PHYS 170 Section 101
Lecture 29
November 19, 2018

Nov 19—Announcements

- Evaluation of teaching assistants during next (final) tutorial, tomorrow, Tuesday, November 20
  - Bring lead pencil to fill out Scantron forms
  - Come prepared with comments as appropriate

Lecture Outline/Learning Goals

- Worked example using conservation of energy
- Begin Chapter 15—Kinetics of a Particle: Impulse and Momentum
- 15.1 Principle of Linear Impulse and Momentum
- 15.2 Principle of Linear Impulse and Momentum for a System of Particles
- 15.3 Conservation of Linear Momentum for a System of Particles

Problem 14-100 (Page 215, 12th edition)

The 1.5 kg collar *C* is released from rest at *A* and travels along the smooth vertical guide. The unstretched length of the spring is 0.1 m. The spring constant is 100 N/m.

Determine the following:

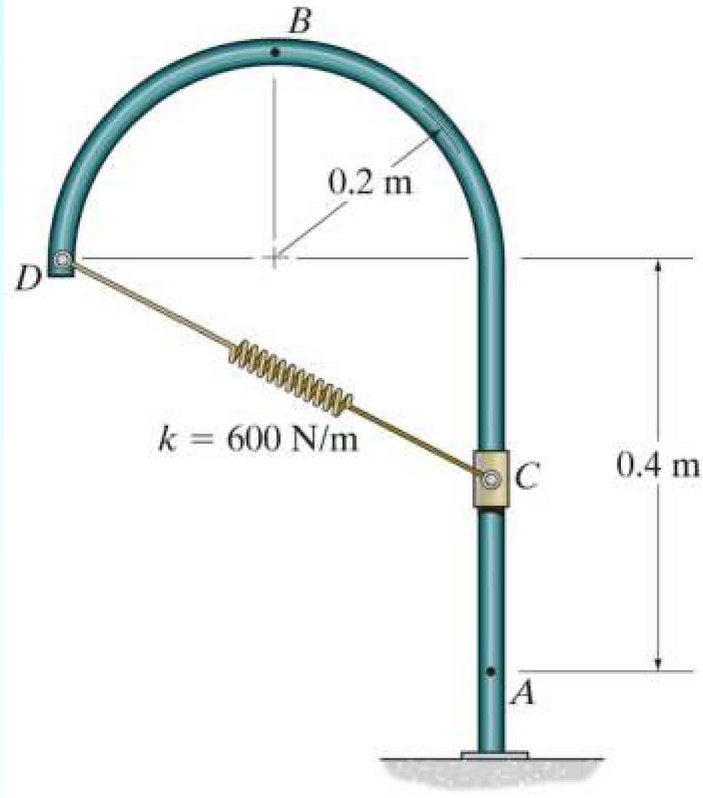
(1) The normal force exerted on the collar at *A*

