Numerical General Relativistic Magnetohydrodynamics

Astronomy Theory Meeting

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March 26, 2008

Outline

- Motivation
- Plasma
- Astrophysical Phenomenon
- Magnetohydrodynamics
- Solver
- Metric
- Waves
- Turbulence
- Current Project

Motivation

- What is General Relativistic Magnetohydrodynamics?
  - The study of compressible, conducting fluids as they exist in curved spacetime.
  - Generally involve the study of three key components
    - Curved Spacetime
    - Hydrodynamic Stress–Energy Tensor
    - Electromagnetism Stress–Energy Tensor
  - Magnetohydrodynamic Stress–Energy Tensor
- Why is magnetohydrodynamics considered important?
  - Magnetic fields are fundamental to many astrophysical phenomenon
  - Currently believed to be a model for the dynamics of an AGN, and Supernovae

Plasma

- Plasma Physics
  - A plasma is a highly ionized gas. When the plasma constituents are sufficiently collisional we use single fluid magnetohydrodynamics to describe the system.
  - If the conductance is sufficiently large we can neglect the resistive terms, thus have an ideal conductor
  - If the system has isotropic pressure, we assume heat flow is small and can neglect the viscous terms, thus have an ideal fluid

- dimensionless plasma parameters
  - Λ - the number of particles in the Debye sphere
  - β - the ratio of the thermal pressure to the magnetic pressure
  - δ - the ratio of the rest mass energy to the magnetic energy

Astrophysical Phenomenon

- Accretion Disks
  - Material that is caught in the gravitational field of a massive body
  - May come from a binary star
  - accretion phenomenon is thought to be much more efficient than nuclear fusion
- Relativistic Jets
  - A steady collimated beam of material emanating from a central body
  - Still not well understood
  - Mechanisms for collimation of the jet is still an open question
- Active Galactic Nuclei
  - This astrophysical phenomenon is said to be at the centre of an active galaxy, understanding this may lead to a better understanding of the origins of our own galaxy
- Angular Momentum Transport
  - There is still ongoing debate as to the mechanism that allows for angular momentum transport within accretion disks.

Magnetohydrodynamic Dimensionless Numbers

- Hydrodynamic dimensionless numbers
  - Knudsen number - $\lambda/L \ll 1$
  - Reynolds number (Re) - VL/ν
  - Mach number (M) - V/c_s
- Where
  - λ - mean free path between particles
  - L - characteristic length scale $\sim R_{EventHorizon}$
  - V - characteristic velocity scale $\sim c$
  - ν - fluid viscosity $\rightarrow 0$
  - c_s - speed of sound $\sim \sqrt{\frac{P}{\rho_o}}$

Magnetohydrodynamic Dimensionless Numbers

- Additional dimensionless numbers when considering magnetic field contributions
  - Magnetic Reynolds number (R_M) - VL/η
  - beta (β) - $P_{thermal}/P_{magnetic}$
  - Alfven number (a) - V/v_A
  - Ratio of energy densities (δ) - E_B/E_{rest}
- where
  - v_A - the Alfvén velocity $= B/\sqrt{\rho_o}$
  - η - the resistivity $\rightarrow 0$
  - $E_B \sim |B|^2$
  - $E_{rest} \sim \rho_o$
- The free parameters to survey are
  - β, M, a, and δ

General Relativistic Magnetohydrodynamics

- The Conservative Equations
  - General conservative form

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla f(\mathbf{q}) = \mathbf{\Sigma}(\mathbf{q}) \neq \mathbf{\Sigma}(\mathbf{q}, \nabla \mathbf{q})$$

  - This is a system of hyperbolic differential equations
  - Conservative variables $\mathbf{q}$
  - Finite volume techniques have been employed to solve these equations
  - Integral form - allows for shock capturing
  - Typically the conservative variables $(\mathbf{q})$ will be functions of what are known as primitive variables $(\mathbf{p})$, $\mathbf{q} \rightarrow \mathbf{q}(\mathbf{p})$
  - Primitive variable conversion is required to obtain the required physical variables, in GR there is no closed form solution to this inversion
  - Equation of State also requires primitive variables

