Status Report

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1 Description of Project

In my thesis I will numerically solve the magnetohydrodynamic equations for a fluid coupled to the Einstein equations in the Kerr spacetime. The astrophysical phenomenon that are thought to obey this type of setup are numerous. Such examples are the active galactic nuclei, accretion disks, and supernovae explosions. Since I am focusing on a stationary spacetime background I will be most interested in accretion phenomenon. There has been a lot of active research on the magnetorotational instability (MRI).

The Einstein equations can be written as:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1}$$

where we adopt geometric units such that $G = c = 1$. Here $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ is the stress energy tensor. In particular for this case the stress energy tensor is for a magnetic fluid, which will be put together below.

We will assume ideal MHD, where the Electric field E^{μ} in the co-moving frame is zero, or $F_{\mu\nu}u^{\nu} = 0$ (the Lorentz force in the rest frame is zero). This can be achieved through the observation that in astrophysical phenomenon the magnetic Reynolds number is comparatively large, which then leads us to the assumption that the conductivity of the system tends to infinity. Further, the electric current J^{μ} is not necessary for evolving the field variables, and one can use Ohm's law to write it in terms of the magnetic field alone.

There are fundamental conserved quantities that are required for the evolution of such a fluid. For the ideal case we have;

$$\nabla_{\alpha} T^{\alpha\beta} = 0 \tag{2}$$

As well as the conservation of the baryon number

$$\nabla_{\alpha} J^{\alpha} = 0 \tag{3}$$

where $J^{\alpha} = \rho_o u^{\alpha}$. Unlike the pure hydrodynamical case, we cannot end there. We need one more condition to be satisfied;

$$\nabla^{*}_{\mu} F^{\mu\nu} = 0 \tag{4}$$

where $F^{\mu\nu}$ is the Faraday tensor for electromagnetism.

The last four equations are highly nonlinear and require the use of a computer for their solution. Along the way, I will attempt to utilize as many symmetries as possible to attempt to reduce the computational load expected. Ultimately I expect to be able to generalize these equations to the beginning stages of plasma evolutions where we have to consider more than just one type of particle as it evolves through time. However before diving into that part of the project I must study the MHD approximation, which will allow for a better understanding of the physics behind the plasma evolution. Furthermore I also expect to observe as many conserved quantities as possible such as the divergence of the magnetic field. From here I will list the work done on MHD, a lot of which came from the relativistic hydrodynamical work by Smarr, Hawley, Evans, Gammie, Olabarrieta, and many more...[4, 5, 6, 8, 7, 9, 10]

2 Previous work

Numerical hydrodynamics dates back to the work done in spherical symmetry by May and White[2]. They developed dynamic coordinate analysis (The Lagrangian approach) using finite differencing. Artificial viscosity was introduced as a higher order phenomenon, so as to smooth out "shock" regions numerically without having a significant effect on the physical meaning of the data. They studied neutron star collapse and supernovae explosions in spherical symmetry. Difficulties in generalization arise from the use of dynamic coordinates. To resolve this issue Wilson, produced what are now known as Eulerian coordinates, where the fluid moves in a fixed coordinate base. Now the equations could be solved using advective equations, however they still required the use of artificial viscosity to circumvent the shock regions, and allow the code to maintain stability. This approach is still used today, for the study of many astrophysical phenomenon, such as the merger of two neutron stars.

All of these methods are equally applicable to the magnetohydrodynamical equations. A fundamental property of the (magneto)hydrodynamic equations is that they may be all written in a conservative form. IE they may be written as:

$$\partial_t q + \partial_i f^i(q) = \Sigma(q) \tag{5}$$

where $q \rightarrow q(x_i, t)$ is a state vector of the fluid. f^i is the flux vector, and Σ is a source. The important property of the source term is that it does not contain any derivatives of the fluid variables. Any such terms would be written into the flux term. This particular form allows us to use Godunov methods also know as high resolution shock capturing methods. This approach will allow us to dismiss the use of artificial viscosity, and consequently allow us to run code for more extreme shocks, in the case of relativistic fluids, much larger γ terms may be observed. These methods allow for up to fully 3D simulations to be run.

3 Hydrodynamic Equations

We start with the stress-energy conservation equation 2;

$$T^{\mu}_{\nu;\mu} = 0$$

we expand this into expressions involving a standard derivative and the Christoffel symbols

$$(T^{\mu}_{\nu})_{,\mu} + T^{\alpha}_{\nu}\Gamma^{\mu}_{\alpha\mu} - T^{\mu}_{\alpha}\Gamma^{\alpha}_{\mu\nu} = 0$$

There is a well know relation for a Christoffel symbol contracted over these two indices;

$$\Gamma^{\mu}_{\alpha\mu} = \frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} \tag{6}$$

so we are left with;

$$(T^{\mu}_{\nu})_{,\mu} + T^{\alpha}_{\nu}\frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} - T^{\mu}_{\alpha}\Gamma^{\alpha}_{\mu\nu} = 0$$

which after multiplying by the scalar $\sqrt{-g}$, reduces to;

$$(\sqrt{-g}T^{\mu}_{\nu})_{,\mu} = \sqrt{-g}T^{\mu}_{\alpha}\Gamma^{\alpha}_{\mu\nu}$$

where now to put it into a flux conservative form we expand the contraction in the left hand side into a time term and a space term

$$(\sqrt{-g}T^{t}_{\nu})_{,t} + (\sqrt{-g}T^{i}_{\nu})_{,i} = \sqrt{-g}T^{\mu}_{\alpha}\Gamma^{\alpha}_{\mu\nu} \tag{7}$$

so we see our conserved variables in these equations are going to involve the term T^{t}_{ν}. We also note that there is an immediate occurrence of a source term if we look at any space other than Minkowski. This will be further broken down when I look at the actual form of the stress energy tensor for ideal MHD.

Next we look at the Baryon conservation equation 3;

$$J^{\mu}_{;\mu} = 0 \tag{8}$$

$$(\rho_o u^\mu)_{;\mu} = 0$$

expanding returns

$$(\rho_o u^\mu)_{,\mu} + (\rho_o u^\alpha)\Gamma^\mu_{\alpha\mu} = 0$$

where we use equation 6 again;

$$(\rho_o u^\mu)_{,\mu} + (\rho_o u^\alpha)\frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} = 0$$

so ultimately we get;

$$(\sqrt{-g}\rho_o u^t)_{,t} + (\sqrt{-g}\rho_o u^i)_{,i} = 0$$

where we obtain another conservative variable $D = \rho_o u^t$.

