PHYS 410 Finite Difference Methods Oct 19, 2000
Expected Behaviour of Finite-Difference Approximations
(Convergence and Accuracy)

We have already seen that the basic "control parameter" of a typical FDA (finite-difference approximation) is the mesh-spacing, h. Fundamentally, for sensibly constructed FDA's, we expect the error in the approximation to go to 0 as h goes to 0.

Notation and Definitions

Let

$$L\,u = f \tag{1}$$

denote a general *differential* system. For simplicity and concreteness, you can think of $u = u(x)$ as a single function of one variable, but the following discussion also applies to cases in multiple independent variables ($u(x,t), u(x,y,t), \cdots$ etc.), as well as multiple *dependent* variables ($u = \mathbf{u} = [u_1, u_2, \cdots, u_n]$).
In (1), L is some differential operator ($L = d^2/dx^2$ in the BVP example we've studied previously), and f is some specified function. We will generically denote an FDA of (1) by

$$\hat{L}\hat{u} = \hat{f} \tag{2}$$

where $\hat{u}$ is the discrete solution, $\hat{f}$ is the specified function evaluated on the finite-difference mesh, and $\hat{L}$ is the finite-difference approximation to L.

Truncation Error: The truncation error, $\hat{\tau}$, of an FDA is defined by

$$\hat{\tau} \equiv \hat{L}u - \hat{f} \tag{3}$$

where u satisfies the continuum system (1). We note that the *form* of the truncation error can always be computed (typically using Taylor series expansions) from the finite difference approximation and the differential equations—we will see an example shortly.

Order of a Finite-Difference Approximation: Assuming that the FDA is characterized by a *single* discretization scale, h, we say that the FDA is *p-th order accurate* or simply *p-th order* if

$$\lim_{h\to 0} \hat{\tau} = O(h^p) \qquad \text{for some integer } p \tag{4}$$

Solution Error: The solution error, $\hat{e}$, associated with an FDA is defined by

$$\hat{e} \equiv u - \hat{u} \tag{5}$$

Relation Between Truncation Error and Solution Error

Is is common to tacitly assume that

$$\hat{\tau} = O(h^p) \qquad \text{implies} \qquad \hat{e} = O(h^p)$$

This assumption is often warranted but it is extremely instructive to consider *why* it is warranted and to investigate in some detail the *nature* of the solution error, $\hat{e}$, for a simple yet representative example.

Example

Consider one of the simplest possible differential systems:

$$L\,u(x) \equiv \left(\frac{d}{dx} - 1\right)\, u(x) = 0 \quad \text{on} \;\; 0 \le x \le 1 \quad \text{with} \;\; u(0) = 1 \tag{6}$$

Clearly, the unique solution of this DE is

$$u(x) = e^x.$$

We introduce a uniform finite-difference mesh, x_j,

$$x_j = jh, \quad j = 0, 1, \cdots n-1 \qquad \text{where} \;\; h = \frac{1}{n-1}$$

and consider the following difference *operators:*

$$\Delta_+^x u_j \equiv h^{-1}\left(u_{j+1} - u_j\right) \tag{7}$$

$$\mu_+^x u_j \equiv \frac{1}{2}\left(u_{j+1} + u_j\right) \tag{8}$$

Now consider Taylor series expansions of u_j and u_{j+1} about

$$u_{j+\frac{1}{2}} \equiv u(x_{j+\frac{1}{2}}) \equiv u\left(\frac{1}{2}\left(x_j + x_{j+1}\right)\right) \equiv \bar{u}$$

We have

$$\begin{aligned} u_{j+1} &= \bar{u} + \left(\frac{1}{2}h\right)\bar{u}' + \frac{1}{2}\left(\frac{1}{2}h\right)^2\bar{u}'' + \frac{1}{6}\left(\frac{1}{2}h\right)^3\bar{u}''' + \frac{1}{24}\left(\frac{1}{2}h\right)^4\bar{u}'''' + O(h^5) \\ u_j &= \bar{u} - \left(\frac{1}{2}h\right)\bar{u}' + \frac{1}{2}\left(\frac{1}{2}h\right)^2\bar{u}'' - \frac{1}{6}\left(\frac{1}{2}h\right)^3\bar{u}''' + \frac{1}{24}\left(\frac{1}{2}h\right)^4\bar{u}'''' + O(h^5) \end{aligned}$$

so

$$\Delta_+^x u_j \equiv h^{-1}\left(u_{j+1} - u_j\right) = \bar{u}' + \frac{1}{24}h^2\bar{u}''' + O(h^4) \tag{9}$$

$$\mu_+^x u_j \equiv \frac{1}{2}\left(u_{j+1} + u_j\right) = \bar{u} + \frac{1}{8}h^2\bar{u}'' + O(h^4) \tag{10}$$