Conservation Form

- To consider the evolution of a magnetohydrodynamic system we use the conservation equations for baryons, energy–momentum, and the magnetic induction equation, as the fluid flows through some volume of space

- Baryon: $\nabla_\mu (J^\mu) = 0$, $J^\mu = \rho_o u^\mu$
  - u^μ is the fluid's 4-velocity
  - This is an expression for the conservation of matter

- Energy–Momentum: $\nabla_\mu T^{\mu\nu} = 0$
  - Conservation of Mass
  - Conservation of Momentum

- Induction Equation: $\nabla_\mu {}^*F^{\mu\nu} = 0$
  - ${}^*F^{\mu\nu}$ is the dual Faraday tensor
  - Using differential geometry terminology, just to say we use Maxwell's relations

Equation of State

- To close the system of equations, we also need an equation of state
- Ideal Gas Equation of state
  - $P = (\Gamma - 1)\rho_o \epsilon$
    - P - Pressure
    - $\rho = \rho_o(1+\epsilon)$ - total mass energy density
    - Γ - adiabatic constant
    - ϵ - specific energy
    - ρ_o - rest mass density
    - $h = 1 + \epsilon + \frac{P}{\rho}$ - specific enthalpy
- Realistic Gas Equation of state
  - $P = \frac{\rho\epsilon[\rho\epsilon+2\rho]}{3(\rho\epsilon+\rho)}$
    - $h = \frac{5P}{2\rho} + \sqrt{\frac{9P^2}{4\rho^2} + 1}$ - specific enthalpy

Relativistic Magnetohydrodynamics

- The Relativistic MHD Equations in Conservative Form

$$
\begin{aligned}
\partial_t(\sqrt{-g}\rho_o u^t) + \partial_i(\sqrt{-g}\rho_o u^i) &= 0 \\
\partial_t(\sqrt{-g}T^t_\nu) + \partial_i(\sqrt{-g}T^i_\nu) &= \sqrt{-g}T^{\mu\kappa}\partial_\mu g_{\nu\kappa} - \sqrt{-g}T^\kappa_\lambda \Gamma^\lambda_{\nu\kappa} \\
\partial_t(\sqrt{-g}B^i) + \partial_j(\sqrt{-g}\left(u^j b^i - u^i b^j\right)) &= 0 \\
\partial_i(\sqrt{-g}B^i) &= 0
\end{aligned}
$$

where g is the determinant of the metric and

$$
\begin{aligned}
T^{\mu\nu} &= (\rho + P + b^2)u^\mu u^\nu + (P + \frac{1}{2}b^2)g^{\mu\nu} - b^\mu b^\nu \\
b^\mu &\equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}u_\nu F_{\lambda\kappa}
\end{aligned}
$$

Metrics

- Life would be considerably easier if all matter was too small to distort space and time surrounding them
- However this would be boring so we are left with different means to measure the distance between points on the spacetime "grid"
  - Minkowski
  - Schwarzschild - static and stationary
  - Kerr - just stationary
  - Fully dynamic Spacetimes - neither static nor stationary
  - Others do exist but they pertain more to cosmology than surrounding individual massive bodies
- Minkowski is used when the massive body has no real effect on the spacetime itself, the masses are too small, or too far away.
- Schwarzschild, first let us consider a cow, next we assume it is a sphere...
- Kerr, let us take that cow and assume it is elliptical... then make it spin?
- Fully dynamic spacetimes are complicated and would need to be considered if the accretion mass was sufficient to significantly modify the mass of the accretor.