Also for notational simplicity;

$$h = 1 + \epsilon + \frac{P}{\rho_o}$$
$$\epsilon = (\Gamma - 1)P/\rho_o \tag{9}$$

To further specify the system we are also required to select an equation of state. $P \rightarrow P(\rho)$. For the relativistic gases I chose the ideal gas $P = (\Gamma - 1)\rho\epsilon$

4 Electromagnetic Equations

Finally we consider the Faraday condition 4

$${}^*F^{\mu\nu}_{;\nu} = 0$$

which will break down exactly like that of the stress energy tensor

$$({}^*F^{\mu\nu})_{,\mu} + {}^*F^{\alpha\nu}\Gamma^\mu_{\alpha\mu} + {}^*F^{\mu\alpha}\Gamma^\nu_{\mu\alpha} = 0$$

again using 6 so we are left with;

$$({}^*F^{\mu\nu})_{,\mu} + {}^*F^{\alpha\nu}\frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} + {}^*F^{\mu\alpha}\Gamma^\nu_{\mu\alpha} = 0$$

which after multiplying by the scalar $\sqrt{-g}$, reduces to;

$$(\sqrt{-g}\,{}^*F^{\mu\nu})_{,\mu} = \sqrt{-g}\,{}^*F^{\alpha\mu}\Gamma^\nu_{\mu\alpha}$$

where now to put it into a flux conservative form we expand the contraction in the left hand side into a time term and a space term

$$(\sqrt{-g}\,{}^*F^{t\nu})_{,t} + (\sqrt{-g}\,{}^*F^{i\nu})_{,i} = \sqrt{-g}\,{}^*F^{\alpha\mu}\Gamma^\nu_{\mu\alpha}$$

This can be further broken down into 2 equations, when we consider the ν index alone, we separate space and time (t,j);

$$(\sqrt{-g}\,{}^*F^{tt})_{,t} + (\sqrt{-g}\,{}^*F^{it})_{,i} = \sqrt{-g}\,{}^*F^{\alpha\mu}\Gamma^t_{\mu\alpha}$$
$$(\sqrt{-g}\,{}^*F^{tj})_{,t} + (\sqrt{-g}\,{}^*F^{ij})_{,i} = \sqrt{-g}\,{}^*F^{\alpha\mu}\Gamma^j_{\mu\alpha}$$

Unlike the Stress-energy tensor, we know that the Faraday tensor is antisymmetric. The terms on the right hand side are now the fully contracted product of a symmetric and antisymmetric tensor. (The Christoffel symbols are symmetric in the lower indices.) Thus the right hand sides are identically zero. Furthermore the diagonal elements of $F^{\mu\nu}$ are also zero. Thus we are left with;

$$(\sqrt{-g}\,{}^*F^{it})_{,i} = 0$$
$$(\sqrt{-g}\,{}^*F^{tj})_{,t} + (\sqrt{-g}\,{}^*F^{ij})_{,i} = 0$$

We know from classical electrodynamics that the elements of the Faraday tensor of the form ${}^*F^{it}$ will be proportional to the magnetic field B^i so the first expression looks like the no-divergence theorem. The second equation, having similar first term is expected to be an evolution equation for the magnetic field.

We are considering a perfect conductor so $\mathcal{J}^\alpha = \rho u^\alpha + \sigma F^{\alpha\nu} u_\nu$ must remain finite with $\sigma \to \infty$, so $F^{\mu\nu} u_\nu = 0 \to E_\mu = 0$ for a co-moving observer $u_\alpha = (1, 0, 0, 0)$. With this and the help of Jackson we have a transformation of the Faraday tensor between two frames of reference.

$$\vec{E}' = \gamma \left(\vec{E} - \vec{v} \times \vec{B} - \frac{\gamma \left(\vec{v} \cdot \vec{E} \right) \vec{v}}{\gamma + 1} \right)$$

So we get

$$\vec{E}' = \gamma(-\vec{v} \times \vec{B})$$

Thus allowing us to eleminate the electric field from the Faraday tensor, leaving us with a set of equations involving the magnetic field and the velocity field alone. Thus from the second line above, we have our last conserved variables; B^j.

5 3+1 decomposition

Mostly you can refer to Matt's notes on this topic. What I need now;

$$\begin{bmatrix} g_{tt} & g_{jt} \\ g_{ti} & g_{ij} \end{bmatrix} = \begin{bmatrix} \beta_s \beta^s - \alpha^2 & \beta_j \\ \beta_i & \gamma_{ij} \end{bmatrix}$$

and

$$\begin{bmatrix} g^{tt} & g^{jt} \\ g^{ti} & g^{ij} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{bmatrix}$$

With this in mind we can define

$$W = \frac{1}{\sqrt{1 - g_{ij} v^i v^j}} = \frac{1}{\sqrt{1 - \gamma_{ij} v^i v^j}} \tag{10}$$

We define the normal vector;

$$n_\mu = (-\alpha, 0, 0, 0) \tag{11}$$
$$n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right) \tag{12}$$

With this in mind we define our velocity 4-vectors as;

$$v^i = \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha} = \frac{u^i}{W} + \frac{\beta^i}{\alpha} \tag{13}$$
$$u^0 = \frac{W}{\alpha} \tag{14}$$
$$v_j = \frac{u_j}{W} \tag{15}$$

We can also define $\perp_{\mu\nu}$ to be

$$\perp_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{16}$$

We further define (as per Gammie & Noble Paper)

$$b^\mu = \frac{1}{W} h^\mu_\nu B^\nu \tag{17}$$
$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \tag{18}$$
$$b_\mu = g_{\mu\nu} b^\nu \tag{19}$$

$h_{\mu\nu}$ is the projection operator to take a vector into the space normal to the fluid 4-velocity u^μ

6 MHD Stress Energy

We define the stress energy tensor for ideal MHD to be a linear combination of the stress energies for both the electromagnetic and hydrodynamic systems:

$$T^{EM}_{\mu\nu} = F_{\mu\sigma} F^\sigma_\nu - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = (\frac{1}{2} g_{\mu\nu} + u_\mu u_\nu) b^2 - b_\mu b_\nu \tag{20}$$
$$T^{Hydro}_{\mu\nu} = (\rho_o + \rho + P) u_\mu u_\nu + P g_{\mu\nu} = \rho_o h u_\mu u_\nu + P g_{\mu\nu} \tag{21}$$
$$T^{MHD}_{\mu\nu} = (\rho_o h + b^2) u_\mu u_\nu + (P + \frac{1}{2} b^2) g_{\mu\nu} - b_\mu b_\nu \tag{22}$$

where our definition of $b^\mu = \left(\frac{W}{\alpha}(\vec{v} \cdot \vec{B}), \frac{B^i}{W} + W(\vec{v} \cdot \vec{B}) \left(v^i - \frac{\beta^i}{\alpha}\right)\right) = \left(\frac{W}{\alpha}(\vec{v} \cdot \vec{B}), \frac{B^i}{W} + \alpha b^0 \left(v^i - \frac{\beta^i}{\alpha}\right)\right)$