(2) The acceleration of the collar at *A*

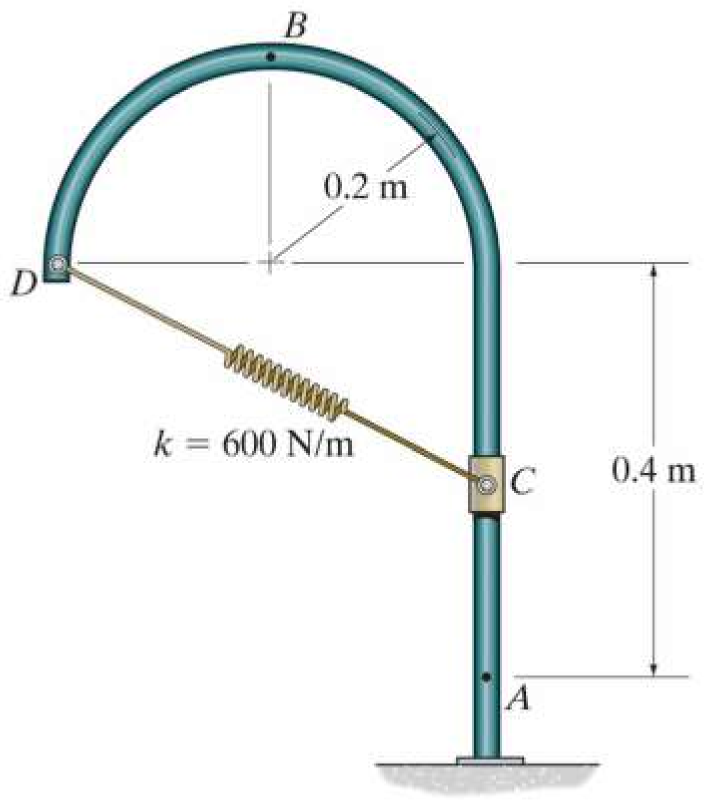
(3) The speed of the collar at *B*

(4) The tangential and normal components of the acceleration at *B*

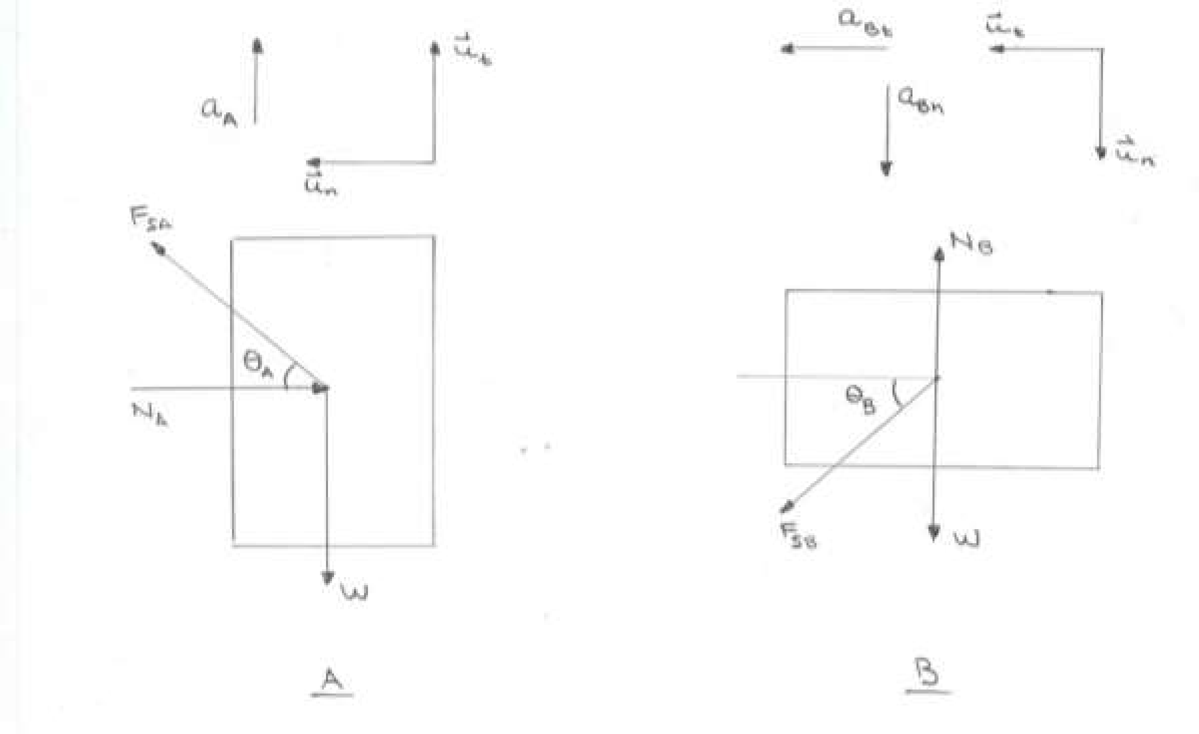
(5) The normal force exerted on the collar at *B*