Kerr

- Since more "realistic" systems are expected to at a minimum have angular momentum, we use the Kerr metric

- The Kerr metric requires the following assumptions
  - Accretion rate is insufficient to modify the spacetime considerably
  - The massive bodies are axisymmetric
  - The massive bodies in space are rotating

- Important features of Kerr
  - Maximum angular momentum, J, is equal to the mass M (geometric units)
  - Has an event horizon
  - Has an ergosphere

Vorticity and Helicity

- Vorticity, ω
  - Newtonian: $\vec{\omega} = \nabla \times \vec{v}$
  - Relativistic: $\omega^{\mu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\lambda\eta} u_{\lambda;\eta} u_{\nu}$

  Using analogies between fluid dynamics and magnetodynamics we find the magnetic field can be treated in a similar way

  - Newtonian: $\vec{B} = \nabla \times \vec{A}$
  - Relativistic: $B^{\mu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\lambda\eta} A_{\lambda;\eta} A_{\nu}$

- Helicity
  - $H = \int_V \omega^{\mu} u_{\mu}$ - Helicity
  - $H_m = \int_V A^{\mu} B_{\mu}$ - Magnetic Helicity
  - $H_c = \int_V B^{\mu} \omega_{\mu}$ - Cross Helicity

- Helicity is regarded as a constant of motion, thus in numerical codes, these can be monitored for the validity of the solution.

Waves

- Characteristic velocities
  - Magnetohydrodynamics involves three type of characteristic velocities
    - Alfven Velocity
    - Fast Magnetosonic Velocity
    - Slow Magnetosonic Velocity
  - Alfven velocity comes from the magnetic field wave equation
  - The Fast and slow magnetosonic velocities come from the velocity wave equation
  - Fast waves have characteristic velocities greater than the Alfven velocity
  - Slow waves have characteristic velocities less than the Alfven velocity
  - Relativistically these must be solved numerically

Real Parameters

- For real accretion disks I will have to attempt to setup the initial conditions as seen from astronomical data

- For these systems we expect the following:
  - $Re > 10^4$
  - $R_M \sim 10^{10}$ (Solar Corona)
  - $\rho_o \sim 10^{-5} - 10^{32} cm^{-3}$
  - $|B| \sim 10^{-12} - 10^{20} G$
  - $\beta \sim 10^{-5} - 10^{20}$

- these ranges include the Newtonian as well as relativistic cases

Typical Setup

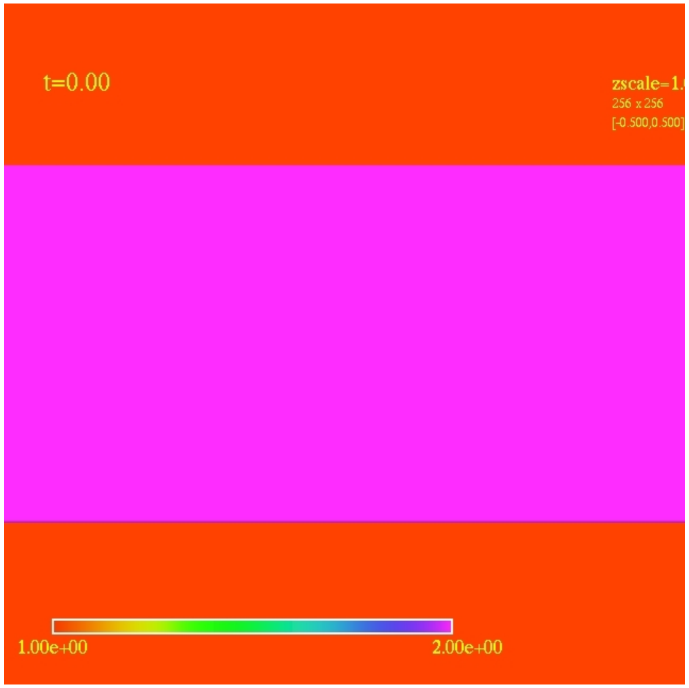
- So for my research the typical parameters for an accretion disk are
  - Radius $\rightarrow$ 0.26a - 0.3a
  - Accretion rate $\rightarrow 10^{-4} M_{\odot}/y$
  - Average flow rate $\rightarrow$ 2.3E5 - 6.0E5 m/s
  - Average pressure $\rightarrow 10^{-10} Pa$
  - Average density $\rightarrow 10^{32} cm^{-3}$
  - Magnetic field strength $\rightarrow 10^{10} - 10^{15}$G
  - Magentic field direction $\rightarrow$ axisymmetric
  - Mass of accreting body $\rightarrow$ 3-10$M_{\odot}$
  - Tempurature $\rightarrow 10^{4} - 10^{8}$K