7 Putting it all together

The final task for the EOM is to use the 3+1 formalism. At first glance this seems like an easy thing to do, involving just 3 transformations of the Stress-Energy tensor, and the magnetic field;

$$E = T^{\mu\nu} n_\mu n_\nu \tag{23}$$
$$S_i = T^{\mu\nu} n_\mu \perp_{\nu i} \tag{24}$$
$$B^\mu = n^*_\nu F^{\mu\nu} \tag{25}$$

since $n_\mu = (-\alpha, 0, 0, 0)$;

$$E = T^{\mu\nu} n_\mu n_\nu = T^{tt}(-\alpha)^2 \tag{26}$$
$$= \alpha^2 \left[(\rho_o h + b^2)(u^t)^2 - \left(P + \frac{1}{2} b^2\right) \frac{1}{\alpha^2} - (b^t)^2 \right]$$
$$= (\rho_o h + b^2) W^2 - (P + \frac{1}{2} b^2) - (\alpha b^t)^2$$
$$\tag{27}$$

$$S_i = -T^{\mu\nu} n_\mu \perp_{\nu i} = \alpha T^{t\nu} \perp_{\nu i} = \alpha T^{t\nu} (g_{\nu i} + n_\nu n_i) \tag{28}$$
$$= \alpha T^t_i = \alpha(\rho_o h + b^2) u^t u_i - \alpha b^t b_i$$
$$= (\rho_o h + b^2) W^2 v_i - \alpha b^t b_i$$

$$B^\mu = -n_\nu F^{\nu\mu} \tag{29}$$

$$\tag{30}$$

$$= \alpha F^{0\mu} \tag{31}$$

From here we get the conserved quantities:

$$q = \begin{bmatrix} D \\ S_j \\ \tau \\ B^k \end{bmatrix} \tag{32}$$

where $\tau = E - D$ which is used for the proper recovery of the Newtonian limit. We have the corresponding flux terms:

$$f(q) = \begin{bmatrix} D\hat{v}^i \\ S_j\hat{v}^i + P\delta^i_j - \frac{b_j B^i}{W} \\ \tau\hat{v}^i + Pv^i - \alpha\frac{b^t B^i}{W} \\ B^k\hat{v}^i - \hat{v}^k B^i \end{bmatrix} \tag{33}$$

where $\hat{v}^i = v^i - \frac{\beta^i}{\alpha}$
This leaves us with the generalized Conservative form;

$$\frac{\partial}{\partial x^0}\sqrt{\gamma}\vec{q} + \frac{\partial}{\partial x^i}\sqrt{-g}\vec{f^i}(\vec{q}) = S(\vec{q}; x^\mu) \tag{34}$$

8 The Solver

For this project I will be using the Roe Solver, which is a specialized Godunov solver, usually used to solve the fluid equations. I describe The basics behind this solver, for full details I refer you to LeVeque.

The Roe solver is used for a hyperbolic system (signified by real eigenvalues of the Jacobian of the flux) For a nonlinear conservation law 5 we have:

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{\gamma}q}{\partial t} + \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}f^i(q)}{\partial x^i} = S(q)$$

We desire the integral form of this equation, so we integrate both parts over some small cell or volume in spacetime. The integral can be taken over a sufficiently small amount of spacetime, so that:

$$\int \frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}\sqrt{\gamma}q dV^{(4)} + \int \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\sqrt{-g}f^i(q)dV^{(4)} = \int S(q)dV^{(4)} \tag{35}$$

where $dV^{(4)} = \sqrt{-g}dx_0dx_1dx_2dx_3$, $\Delta V^{(4)} = \sqrt{-g}\Delta x_1\Delta x_2\Delta x_3$. We will also use $\Delta V^{(3)} = \sqrt{\gamma}\Delta x_1\Delta x_2\Delta x_3$ break up the integrals;

$$\int_{\Delta x_0}\int_{\Delta x^i}\frac{\partial}{\partial t}\sqrt{\gamma}q dx_1dx_2dx_3dx_0 + \int_{\Delta x_0}\int_{\Delta x^i}\frac{\partial}{\partial x^i}\sqrt{-g}f^i(q)dx_1dx_2dx_3dx_0 = \int_{\Delta x_0}\int_{\Delta x^i}S(q)\sqrt{-g}dx_1dx_2dx_3dx_0 \tag{36}$$

I define a cell average to be:

$$Q \approx \frac{1}{\Delta V^{(3)}}\int_{\Delta x^i}\sqrt{\gamma}q dx_1dx_2dx_3 \tag{37}$$

so

$$\int_{\Delta x_0}\frac{\partial}{\partial t}Q\Delta V^{(3)}dx_0 + \int_{\Delta x_0}\int_{\Delta x^i}\frac{\partial}{\partial x^i}\sqrt{-g}f^i(q)dx_1dx_2dx_3dx_0 = \int_{\Delta x_0}\int_{\Delta x^i}S(q)\sqrt{-g}dx_1dx_2dx_3dx_0 \tag{38}$$

and using the divergence theorem on the flux term, and approximating the spatial integral on the source term;

$$\int_{\Delta x_0} \frac{\partial}{\partial x_0} Q\Delta V^{(3)} dx_0 + \int_{\Delta x_0}\int_{\partial V} f(q) d\Sigma dx_0 = \int_{\Delta x_0} \hat{S}(q)\Delta V^{(4)} dx_0 \tag{39}$$

$\hat{S}(q)\Delta V^{(4)} = \int_{\Delta V^{(3)}} S(q)\sqrt{-g}dx_1 dx_2 dx_3$

approximate the integral over the flux terms;

$$\int_{\Delta x_0} \frac{\partial}{\partial x_0} Q\Delta V^{(3)} dx_0 + \int_{\Delta x_0} f^i(q)\Delta\Sigma_i dx_0 = \int_{\Delta x_0} \hat{S}(q)\Delta V^{(4)} dx_0 \tag{40}$$

where $\Delta\Sigma_j = \sqrt{-g}dx_i dx_k \epsilon_{ijk}$ such that the direction of the surface element is orthogonal to the surface area in question.

Now we perform the time integration

$$(Q\Delta V^{(3)})^{n+1} - (Q\Delta V^{(3)})^n + \Delta x_0 \sum_j \Delta\Sigma_i F^i_j(q) = \hat{S}\Delta V^{(4)} \tag{41}$$

where we have defined $F^i(q) = \frac{1}{\Delta x_0}\int_{\Delta x_0} f^i(q)dx_0$, and assumed that the evolution is over such a short period of time that the sources remain constant in time (as with the surfaces through which the flux travels. The sum over j is simply the sum of the fluxes over all cell faces.

Now we have;

$$Q^{n+1} = Q^n - \frac{\Delta x_0}{\Delta V^{(3)}} \sum_j \Delta\Sigma_i F^i_j(q) + \hat{S}\frac{\Delta V^{(4)}}{\Delta V^{(3)}} \tag{42}$$

To handle the flux time integral without explicitly performing the integral, we rely on approximations sure as the Roe solver, Tadmor, Marquina, HLLE, HLLC etc...

I explain the Roe solver below, the others mentioned may be found in papers on my references webpage (see bibliography).