Thus, we can identify the *difference operators*, Δ_+^x and μ_+^x, with (formal) power series (in h) of *differential operators*:

$$\Delta_+^x \equiv \frac{d}{dx} + \frac{1}{24}h^2\frac{d^3}{dx^3} + O(h^4) \tag{11}$$

$$\mu_+^x \equiv 1 + \frac{1}{8}h^2\frac{d^2}{dx^2} + O(h^4) \tag{12}$$

We now write down our FDA of (6):

$$\left(\frac{d}{dx} - 1\right)\, u(x) = 0 \quad \longrightarrow \quad \left(\Delta_+^x - \mu_+^x\right)\, u_j = 0 \tag{13}$$

(i.e. $\hat{L} = \Delta_+^x - \ \mu_+^x$). Explicitly, we have

$$\frac{u_{j+1} - u_j}{h} = \frac{u_{j+1} + u_j}{2} \qquad j = 0, 1, \cdots n-2 \qquad \text{with} \ \ u_0 = u(0) = 1 \tag{14}$$

Let us first consider the *truncation error* associated with (13,14):

$$\begin{aligned}
\hat{\tau} \equiv \hat{L}u - \hat{f} = \hat{L}u &= (\Delta_+^x - \ \mu_+^x)\ u \\
&= \left(\frac{d}{dx} + \frac{1}{24}h^2 \frac{d^3}{dx^3} - 1 - \frac{1}{8}h^2 \frac{d^2}{dx^2} + O(h^4) \right) \ \bar{u} \\
&= \left(\frac{d}{dx} - 1 \right) \ \bar{u} + h^2 \left(\frac{1}{24} \frac{d^3}{dx^3} - \frac{1}{8} \frac{d^2}{dx^2} \right) \ \bar{u} + O(h^4) = O(h^2)
\end{aligned}$$

where we have used the original differential equation (6) in the form

$$\left(\frac{d}{dx} - 1 \right) \ \bar{u} = 0.$$

Thus, since $\hat{\tau} = O(h^2)$, our FDA is *second order*. Note, however, that we had to be careful about choosing the x value about which we performed our Taylor series expansions—in general we must choose the expansion point which gives the *highest-order* truncation error or, intuitively, the point about which the scheme is naturally "centred".

The FDA (14) is sufficiently simple (and linear) that we can solve it explicitly:

$$\frac{u_{j+1} - u_j}{h} = \frac{u_{j+1} + u_j}{2} \quad \longrightarrow \quad u_{j+1} = \left(\frac{h^{-1} + \frac{1}{2}}{h^{-1} - \frac{1}{2}} \right) \ u_j = \left(\frac{1 + \frac{h}{2}}{1 - \frac{h}{2}} \right) \ u_j \equiv \rho \, u_j \quad \text{where } \rho \equiv \frac{1 + \frac{h}{2}}{1 - \frac{h}{2}}$$

Thus we have

$$u_j = \rho^j \, u_0 = \rho^j \tag{15}$$

where ρ^j denotes the j-th power of ρ and we have used the initial condition $u_0 = 1$. Now, we can write ρ^j as follows:

$$\rho^j = \exp\left(j \ln \rho\right) = \exp\left(j \left(\ln\left(1 + \frac{h}{2}\right) - \ln\left(1 - \frac{h}{2}\right) \right) \right).$$