Turbulence & Instabilities

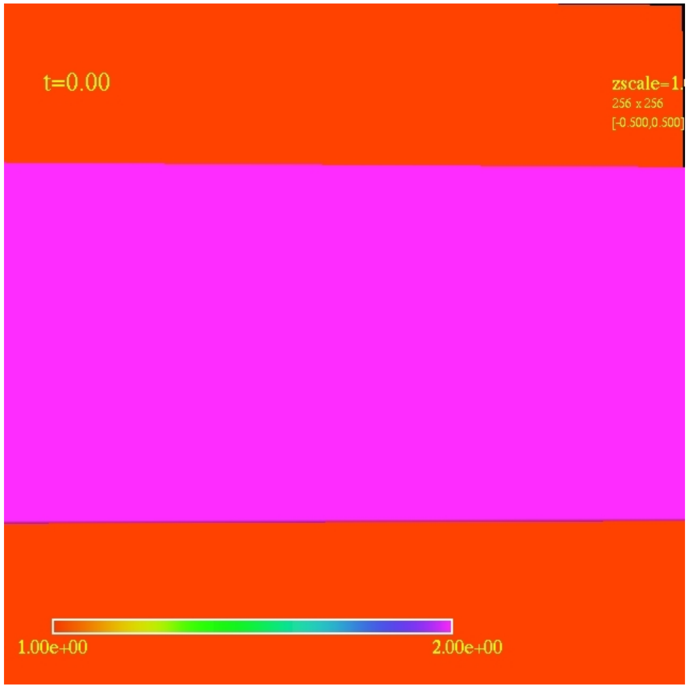
- Instabilities and Turbulence
  - Instabilities arise when the system undergoes a small perturbation and the dispersion relation leads to complex frequencies, thus the amplitudes either expand uncontrollably.
  - Turbulence arises when energy differences in the system undergo a transition from one energy type to another. Typically this energy will cascade from one length scale to the next, until viscous terms become non-negligible
  - There are a two types of turbulence that are of immediate concern in magnetohydrodynamic systems
    - Kelvin Helmholtz Instability
    - Magnetorotational Instability

Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented the Kelvin-Helmholtz instability
- The Kelvin-Helmholtz instability comes from the over abundance of kinetic energy in a localized region. This physically comes from differential rotation which is expected to be found in accretion disks
- There is slipping between the two layers of the fluid. Relative velocity between the two streams is 0.8c.



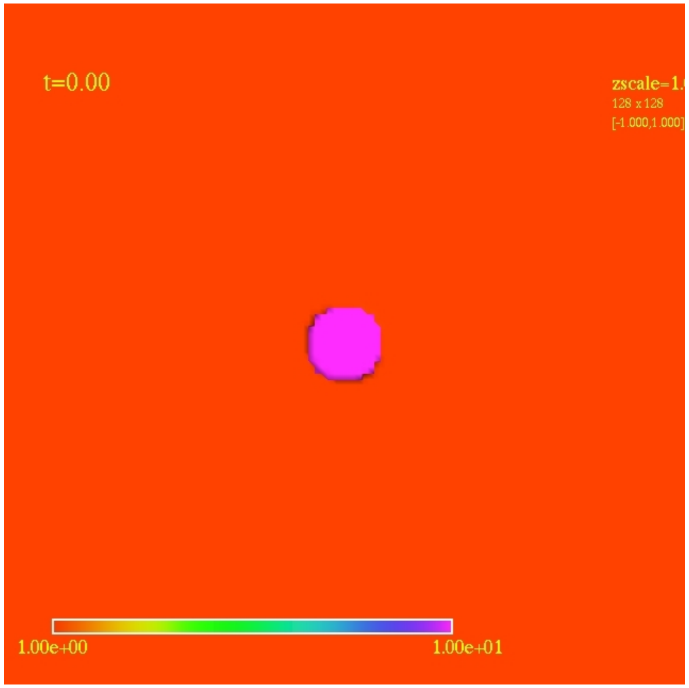
KHI No Magnetic field.