In this form, one can handle discontinuities rather easily by breaking up the integrals to integrate over the continuous parts of space. Here is where we implement Godunov's idea. We think of the discritization Q^n_i as being a piecewise constant reconstruction of the solution q(x). Then at every cell boundary we have a Riemann problem (the discontinuity). To estimate the flux in the above equation we write the flux as $f(q^*)$ where q^* is the solution at the cell boundary to the problem given by:

$$\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} = 0 \tag{43}$$

with the Jacobian $A = \partial F/\partial q$ is constant. Essentially we have linearized the equation. The following refers to the figure 1. From here we want the characteristics of 43 which are calculated by $x - At = k$ for constant k. Thus q^*;

For $x = 0$: If $A > 0$ the flow is to the right, so $q^* = q_L$, this corresponds to $f(q^*) = Aq_L$. If $A < 0$ the flow is to the right, so $q^* = q_R$, which corresponds to $f(q^*) = Aq_R$. Rather than using an if/then approach we use a general form:

$$f_{i+1/2} = f(q^*_{i+1/2}) = \frac{1}{2}\left(Aq_L + Aq_R - |A|(q_R - q_L)\right) \tag{44}$$

So if $A > 0$ then $f = \frac{1}{2}(Aq_L + Aq_L) = Aq_L$

This follows directly from the method of characteristics which will be included in these notes at a later time.

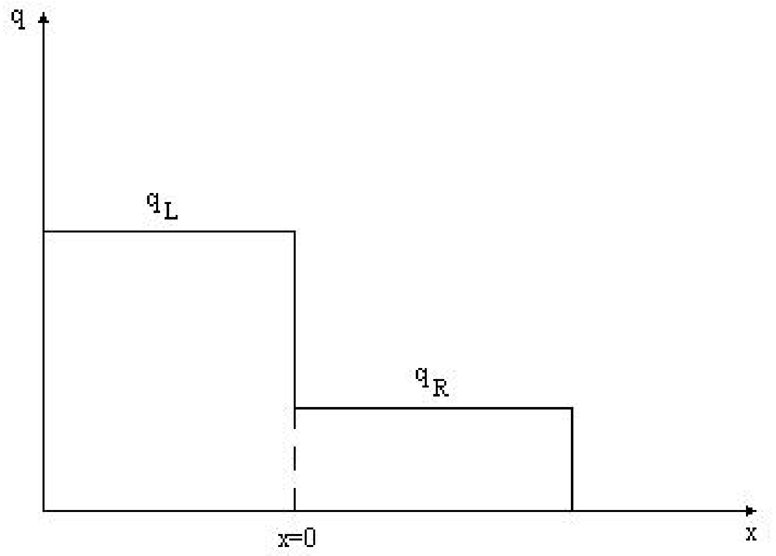

Figure 1: The setup for the Riemann Problem

9 The $\nabla \cdot B = 0$ constraint

Regular Riemann solvers when applied to the magnetic field components will disobey the magnetic field constraints. This is of course impossible physically and exposes flaws in the flux calculations. The remedy has been investigated by Hawley and Evans (1978) using what they called Constrain Transport. The difficulty in this method is that they use a staggered grid approach where one evaluates the magnetic field components at the face centred locations, and the flux is calculated in the corners. This adds to the computation time, as well as the overall complexity in the implementation of the code.

This idea has been revisited several times by several authors, however what I view as the biggest contributor is Gabor Toth (2000) where he cross compares several methods, including Powell's 8-wave solver. His conclusions are that the constrained transport can be realized using cell centred (natural to the problem) coordinates and values. This method involves a flux averaging using neighbouring cells. It is explained very well in "HARM ..." by Gammie et al.

Essentially the idea behind this was to extend the stencil for the higher dimensional magnetic field contributions. This allows for a complete cancellation of the approximated terms.

$$\nabla \cdot \vec{B} = 0 \tag{45}$$

in 2D takes on the form

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \tag{46}$$

or in finite difference form

$$\frac{B_x(i+1,j) - B_x(i,j)}{\Delta x} + \frac{B_y(i,j+1) - B_y(i,j)}{\Delta y} = 0 \tag{47}$$

or in the extended stencil

$$\frac{B_x(i,j) + B_x(i,j-1) - B_x(i-1,j) - B_x(i-1,j-1)}{2\Delta x} + \\ \frac{B_y(i,j) + B_y(i-1,j) - B_y(i,j-1) - B_y(i-1,j-1)}{2\Delta y} = 0 \tag{48}$$

Now if we look at the expansion of the magnetic flux term in the Godunov scheme;

$$\partial_t(\sqrt{\gamma}B^k)(i,j) = \partial_l(\sqrt{-g}F^{kl})(i,j)$$

Which became (after FVM)

$$B^{k,n+1} = B^{k,n} - \frac{\Delta x_0}{\Delta V^{(3)}} \sum_i \Delta\Sigma_j F_i^{k,j}(q) \tag{49}$$

NOTE: No source terms are expected on the induction equation (see later in this set of notes)
Let us focus on 2D $i \in [1,2]$

$$B^{k,n+1} = B^{k,n} - \frac{\Delta x_0}{\Delta V^{(2)}} \left(\Delta\Sigma_j F_1^{k,j} + \Delta\Sigma_j F_2^{k,j}\right) \tag{50}$$

$\Delta V^{(2)} = dx_1 dx_2$

$$B^{k,n+1} = B^{k,n} - \frac{\Delta x_0}{\Delta V^{(2)}} \left(\Delta\Sigma_{x_1 R} F_R^{k,x1} - \Delta\Sigma_{x_1 L} F_L^{k,x1} + \Delta\Sigma_{x_2 R} F_R^{k,x2} - \Delta\Sigma_{x_2 L} F_L^{k,x2}\right) \tag{51}$$

where the R and L denote the left and right evaluation of the flux in the cell. F^{kl} represents the flux for the magnetic field.

What is important here is that the approximation $\Delta\Sigma_{x_i} \sim \sqrt{-g}dx_j dx_k \epsilon^{ijk}$ carries a $\sqrt{-g}$ term which is to be evaluated on the surface of the cell interface. Thus it takes on the R, L notation used for the actual flux term. If we group this term in with the flux ie let $\mathcal{F} = \sqrt{-g}F$ then we have;

$$B^{k,n+1} = B^{k,n} - \frac{\Delta x_0}{\Delta V^{(2)}} \left(\Delta x_2 \mathcal{F}_R^{k,x1} - \Delta x_2 \mathcal{F}_L^{k,x1} + \Delta x_1 \mathcal{F}_R^{k,x2} - \Delta x_1 \mathcal{F}_L^{k,x2}\right) \tag{52}$$

Now, to prove something with the constraint, we take the divergence of the above equation. Since we assume the initial conditions are in agreement with the constraint (ie for $n = 0$ we demand that this be true) we expect

$$\partial_k(\sqrt{\gamma}B^k)^n = 0$$

identically. Also note:

$$\partial_0(\sqrt{\gamma}B^k) + \partial_i(\mathcal{F}^{ki}) = 0 \tag{53}$$
$$\partial_k\partial_0(\sqrt{\gamma}B^k) + \partial_k\partial_i(\mathcal{F}^{ki}) = 0 \tag{54}$$
$$\partial_0\partial_k(\sqrt{\gamma}B^k) + \partial_k\partial_i(\mathcal{F}^{ki}) = 0 \tag{55}$$
$$\partial_k\partial_i(\mathcal{F}^{ki}) = 0 \tag{56}$$

Where the last line is true simply due to antisymmetry of the Flux term, and the symmetry of the partial derivatives.

