservation law

$$\begin{aligned} \nabla_\mu \left(n_\nu T^{\mu\nu} \right) &= 0 \\ \nabla_\mu \left(\gamma_{\nu\lambda} T^{\mu\nu} \right) &= 0 \end{aligned}$$

  with

$$T^{\mu\nu} = (\rho + P + b^2) u^\mu u^\nu + (P + \frac{1}{2} b^2) g^{\mu\nu} - b^\mu b^\nu$$

Relativistic Magnetohydrodynamics

- The Conserved Relativistic MHD Equations

Finally we have

$$\partial_t\left(\sqrt{-g}D\right) + \partial_i\left(\sqrt{-g}D\left(v^i - \frac{\beta^i}{\alpha}\right)\right) = 0$$

$$\partial_t(\sqrt{-g}S_j) + \partial_i\left(\sqrt{-g}\frac{S_j}{\alpha} - b^i b^j + b^t b^i\right) = \frac{\sqrt{-g}}{2}T^{\mu\nu}\left(\partial_i g_{\mu\nu} - \Gamma^{\lambda}_{\mu\nu}g_{\lambda i}\right)$$

$$\partial_t(\sqrt{-g}E) + \partial_i\left(\sqrt{-g}\left(E\left(v^i - \frac{\beta^i}{\alpha}\right) + \left(P + \frac{b^2}{2}\right)v^i + \left(\alpha b^t\right)^2 - b^t b^i \alpha\right)\right) =$$
$$-\alpha\left(T^{\mu t}\partial_\mu(\ln\alpha) - T^{\mu\nu}\Gamma^t_{\mu\nu}\right)$$

$$\partial_t(\sqrt{-g}B^i) + \partial_j(\sqrt{-g}\left(u^j b^i - u^i b^j\right)) = 0$$

$$\partial_i(\sqrt{-g}B^i) = 0$$

With

$$T^{\mu\nu} = (\rho + P + b^2)u^\mu u^\nu + (P + \frac{1}{2}b^2)g^{\mu\nu} - b^\mu b^\nu$$

Relativistic Magnetohydrodynamics

- The Conserved Relativistic MHD Equations
  - General conservative form

$$\frac{\partial q}{\partial t} + \nabla f(q) = \Sigma(q)$$

  - This is a system of hyperbolic differential equations
  - Conservative variables $q = \sqrt{-g}\,\{D, S_j, E, B_i\}$
  - Finite volume techniques will be employed to solve these equations
  - Integral form - allows for shock capturing
  - Primitive variable conversion will be required to obtain the required physical variables
  - Primitive variables $p = \{\rho_o, v_j, P, B_i\}$

Previous Work

- C. Gammie, J. McKinney, G. Tóth, (2003) ApJ, 589, 444
  - *HARM: A Numerical Scheme for General Relativistic Magnetohydrodynamics*
  - They study a conservative HRSC method for GRMHD
  - They implement a variation of the contrained transport technique developed by Evans, and Hawley in 1983.
  - This technique is necessary for maintenance of the constraint $\nabla \cdot \vec{B} = 0$
- S. Noble, C. Gammie, J. McKinney, (2005) submitted ApJ
  - *Primitive Variable Solvers For Conservative General Relativistic Magnetohydrodynamics*
  - Study optimization techniques for solving the general relativistic magnetohydrodynamic codes using finite volume techniques
  - They study various for converting the conservative variables to primitive variables

Current Project - Numerical Magnetohydrodynamics

- Motivation
  - Existing theories and codes appear to work well, however they are too idealistic
  - To study more realistic systems one will have to remove the idealizations
    - Viscous terms in hydrodynamic contributions
    - Dissipative terms in electromagnetic contributions
  - More realistic systems also consist of more than one fluid type
  - To better understand the complex dynamics of accretion disks and jets around compact object such as black holes

What Has Been Done Thus Far

- September 03 2004 - Arrived at UBC
  - Completed my BSc in physics in 2002
  - Completed my MSc in nonlinear physics in 2004
  - Completed required coursework in 2004-2005
  - To date I have completed the 1D special relativistic hydrodynamics
  - I have completed the Cartesian 2D Newtonian hydrodynamic codes, along with consistency/convergence tests
  - I have initiated the Newtonian MHD programming, need to implement proper magnetic field initial conditions
  - I have become comfortable with the use of the necessary diagnostic programs needed to further this research
    - This includes XVS the 1D visualization program, with all the necessary diagnostic functions
    - DV the 1, 2, and 3D visualization program written by Frans Pretorius which also contains essential diagnostic functions
    - I have also learned the finite volume PDE solver technique, along with several finite difference approximations