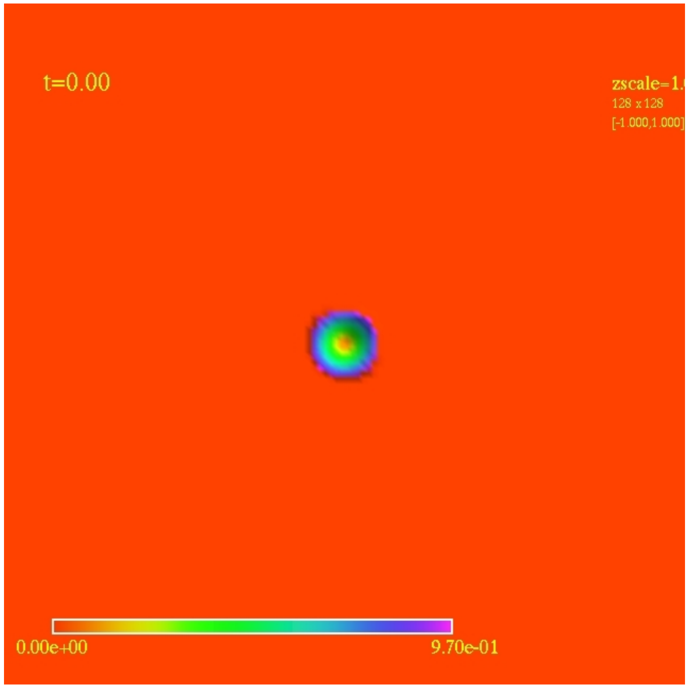
KHI With Magnetic Field, $B_x = 0.5$

Preliminary Results - Minkowski spacetime

- In the Minkowski spacetime I have implemented a rotor, edge angular velocity $\Omega = 9.75$
- Pressure $= 1.0$
- Density $= 10$ inside the disk, 1 outside of the disk