Thus we have that the divergence of the induction equation leads to a new form of the divergence constraint. This form is easily applied to the finite volume method since the fluxes are calculated at the cell interfaces.

Since the integral form of the equation preserves symmetry one expects that taking an average over flux interfaces will also preserve the symmetry.

Multiply the FVM treatment of the variables by the geometric term $\sqrt{\gamma}$;

$$\sqrt{\gamma}B^{k,n+1} = \sqrt{\gamma}B^{k,n} - \sqrt{\gamma}\frac{\Delta x_0}{\Delta V^{(2)}} \left(\Delta x_2 \mathcal{F}_R^{k,x1} - \Delta x_2 \mathcal{F}_L^{k,x1} + \Delta x_1 \mathcal{F}_R^{k,x2} - \Delta x_1 \mathcal{F}_L^{k,x2}\right) \tag{57}$$

Recall that the definition $\Delta V^{(3)} \sim \sqrt{\gamma}dx_1 dx_2 dx_3$ (in 3D) expanding this in the above expression cancels the "extra" geometric factor introduced with the generalized divergence. We are left with (again focusing on 2D);

$$\sqrt{\gamma}B^{k,n+1} = \sqrt{\gamma}B^{k,n} - \frac{\Delta x_0}{\Delta x_1 \Delta x_2} \left(\Delta x_2 \mathcal{F}_R^{k,x1} - \Delta x_2 \mathcal{F}_L^{k,x1} + \Delta x_1 \mathcal{F}_R^{k,x2} - \Delta x_1 \mathcal{F}_L^{k,x2}\right) \tag{58}$$

Now take the divergence of the above equation, keeping in mind that $\mathcal{F}^{ii} = 0$;

$$\partial_k(\sqrt{\gamma}B^k)^{n+1} = \partial_k(\sqrt{\gamma}B^k)^n - \frac{\Delta x_0}{\Delta x_1}\partial_{x_2}\left((\mathcal{F}^{x_1x_2})_R - (\mathcal{F}^{x_1x_2})_L\right) + \frac{\Delta x_0}{\Delta x_2}\partial_{x_1}\left((\mathcal{F}^{x_2x_1})_R - (\mathcal{F}^{x_2x_1})_L\right) \quad (59)$$

Now if we assume $\partial_k(\sqrt{\gamma})^n = 0$ which is demanded for $n = 0$ we are left with

$$\partial_k(\sqrt{\gamma}B^k)^{n+1} = \frac{\Delta x_0}{\Delta x_1}\partial_{x_2}\left((\mathcal{F}^{x_1x_2})_R - (\mathcal{F}^{x_1x_2})_L\right) + \frac{\Delta x_0}{\Delta x_2}\partial_{x_1}\left((\mathcal{F}^{x_2x_1})_R - (\mathcal{F}^{x_2x_1})_L\right) \quad (60)$$

Further defining $\mathcal{B} = \sqrt{\gamma}B$ the extended stencil becomes;

$$\frac{\mathcal{B}_{x_1}(i,j) + \mathcal{B}_{x_1}(i,j-1) - \mathcal{B}_{x_1}(i-1,j) - \mathcal{B}_{x_1}(i-1,j-1)}{2\Delta x_1} + \frac{\mathcal{B}_{x_2}(i,j) + \mathcal{B}_{x_2}(i-1,j) - \mathcal{B}_{x_2}(i,j-1) - \mathcal{B}_{x_2}(i-1,j-1)}{2\Delta x_2} = 0 \quad (61)$$

Where it is clear that if we group the geometric terms with the dynamic variables $\mathcal{B} = \sqrt{\gamma}B$ we recover the stencil for the cartesian coordinates in Minkowski space. Further noting that the flux terms are really unaffected by the new geometric factor. Thus it appears that the only averaging that needs to be performed in over the $\mathcal{F}$ fluxes and not any geometric terms associated with taking the divergence. (ie do not include an extra factor of $\sqrt{\gamma}$ in the flux terms before averaging.)

Performing the finite difference on the expanded stencil we have,

$$\begin{aligned}\partial_k(\sqrt{\gamma}B^k)^{n+1} = & \frac{\Delta t}{\Delta x}\left[\frac{1}{2\Delta y}\left(\mathcal{F}^{xy}_{i+\frac{1}{2},j} + \mathcal{F}^{xy}_{i-\frac{1}{2},j} - \mathcal{F}^{xy}_{i+\frac{1}{2},j-1} - \mathcal{F}^{xy}_{i-\frac{1}{2},j-1}\right)\right] - \\ & \frac{\Delta t}{\Delta x}\left[\frac{1}{2\Delta y}\left(\mathcal{F}^{xy}_{i-\frac{1}{2},j} + \mathcal{F}^{xy}_{i-\frac{3}{2},j} - \mathcal{F}^{xy}_{i-\frac{1}{2},j-1} - \mathcal{F}^{xy}_{i-\frac{3}{2},j-1}\right)\right] + \quad (62)\\ & \frac{\Delta t}{\Delta y}\left[\frac{1}{2\Delta x}\left(\mathcal{F}^{yx}_{i,j+\frac{1}{2}} + \mathcal{F}^{yx}_{i,j-\frac{1}{2}} - \mathcal{F}^{yx}_{i-1,j+\frac{1}{2}} - \mathcal{F}^{yx}_{i-1,j-\frac{1}{2}}\right)\right] - \quad (63)\\ & \frac{\Delta t}{\Delta y}\left[\frac{1}{2\Delta x}\left(\mathcal{F}^{yx}_{i,j-\frac{1}{2}} + \mathcal{F}^{yx}_{i,j-\frac{3}{2}} - \mathcal{F}^{yx}_{i-1,j-\frac{1}{2}} - \mathcal{F}^{yx}_{i-1,j-\frac{3}{2}}\right)\right] \quad (64)\end{aligned}$$

Finally we use the flux averaging explained in Gammie et al.