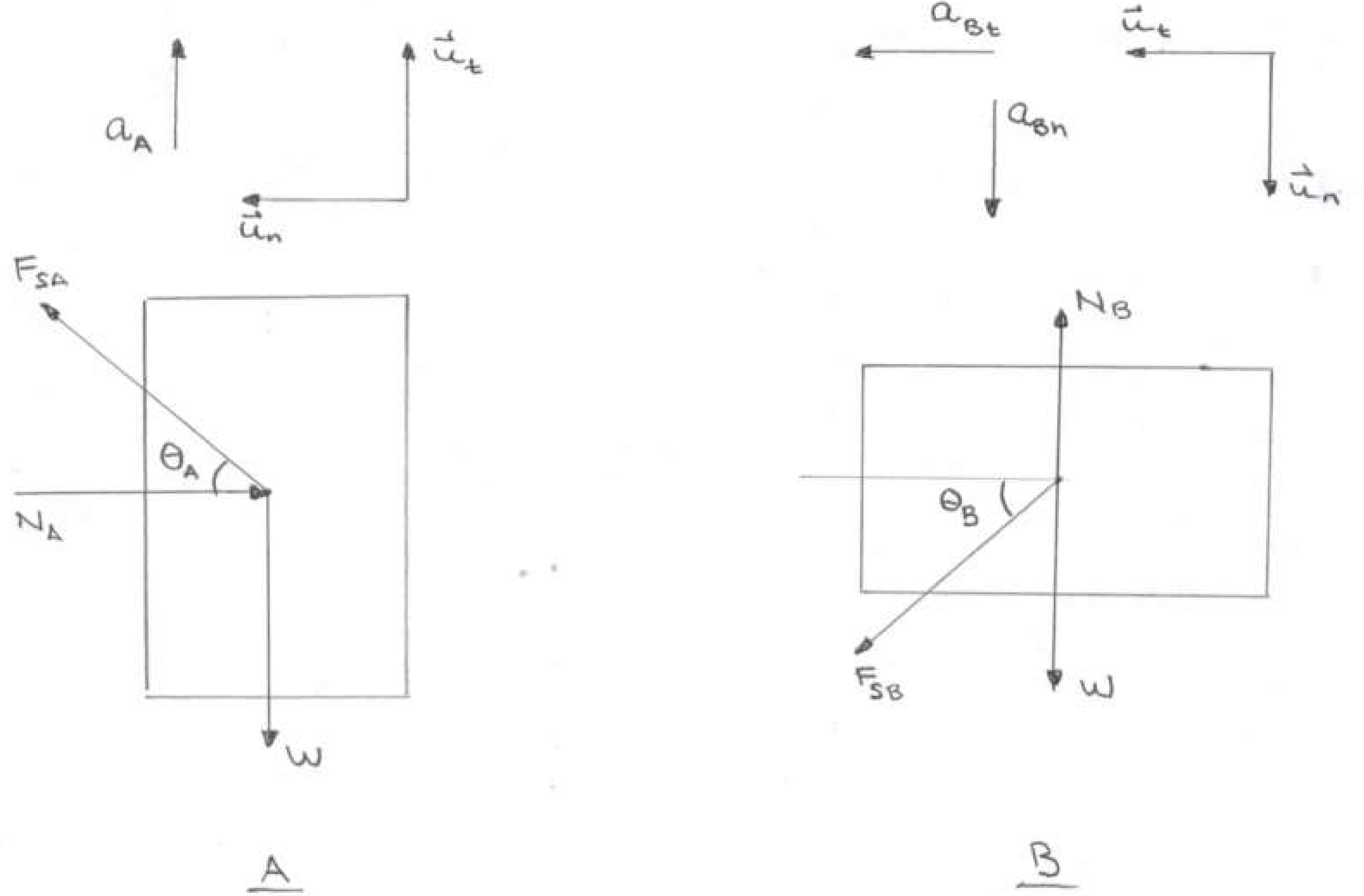


PROB14_100.jpg

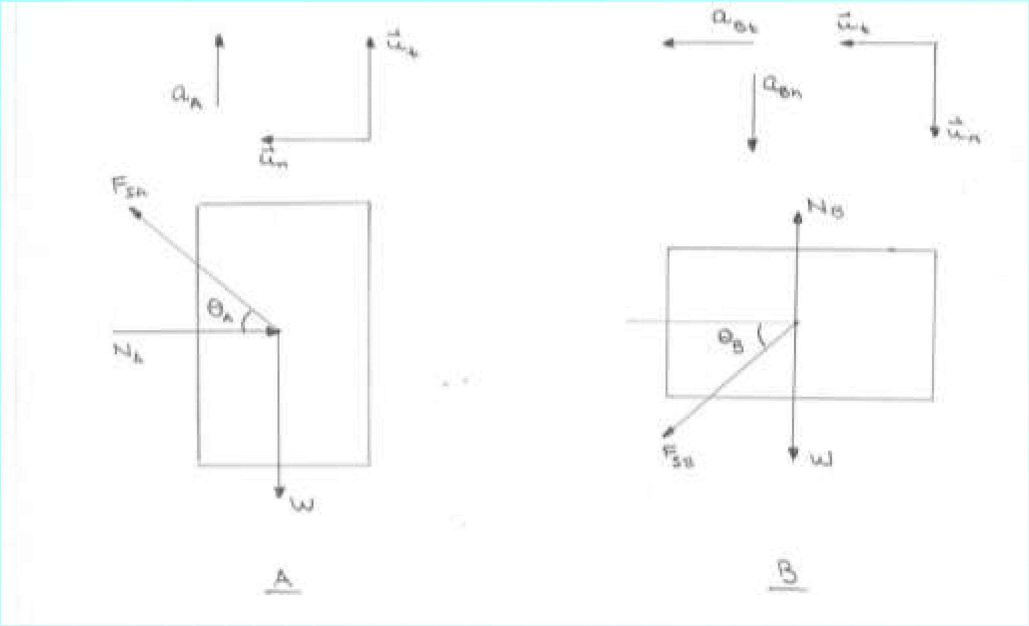


PROB14_100.jpg





a_A
$\vec{u}_t$
$\vec{u}_n$
F_{SA}
θ_A
N_A
W
A
a_{Bt}
$\vec{u}_t$
a_{Bn}
$\vec{u}_n$
N_B
θ_B
F_{SB}
W
B



Solution strategy

Use equations of motion (tangential/normal) at *A* to determine normal force and acceleration

Use conservation of energy to determine speed of block at *B*

Use equations of motion to determine acceleration components and normal force at *B*

Data

$m = 1.5 \text{ kg}$ $\quad r_0 = 0.1 \text{ m}$ $\quad k = 100 \text{ N/m}$

$v_A = 0 \text{ m/s}$ $\quad h_A = 0 \text{ m}$

$\theta_A = 45^\circ$ $\quad r_A = \sqrt{0.4^2 + 0.4^2} \text{ m}$

$R = 0.2 \text{ m}$ $\quad h_B = 0.6 \text{ m}$

$\theta_B = 45^\circ$ $\quad r_B = \sqrt{0.2^2 + 0.2^2} \text{ m}$

Normal force and acceleration at *A*

$$\sum F_n = ma_n : \qquad -N_A + k(r_A - r_0)\cos\theta_A = 0$$

$$\sum F_t = ma_t : \qquad k(r_A - r_0)\sin\theta_A - mg = ma_A$$