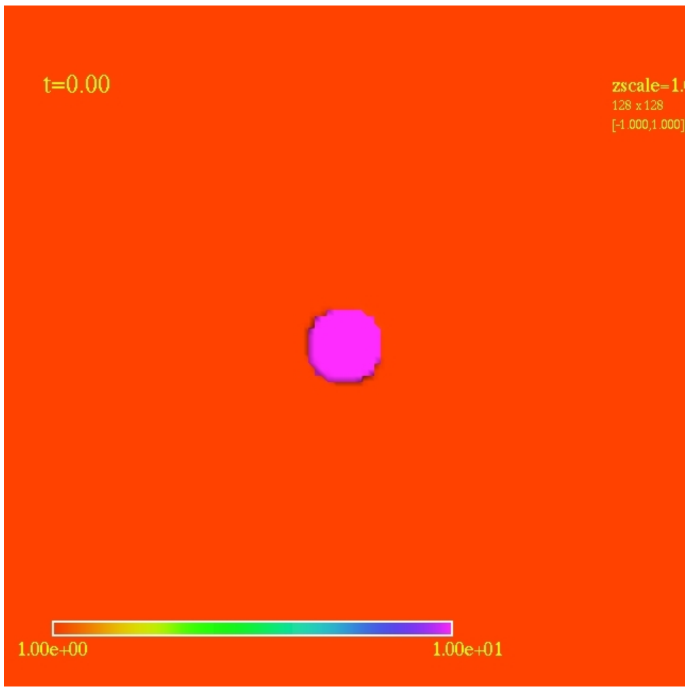
Density



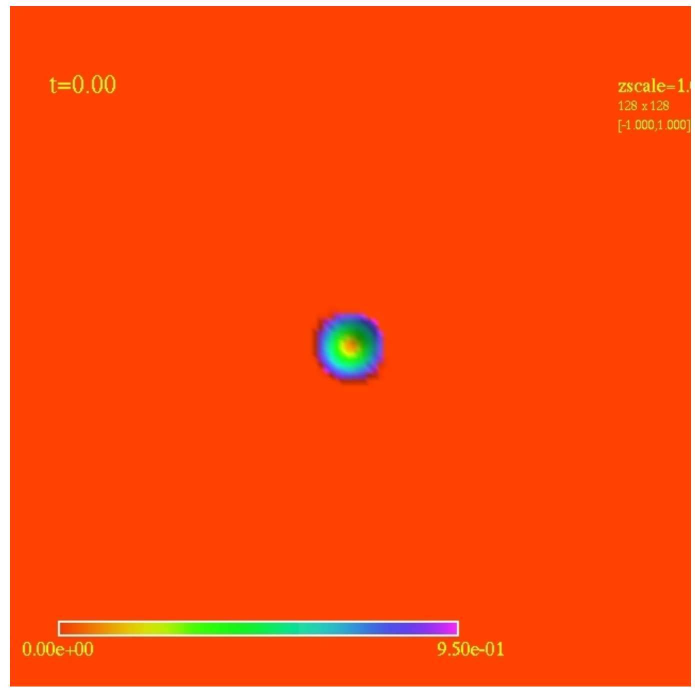
Speed

Preliminary Results - Minkowski spacetime

- Now including the magnetic field, $B_x = 0.5$
- Pressure $= 1.0$
- Density $= 10$ inside the disk, 1 outside of the disk



Density



Speed

Current Research

- Magnetorotational Instability
  - Rotational energy is converted to magnetic energy
  - Thought to underlie viscous dissipation when matter falls onto the compact object from a companion
    - This could answer the angular momentum transport problem
  - Infalling matter has considerable angular momentum
  - This forms the rotating disk around the object, with differential rotation
  - Inner regions lose angular momentum due to viscosity and fall inward, outer regions gain angular momentum and and move outward
  - infalling matter perturbs the outer regions preventing the outflow

Current Project - Numerical Magnetohydrodynamics

- Motivation
  - Existing theories and codes appear to work well, however they all rely on a ideal equation of state, other equations of state may lead to results that come closer to the observed phenomenon.
  - More realistic systems also consist of more than one fluid type, however this is beyond the scope of my project
  - To study the turbulence in accretion disks
  - To study an expansion to the thin disk approximation to accretion flows
  - Ultimately, to better understand the complex dynamics of accretion disks around compact object such as black holes

Current Project - Numerical Magnetohydrodynamics

- Questions to consider

  - How does a more realistic EoS impact the turbulence in MRI?
  - Will this realistic EoS allow for angular momentum transport in the accretion disk?
  - Does turbulence aid in the angular momentum transport by dissipating the energy to smaller scales?
  - What happens if matter infalls as lumps (high density regions) rather than as a steady stream of thin matter?
  - What are the effects of frame dragging for infalling matter to an observer at infinity (earth).

Physical Setup

- The next setup
  - Kerr parameter $a = 0, 0.5, 1$
  - Mass $M = 1$ - scaled
  - Equatorial plane
  - Inside the disk
    - $\vec{v} = \vec{0}$
    - $\vec{B} = (B_r, 0, 0)$
    - $P =$ low
    - $\rho =$ low
  - Periodic BC's at $\phi = 0, \pi/2$
  - Outflow at the event horizon $R_{EH} = M + \sqrt{M^2 - a^2}$ (inner radius)
  - Outflow at outer radius $\sim 6R_{EH}$, except at a single point
  - Single Point
    - $\vec{v} = (v_r, v_\phi, 0)$
    - $\vec{B} = (B_r, 0, 0)$
    - $P =$ high
    - $\rho =$ high and steady