The Plan - Numerical Magnetohydrodynamics

- Phases of the project
  - The final goal is to run detailed 2D simulations of active galactic nuclei with relativistic jets
  - To achieve this several small projects must be completed;
    - Next month
      - Finalize 2D Newtonian magnetohydrodynamics, ensure all constraints are being handled properly
    - Next 4-6 months
      - Parallelize the code
      - Implement a two particle MHD code, thus extending existing algorithms for MHD to the plasma limit
      - Code the relativistic magnetohydrodynamics in Minkowski spacetime
      - Implement Kerr Spacetime
  - Afterwards
    - Modify the code for use of parallel adaptive infrastructure that is being constructed by Martin Snajdr et al.
    - Start investigating relativistic accretion disks with relativistic jets

Committee Contribution

- What I request from you
  - Dr. Heyl - Expert on astrophysical phenomenon, I hope to gain great insight on the physics behind such phenomenon as active galactic nuclei, jets, and supernovae.
  - Dr. Unruh - Expertise on black holes, and relativistic phenomenon, I hope that his contributions to my project will aid in my understanding of different spacetime geometries that are involved with active galactic nuclei and supernovae.
  - Dr. Jones - Expert in experimental physics - I hope to stay grounded in terms of physical limitations of MHD and it's application to plasma physics. I would like to have an evolution that is more than just a toy model.
  - Dr. Choptuik - Expert on the topic of numerical relativity, of course I hope to gain a lot of knowledge in numerical techniques for solving these problems both effectively and efficiently.

Motivation

- Development of a computational infrastructure for 2D and 3D codes
  - 2D & 3D numerical relativistic calculations are computationally very expensive
    - Any calculation done using a 3D uniform finite difference method scales as N^4, where N is the number of grid points along one dimension
    - Moore's law asserts that processing speed is roughly doubled every 1.5 yr
    - Then in order to gain in resolution by a factor of 2 for instance we need about 3 years of processor development
  - Need for more efficient computational techniques
    - Parallelization - Share the numerical task between many processors
    - Adaptive Mesh Refinement (AMR) allows resolution to vary locally in response to solution features
    - This infrastructure is being constructed by Frans Pretorius and Martin Snajdr

Relativistic Magnetohydrodynamics

- The Conserved Relativistic MHD Equations
  - we get the conserved quantities

$$
\begin{aligned}
D &= \rho_o W \\
E &= (\rho + P + b^2)W^2 - (P + \frac{1}{2}b^2) - (\alpha b^t)^2 \\
S_i &= (\rho + P + b^2)W(v_i - \frac{\beta_i}{\alpha}) - (\alpha b^t b_i) \\
B_i &
\end{aligned}
$$

Conservative Variables

- D - rest mass density
- E - energy
- S_i - momentum
- W - Lorentz factor
- ρ - energy density
- P - pressure

Current Project Numerical Magnetohydrodynamics

- Questions to be addressed
  - Can a more realistic equation of state be successfully evolved?
  - Can we learn more about AGN's using less ideal fluids?
  - Will the less ideal MHD equations allow us to better understand observed behaviour of active galactic nuclei?

Current Project Numerical Magnetohydrodynamics

- The Roe Solver

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$

Cell average:

$$Q_i \approx \frac{1}{\Delta V} \int_{\Delta V} q dV$$

Integrate conservative equation over a small volume

$$\int_{\Delta V} \frac{\partial}{\partial t} q dx + \int_{\Delta V} \frac{\partial}{\partial x} f(q) dV = 0$$

which can be re-arranged, using Stokes' theorem

$$\frac{\partial}{\partial t}\int_{\Delta V} q dx + \int_{\partial V} f(q) dS = 0$$

So now using our cell average, we get

$$\frac{\partial}{\partial t}(Q_i) + \frac{1}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) = 0$$

finite difference the time derivative

$$(Q_i)^{n+1} = (Q_i)^n - \frac{\Delta t}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) + S_i$$

Where S_i is a source term

Current Project Numerical Magnetohydrodynamics

- Shock Capturing We think of the discreteization Q_i^n as being a piecewise constant reconstruction of the solution q(x). Then at every cell boundary we have a Riemann problem (the discontinuity). To estimate the flux in the above equation we write the flux as $f(q^*)$ where q^* is the solution at the cell boundary to the problem given by:

$$\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} = 0$$

with the Jacobian $A = \partial F/\partial q$ is constant.

For $x = 0$: If $A > 0$ the flow is to the right, so $q^* = q_L$, this corresponds to $f(q^*) = Aq_L$. If $A < 0$ the flow is to the right, so $q^* = q_R$, which corresponds to $f(q^*) = Aq_R$. Rather than using an if/then approach we use a general form:

$$f_{i+1/2} = f(q^*_{i+1/2}) = \frac{1}{2}\left(Aq_L + Aq_R - |A|(q_R - q_L)\right)$$

So if $v > 0$ then $f = \frac{1}{2}(Aq_L + Aq_L) = Aq_L$