INSERT IMAGE FOR FLUX AVERAGING

We have the flux average for all cell boundaries folows the form:

$$\begin{aligned}{}^*\mathcal{F}^{xy}(i+\tfrac{1}{2},j) = & \\ \frac{1}{2}\Bigg(\frac{1}{4}\left(\mathcal{F}^{xy}(i+\tfrac{1}{2},j) + \mathcal{F}^{xy}(i-\tfrac{1}{2},j) + \mathcal{F}^{xy}(i,j+\tfrac{1}{2}) + \mathcal{F}^{xy}(i,j-\tfrac{1}{2})\right) + & \\ \frac{1}{4}\left(\mathcal{F}^{xy}(i+\tfrac{1}{2},j) + \mathcal{F}^{xy}(i+\tfrac{3}{2},j) + \mathcal{F}^{xy}(i+1,j+\tfrac{1}{2}) + \mathcal{F}^{xy}(i+1,j-\tfrac{1}{2})\right)\Bigg) & \quad (65)\end{aligned}$$

Which reduces to:

$$\begin{aligned}{}^*\mathcal{F}^{xy}(i+\tfrac{1}{2},j) = & \\ \frac{1}{8}\left(2\mathcal{F}^{xy}(i+\tfrac{1}{2},j) + \mathcal{F}^{xy}(i-\tfrac{1}{2},j) - \mathcal{F}^{yx}(i,j+\tfrac{1}{2}) - \mathcal{F}^{yx}(i,j-\tfrac{1}{2}) + \mathcal{F}^{xy}(i-\tfrac{1}{2},j) - \mathcal{F}^{yx}(i,j+\tfrac{1}{2}) - \mathcal{F}^{yx}(i,j-\tfrac{1}{2})\right) & \quad (66)\end{aligned}$$

likewise for ${}^*\mathcal{F}^{yx}(i,j+\frac{1}{2})$

Using Maple I veryified that these two expressions are identically zero, thus we remove the magnetic field constraint violations that occur due to the evolution.

10 Jacobian and Eigenvalues

From Font etal. we have explicit calculations for the eigenvalues for the Jacobian matricies. I list the results for the x-direction waves here:

The matter wave

$$\lambda_M = \alpha v^x - \beta^x$$

The Alfven wave

$$\lambda_{a\pm} = \frac{b^x \pm \sqrt{\rho h + b^2} u^x}{b^t \pm \sqrt{\rho h + b^2} u^t}$$

The magnetosonic waves are the solutions to a fourth order polynomial

$$N_a = \rho h \left(\frac{1}{c_s^2} - 1 \right) a^4 - \left(\rho h + \frac{b^2}{c_s^2} \right) a^2 G + B^2 G = 0$$

with

$$a = \frac{W}{\alpha} \left(-\lambda + \alpha v^x - \beta^x \right)$$

$$B = b^x - b^t \lambda$$

$$G = \frac{1}{\alpha^2} \left(-(\lambda + \beta^x) + \alpha^2 \gamma^{xx} \right)$$

and c_s is the speed of sound. This accounts for 7 of the 8 expected wave velocities. The last wave velocity is expected to be zero, since the magnetic field contributions from the direction in question are non-existant. (IE the B^x contributions in the x-direction do nothing)

What is important to note is that the above code calculations require the use of the given flux variables, and the conservative definitions and the lapse function, so to find these eignevalues we have:

the conservative variables F^0, the flux contributions F^i, and the primitive variables P, as a generalized eigenvalue problem

$$A_0 = \frac{\partial F^0}{\partial P}$$

$$A_1^i = \frac{\partial F^i}{\partial P}$$

$$A_1^i \vec{q} = \lambda \alpha A_0 \vec{q}$$

Note in numerical routines, typical eigenvectors returned need to be multiplied by the A_0 matrix to return the eigenvectors to the overall system.

Also note that although the papers I cite use a Jacobian that includes the lapse function, one can simply multiply the resulting eigenvalues by this lapse if critical, most of the time one does not wish to do this as flux averaging techniques tend to want to divide the eigenvalues by the laspe, leading me to believe it is best to not include them in the first place. It does not help the code much, and certainly does not save any computation time.

11 Iterative Procedure

One of the more challenging problems facing GRMHD was the recovery of the primitive variables. In the Newtonian code these are known in closed form however due to the lorentz factor we are left to use an iterative procedure for the recovery in this system. I will give a brief outline of Del Zanna's 1D conservative to primitive variable solver here. For more details please see his paper.

We set $\Omega = \rho h \gamma^2$, and $\vec{Q} = \gamma_{ij} S^j B^i$

Then we have a new form for the energy equation

$$\tau = E - D = \Omega - P + \left(1 - \frac{1}{2W^2} \right) |B|^2 - \frac{Q^2}{2\Omega^2} - D \qquad (67)$$

and the momentum equation

$$|S|^2 = (\Omega + |B|^2)^2 \left(1 - \frac{1}{\gamma^2}\right) - \frac{Q^2}{\Omega^2}(2\Omega + |B|^2) \tag{68}$$

Re-writing the momentum equation we solve for W

$$W = \left[1 - \frac{Q^2(2\Omega + |B|^2) + |S|^2\Omega}{(\Omega + |B|^2)^2\Omega^2}\right]^{-\frac{1}{2}} \tag{69}$$

We replace the pressure term using the equation of state:

$$P(\Omega) = \frac{(\Omega - DW)(\Gamma - 1)}{\Gamma W^2} \tag{70}$$

Finally we can start using Newton's procedure using the energy equation

$$f(\Omega) = \Omega - P + \left(1 - \frac{1}{2W^2}\right)|B|^2 - \frac{Q^2}{2\Omega^2} - D - \tau = 0 \tag{71}$$

and we require the first derivative

$$\frac{df}{d\Omega} = 1 - \frac{dP}{d\Omega} + \frac{|B|^2}{W^3}\frac{dW}{d\Omega} + \frac{Q^2}{\Omega^2} \tag{72}$$

with

$$\frac{dP}{d\Omega} = \frac{\left(W\left(1 + D\frac{dW}{d\Omega}\right) - 2\Omega\frac{dW}{d\Omega}\right)(\Gamma - 1)}{\Gamma W^3} \tag{73}$$

$$\frac{dW}{d\Omega} = -W^3\frac{2Q^2\left(3\Omega^2 + 3\Omega|B|^2 + |B|^4\right) + |S|^2\Omega^3}{2\Omega^3\left(\Omega + |B|^2\right)^3} \tag{74}$$

Solving this with an initial guess of $\Omega = 500$ appears to work well for all cases (ultra relativistic, zero velocity, moderately relativistic,and Newtonianesque).

With the value of Ω determined we can solve for the primitive variables. This is done by re-arranging the above expressions, and of course using the last value of Ω obtained.

$$v_k = \frac{1}{\Omega + |B|^2}\left(S_k + \frac{Q}{\Omega}B_k\right) \tag{75}$$

$$P(\Omega) = \frac{(\Omega - DW)(\Gamma - 1)}{\Gamma W^2} \tag{76}$$

$$W = \left[1 - \frac{Q^2(2\Omega + |B|^2) + |S|^2\Omega}{(\Omega + |B|^2)^2\Omega^2}\right]^{-\frac{1}{2}} \tag{77}$$

$$\rho_o = \frac{D}{W} \tag{78}$$

Note that the magnetic field components are unaltered. They are both primitive and conservative variables.

What is also convenient and quite expected is that this procedure reduces to the known procedure for the GRhydro codes when the magnetic field is turned off.