Physical Setup

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  - Inside the disk
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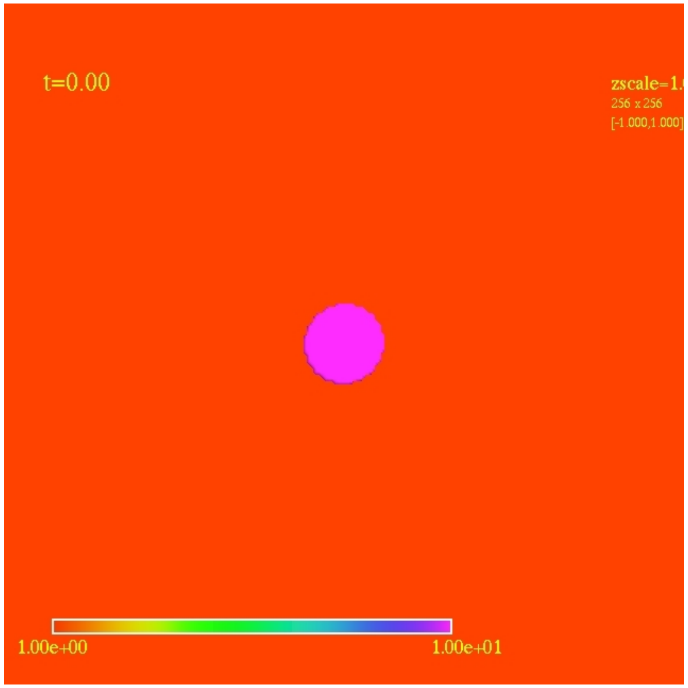
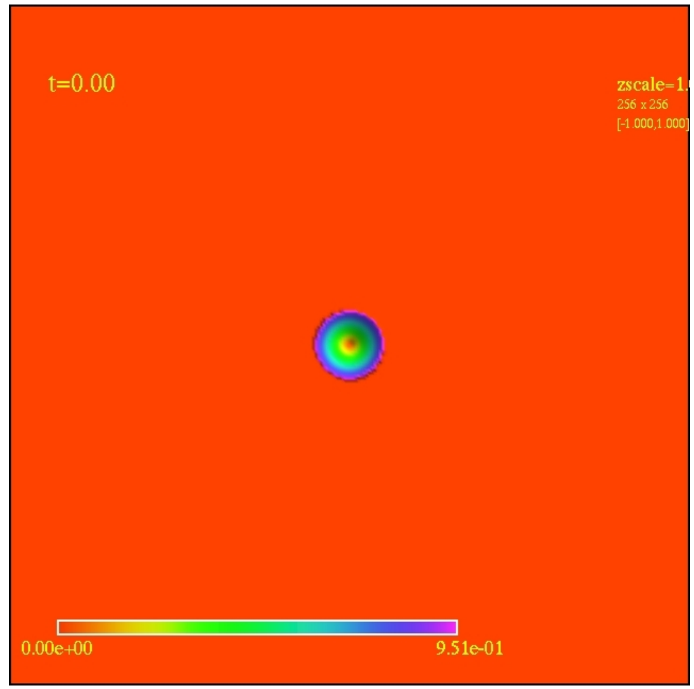
  - Outflow at outer radius $\sim R_{EH}$, except at a single point
  - Single Point
    - $\vec{v} = (v_r, v_\phi, 0)$
    - $\vec{B} = (B_r, 0, 0)$
    - $P =$ high
    - $\rho =$ high and a function of time, periodic or possibly random

C'est Fin

- Questions?

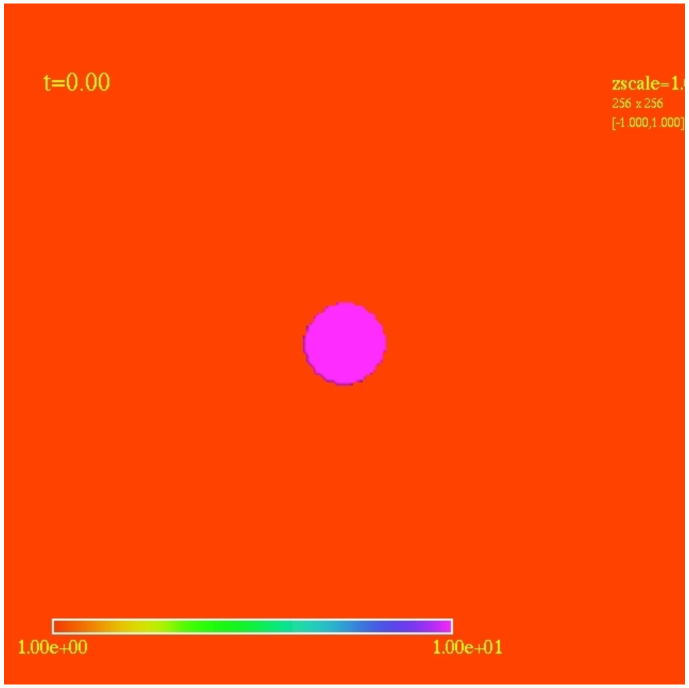
More Preliminary Results - Minkowski spacetime