Then, using the Taylor series expansion

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$$

we have

$$\ln\left(1 + \frac{h}{2}\right) - \ln\left(1 - \frac{h}{2}\right) = h + \frac{1}{12}h^3 + O(h^5)$$

so that

$$\begin{aligned}
\rho^j = \exp\left(jh + (jh)\,\frac{1}{12}h^2 + O(h^4) \right) &= \exp\left(jh\right) \exp\left(jh\frac{h^2}{12} + O(h^4) \right) = e^{x_j} \left(1 + \frac{h^2}{12}\, x_j + O(h^4) \right) \\
&= e^{x_j} + \frac{1}{12}h^2 \, x_j \, e^{x_j} + O(h^4)
\end{aligned}$$

Thus, the *solution error*, $\hat{e}$ is given by

$$\hat{e} = u - \hat{u} = -\frac{1}{12}h^2 x e^x + O(h^4) = O(h^2) \tag{16}$$

Note in particular, the *form* of the solution error; to leading order it is

$$h^2 \times \ -\frac{1}{12}x e^x = -h^2 \times e_2(x)$$

where $e_2(x)$ is a *function* $(\frac{1}{12}xe^x)$ with smoothness (i.e. magnitude of derivatives) comparable to the continuum solution, $u(x) = e^x$.
There is another way to derive (16) which clearly illustrates the fundamental lesson concerning the errors in finite difference approximations.
Let us *assume* that u and $\hat{u}$ are related by the asymptotic ($h \to 0$) expansion,

$$\hat{u} = u + h^2\, e_2 + h^4\, e_4 + \cdots \tag{17}$$

which we call a *Richardson expansion* (after L.F. Richardson, who studied such matters in the early part of this century).
We start with the FDA

$$\hat{L}\,\hat{u} = 0 \quad \longrightarrow \quad (\Delta_+^x - \ \mu_+^x)\ \hat{u} = 0$$

and replace both $\hat{L}$ and $\hat{u}$ by their "continuum expansions" (11,12,17):

$$\left(\frac{d}{dx} - 1 + h^2\left(\frac{1}{24}\frac{d^3}{dx^3} - \frac{1}{8}\frac{d^2}{dx^2}\right) + O(h^4)\right)\left(u + h^2\, e_2 + O(h^4)\right) = 0$$

We now demand that this equation vanish, *order by order,* in h. For the $O(1)$ and $O(h^2)$ cases we have

$$O(1): \qquad \left(\frac{d}{dx} - 1\right) u = 0 \quad \longrightarrow \quad u = e^x \qquad (\textit{consistency} \text{ of FDA})$$

$$O(h^2): \qquad \left(\frac{d}{dx} - 1\right) e_2 = \left(\frac{1}{8}\frac{d^2}{dx^2} - \frac{1}{24}\frac{d^3}{dx^3}\right) u$$

Thus, we see that the error function, e_2, *itself* satisfies a differential equation which is very similar in form to the original DE. Moreover, given that we know $u(x) = e^x$, we can solve this DE for e_2. Specifically, we have

$$\left(\frac{d}{dx} - 1\right) e_2 = \frac{1}{12}\, e^x \quad \longrightarrow \quad e_2 = \frac{1}{12} x e^x \tag{18}$$

and since our original *assumption* (17) was

$$\hat{u} = u + h^2\, e_2 + O(h^4) = u + \frac{1}{12}h^2 x e^x + O(h^4)$$

we get

$$\hat{e} \equiv u - \hat{u} = -\frac{1}{12}h^2 x e^x + O(h^4)$$

as previously.

Comments

Even in cases where we *don't* know the continuum solution (this clearly includes most of the interesting instances where we are likely to apply finite-difference techniques), it is still very useful to think of *difference operators* and *difference solutions* in terms of asymptotic expansions in powers of the mesh spacing, h:

$$\begin{aligned} \hat{L} &= L + h^2 L_2 + h^4 L_4 + O(h^6) \\ \hat{u} &= u + h^2 e_2 + h^4 e_4 + O(h^6) \end{aligned}$$

so that

$$\hat{L}\,\hat{u} = \hat{f}$$

actually represents an (infinite) hierarchy of differential systems

$$\begin{aligned} L\,u &= f \\ L\,e_2 &= -L_2\,u \\ &\ \ \vdots \end{aligned}$$

In other words, in FDA's of continuum systems, the error is not "random"—rather, in principle, it is no less computable than the fundamental solution itself.