12 Das Floor, and other numerical restrictions

For relativistic fluids one is in consistent danger of producing either negative pressures (usually arising from negative energy) or negative baryon densities. To circumvent this issue we use a numerical floor. This floor is typically imposed on the conservative variables, before calculating the primitives. What seems to be the case is that different floors can be used on the energy than the baryon density, by a few orders of magnitude. Several authors have investigated this behaviour and found that the floor used in the hydrodynamic code (on order of 10^{-11} or smaller) had little effect on their results. For the MHD case the problem is a little more apparent due to the constibutions from the magnetic field (obviously I suppose), McKinney et al. inverstigated this problem in their paper (2006 - include in references), and found that the floor is a little more restrictive, on the order of 10^{-7}. Specifically in this paper they also made the floor dependent on the radial distance from the origin, with significan justification. Further, they found that this floor had little bearing on the computed results.

Author's note: It is important to understand that the floor is not a physical quantitiy, but rather a numerical control. It has been observed that this quantity is only necessary in what are considered ultrarelativistic velocities. (very close to the speed of light) In astro papers one must watch out for the language used. In several papers they refer to superluminal speeds, which is the same as saying ultrarelativistic as defined above, in GR type journals superluminal appears to commonly mean "exceed the speed of light".

Numerical techniques may also lead to non-zero values where one clearly expects proper zero. Of course this usually corresponds to rounding errors, and lack of "infinite precision", (such as trying to calculate $\cos \pi/2 = 0$ for a numerical value of pi). This can cause catastrophic effects when trying to perform such tasks as inverting matricies, and trying to calculate a Newton Iterative procedure. The need for a tolerance parameter origninally arose when inverting matricies to calculate the eigenvectors for the Roe approximation in the MHD code. The close to zero values which were well distributed in the matrix (near zero meaning $\sim 10^{-16}$), which when inverting the matrix returned extremely high values ($\sim 10^{200}$) which of course are incorrect. Further inspection indicated that theoretically these small values were zero, but numerically these were "essentially" zero, or at least below machine precision. The introduction of a tolerance parameter to replace the "essentially zero" elements with exactly zero elements, did the trick. This does lead to some troubling concerns, however investigation upon investigation reveals that this tolerance parameter has no effect on the results, unless of course it is too liberal in defining zero, or strict beyond machine precision (in which case the computer will never use the tolerance anyways). I found that an acceptible value of tolerance is around 10^{-13}. This means that the resolution of the results is unknown below this tolerance, however as more numerical techniques are developed, and the precision of the standard compilers (right now 32bit is the standard) increases this tolerance should diminish and ideally zero tolerance will be reached.

As mentioned the tolerance is also useful for defining a limit on the Newton solver. This tolerance can be played with a little, and could in principle be reduced below the precision of the matrix tolerance. There is some advantage to this, first it ensures that the matrix tolerance is the limiting precision. Secondly it has been experimentally verified (by this author) that the lower this tolerance is the more likely the primitive variable solver will return non-superluminal velocities. This is accomplished by scaling the tolerance by a few orders of magnitude as it is applied to the solver. One does have to be careful since if tolerance is too low, one can get trapped in the Newton solver forever.

13 Equation of State

The given MHD equations do not form a closed set, to further specify the system we are required to select an equation of state. $P \to P(\rho)$. For the relativistic gases I chose the ideal gas $P = (\Gamma - 1)\rho\epsilon$. And $h = 1 + \epsilon + \frac{P}{\rho}$

14 Dimensionless Parameters

In hydrodynamics we are faced with several dimensionless parameters such as:

- Reynolds Number
- Prandtl Number

- Mach Number
- Knudsen Number
- Richardson Number
- Strouhal Number

The Reynolds Number is the ratio of the mean fluid velocity, the characteristic Length (cross section). In our system this number tends to infinity, so the fluid viscosity is zero.

The Prandtl number is the ratio of the momentum of diffusivity and the thermal diffusivity. In ideal fluids the Prandtl number is not a factor in the system, since momentum diffusivity is zero.

Mach number is the ratio of the velocity of the fluid and the speed of sound in the fluid. This is a measurable observable

Knudsen number is the ratio of the mean free path and the length scale of the fluid. This length scale could be the diameter of the fluid body (for example). The Knudsen number determines whether continuum mechanics or statistical mechanics dominates for a fluid. In our case the Knudsen number is large since we are assuming tightly packed particles. If this is violated, say in a situation where pressure and density are both far too low, the MHD approximation will breakdown and the results are no longer reliable.

Richardson number is the ratio of the gravitational potential energy to the fluid kinetic energy. Since in our case I treat gravity through the geometry I do not include a source term representing gravity. This number is also zero. Buoyancy is irrelevant. This number can be used to measure turbulence.

Strouhal number is used to measure the oscillating flow of the fluid around some obstacle in the trajectory., Again, since I have not implemented any obstacles, this number is of little value. In the future developement of code, I will reconsider this number.

Likewise magnetohydrodynamic systems have dimensionless parameters:

- Magnetic Reynolds Number
- Beta
- Alfven Number

The magnetic Reynolds number is the ratio of the characteristic velocity times the length scale, with respect to the magnetic diffusivity (inversely proportional to the conductance). In our case the magnetic Reynolds number is infinite, so the diffusive terms are negligible.

The β is the ratio of the magnetic pressure and the hydrodynamic pressure. This is a measurable observable.

The Alfven number is the ratio of the Alfven speed to the characteristic speed of the system. This is also a measurable observable.

15 Trials

15.1 Minkowski Spacetime As a Background using Cartesian coordinates

For this spacetime we have the line element:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \tag{79}$$

so the metric takes on the standard special relativistic form $\eta_{\mu\nu}$, this makes quick work of our equations as seen below. The lapse function is identically unity ($\alpha = 1$), and the shift vector elements are all zero $\left(\beta^i = 0\right)$. Thus 3+1 is fairly straight forward in this spacetime. Further $\sqrt{-g} = \sqrt{\gamma} = 1$, and the Christoffel symbols are zero.

We get the conserved quantities:

$$q = \begin{bmatrix} D \\ S_j \\ \tau \\ B^k \end{bmatrix} \tag{80}$$

along with these one gets the corresponding flux terms:

$$f(q) = \begin{bmatrix} Dv^i \\ S_j v^i + P\delta^i_j - \frac{b_j B^i}{W} \\ \tau v^i + Pv^i - (B^j v_j)B^i \\ b^k v^i - v^k b^i \end{bmatrix} \tag{81}$$

There are no source terms in the case of cartesian coordinates.

With this case I was able to produce turbulence given periodic boundary conditions and with help from Jim Stone's website. I was able to commit to several test initial conditions such as the rigid rotor, the blast wave. All of which were originally produced for Newtonian MHD.

PLACE SNAPSHOTS HERE for all three cases.

There were a few interesting developements when including the magnetic field. The first and most leading was when I applied the rigid rotor test, and included a constant magnetic field in one direction (here the B_x direction). One could easily observe the flattening of the rotating fluid, a dramatic display of the effects of invisible fields.

Primarily this was intended to be a test setup, however there are several known problems with unknown solutions for which the SRMHD code is sufficient to test.