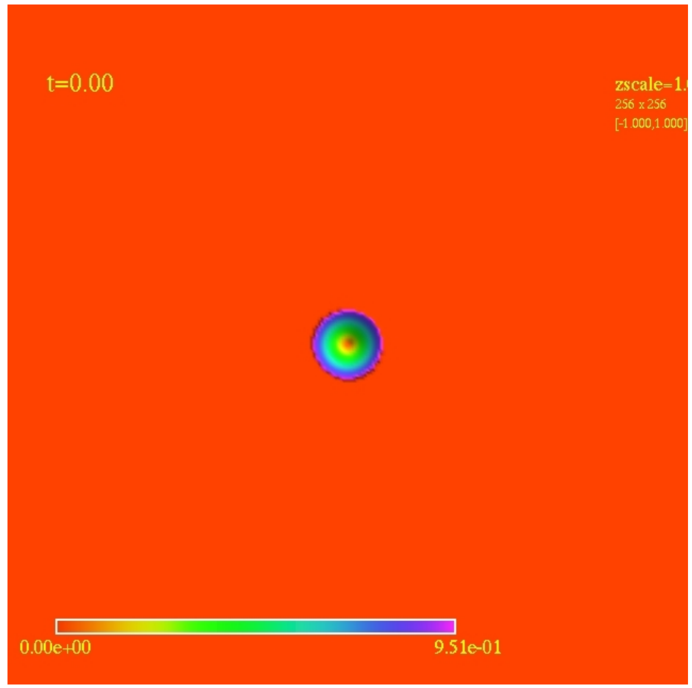




Preliminary Results - Minkowski spacetime

- Now including the magnetic field





The Solver

- The Roe Solver

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = \Sigma(q)$$

Cell average:

$$Q_i \approx \frac{1}{\Delta V} \int_{\Delta V} q dV$$

Integrate conservative equation over a small volume, and divide by that same volume

$$\frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial t} q dx + \frac{1}{\Delta V} \int_{\Delta V} \frac{\partial}{\partial x} f(q) dV = \frac{1}{\Delta V} \int_{\Delta V} \Sigma(q) dV$$

The Solver

which can be re-arranged, using Stokes' theorem

$$\frac{1}{\Delta V}\frac{\partial}{\partial t}\int_{\Delta V} q dx + \frac{1}{\Delta V}\int_{\partial V} f(q) dS = \frac{1}{\Delta V}\int_{\Delta V} \Sigma(q) dV$$

So now using our cell average, and the mean value theorem on the source term we get

$$\frac{\partial}{\partial t}(Q_i) + \frac{1}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) = \Sigma(Q_i)$$

finite difference the time derivative

$$(Q_i)^{n+1} = (Q_i)^n - \frac{\Delta t}{\Delta V}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) + \Delta t \Sigma_i$$

Flux Approximations

- Shock Capturing

We think of the discretization Q_i^n as being a piecewise constant reconstruction of the solution q(x). Then at every cell boundary we have a Riemann problem (the discontinuity). To estimate the flux in the above equation we write the flux as $f(q^*)$ where q^* is the solution at the cell boundary to the problem given by:

$$\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} = 0$$

with the Jacobian $A = \partial F/\partial q$ is constant.

For $x = 0$: If $A > 0$ the flow is to the right, so $q^* = q_L$, this corresponds to $f(q^*) = Aq_L$. If $A < 0$ the flow is to the right, so $q^* = q_R$, which corresponds to $f(q^*) = Aq_R$. Rather than using an if/then approach we use a general form:

$$f_{i+1/2} = f(q^*_{i+1/2}) = \frac{1}{2}(Aq_L + Aq_R - |A|(q_R - q_L))$$

So if $A > 0$ then $f = \frac{1}{2}(Aq_L + Aq_L) = Aq_L$