16 Spacetimes

16.1 Minkowski in Spherical Coordinates

Before moving on to the Kerr metric I implement the curved coordinates, well that is to say, generalized coordinates. This will allow the shift and lapse functions to take on the same values as in the flat space however there are geometric terms that have to come into play. This was by-in-large more of a simplifying step in code developement than anything ground breaking. But as expected in the SRMHD code this may be useful for the developement of routines to check conserved or convergent quantities.

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2 \tag{82}$$

For simplicity I look at $\theta = \pi/2$

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 \tag{83}$$

The Christoffel symbols are not all zero in general.

From here we get the conserved quantities:

$$q = \begin{bmatrix} D = \rho W \\ S_i = (\rho_o h + b^2)W^2 v_i - b^t b_i \\ \tau = (\rho_o h + b^2)W^2 - (P + \frac{1}{2}b^2) - (b^t)^2 - D \\ B^k = B^k \end{bmatrix}$$

We are then left with the following flux vector elements:

$$f(q) = \begin{bmatrix} Dv^i \\ S_j v^i + P\delta^i_j - \frac{b_j B^i}{W} \\ \tau v^i + Pv^i - \frac{b^t B^i}{W} \\ B^k v^i - v^k B^i \end{bmatrix} \tag{84}$$

And the source terms

$$\Sigma(q) = \begin{bmatrix} 0 \\ T^{\mu\nu}\left(g_{\nu j,\mu} - \Gamma^\lambda_{\mu\nu} g_{\lambda j}\right) \\ -\Gamma^t_{\mu\nu} T^{\mu\nu} \\ 0 \end{bmatrix} \tag{85}$$

Where the Christoffel symbols are close to trivial.

Although, due to symmetry considerations, this is a 1D problem, it allowed for a more robust test of the code since the two dimensions considered in this problem (r,ϕ) were not interchangeable as they are in cartesian coordinates.

16.2 Schwarzschild Spacetime As a Background

For the Schwarzschild spacetime in spherical coordinates we have the line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{86}$$

M is the mass of the static object in geometric units. This metric contains the well known singularities at the origin and the event horizon $R = 2M$. Clearly this will become an issue in the future, however if one avoids the event horizon, this provides the simplest metric which has a nontrivial lapse, and a trivial shift.

From here we get the conserved quantities:

$$q = \begin{bmatrix} D = \rho W \\ S_i = (\rho_o h + b^2)W^2 v_i - \alpha b^t b_i \\ \tau = (\rho_o h + b^2)W^2 - (P + \frac{1}{2}b^2) - (\alpha b^t)^2 - D \\ B^k \end{bmatrix}$$

We have the corresponding flux terms:

$$f(q) = \begin{bmatrix} Dv^i \\ S_j v^i + P\delta^i_j - \frac{b_j B^i}{W} \\ \tau v^i + Pv^i - \alpha\frac{b^t B^i}{W} \\ B^k v^i - v^k B^i \end{bmatrix} \tag{87}$$

$$\Sigma(q) = \begin{bmatrix} 0 \\ T^{\mu\nu}\left(g_{\nu j,\mu} - \Gamma^\lambda_{\mu\nu} g_{\lambda j}\right) \\ \alpha\left(T^{\mu t}\left(\ln\alpha\right)_{,\mu} - \Gamma^t_{\mu\nu} T^{\mu\nu}\right) \\ 0 \end{bmatrix} \tag{88}$$

This spacetime has a few interesting features even well outside of the event horizon. If one wishes to demonstrate the effects of a black hole on the surrounding fields, just setup a static system with zero initial velocity (IE pressure and density are the only non-zero fields). As the system evolves one can watch the velocity of the system increase in the radial direction, and if all is working properly, there will be no evolution of any other element of the velocity vector field. Again, I am walking you through the basic tests for developing fluid code. Another good test is if we set $M = 0$, we had better recover the Minkowski spacetime evolution.

As mentioned this metric has a coordinate singularity at $R = 2M$ so a more appropriate set of coordinates will need to be selected for evolution up to and including the event horizon. Coding issues definitely arise as one tends near the event horizon, especially if one performs the given magnetic field averaging, where the use of ghost cells is required. For the sake of credibility the Eddington Finkelstein coordinates will need to be implemented.

The Eddington-Finkelstein line element is $ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\theta + r^2 \sin^2\theta d\phi$

16.3 Kerr Spacetime As a Background

For the Kerr spacetime in Boyer-Linquist coordinates we have the line element:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin(\theta)^2}{\Sigma} dtd\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2a^2 Mr\sin(\theta)^2}{\Sigma}\right)\sin(\theta)^2 d\phi^2 \tag{89}$$

Where we use

$$\Sigma = r^2 + a^2\cos(\theta)^2$$

$$\Delta = r^2 - 2Mr + a^2$$

M is the mass of the rotating object $a = \frac{J}{M}$ is the rotation of the black hole, with angular momentum J as before we use Geometric units.

This metric requires all elements of the 3+1 decomposition, so I will not restate the fluid equations.

As with the Schwarzschild metric we have signularities at the event horizon that can be circumvented by use of the Kerr-Schild coordinates. This metric leaves us with a line element

$$ds^2 = -\left(1 - \frac{2M}{r_s}\right) dt^2 + \frac{4M}{r_s^2} dtdr_s + \left(1 + \frac{2M}{r_s}\right) dr_s^2 + r_s^2 d\theta + r_s^2 \sin^2\theta d\phi$$

with

$$r_s = r\left(1 + \frac{M}{2r}\right)$$

clearly the singularity at the origin still exists, but since that is hidden by the event horizon, it is of little concern.

17 Vorticity and Helicity

The 4-vorticity of a hydrodynamic system is defined to be

$$\omega^\alpha = \epsilon^{\alpha\beta\gamma\delta} u_\beta u_{\delta;\gamma}$$

For the magnetic field case, the vorticity analogue is the magnetic field itself.

$$B^\alpha = \epsilon^{\alpha\beta\gamma\delta} u_\beta A_{\delta;\gamma}$$

With these (typically proven the other way around) we have the vorticity pseudotensors.

$$\omega^{\alpha\beta} = (hu_\beta)_{,\alpha} - (hu_\alpha)_{,\beta}$$

and

$$F^{\alpha\beta} = (A_\beta)_{,\alpha} - (A_\alpha)_{,\beta}$$

From Bekenstein we know of three helicity relations that are expected to be globally conserved:

The fluid helicity

$$H_f^\alpha = {}^*\omega^{\alpha\beta}(hu_\beta)$$

The magnetic helicity

$$H_m^\alpha = {}^*F^{\alpha\beta} A_\beta$$

The magnetic fluid helicity (cross helicity)

$$H_{fm}^\alpha = {}^*F^{\alpha\beta}(hu_\beta) = hB^\alpha$$

where h is the relativistic enthalpy as defined in an earlier section. He follows each of these with a proof of their conservation which is trivially true in the case of isentropic evolution, fortunately my system is treated as isentropic.

18 Turbulence

I will get to this material as soon as I can.

Here are a list of a few of the papers I used to get my research moving. All of these (minus the textbooks) are available on my webpage http://laplace.physics.ubc.ca/Members/ajpenner/MHD.html

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