Solving: $N_A = 32.9 \text{ N}$ $\qquad a_A = 12.1 \text{ m/s}^2$

Data

$m = 1.5$ kg $r_0 = 0.1$ m $k = 100$ N/m

$v_A = 0$ m/s $h_A = 0$ m

$\theta_A = 45^\circ$ $r_A = \sqrt{0.4^2 + 0.4^2}$ m

$R = 0.2$ m $h_B = 0.6$ m

$\theta_B = 45^\circ$ $r_B = \sqrt{0.2^2 + 0.2^2}$ m

Speed at *B*

Conservation of Energy

$$\frac{1}{2}mv_A^2 + mgh_A + \frac{1}{2}k(r_A - r_0)^2 = \frac{1}{2}mv_B^2 + mgh_B + \frac{1}{2}k(r_B - r_0)^2$$

Solving: $v_B = 0.676$ m/s

Accelerations and normal force at B

$$\sum F_t = ma_t \; : \qquad k(r_B - r_0)\cos\theta_B = ma_{Bt}$$

$$\sum F_n = ma_n \; : \qquad -N_B + mg + k(r_B - r_0)\sin\theta_B = ma_{Bn}$$

Normal component of acceleration

$$a_{Bn} = \frac{v_B^2}{R}$$

Solving: $a_{Bt} = 8.62 \text{ m/s}^2$ $\quad a_{Bn} = 2.28 \text{ m/s}^2$ $\quad N_B = 24.2 \text{ N}$

What's 2 + 2?

ENGINEER: "It lies between 3.98 and 4.02."

MATHEMATICIAN: "In 2 hours I can demonstrate it equals 4 with the following proof."

PHYSICIST: "It's in the magnitude of 1×10^1."

LOGICIAN: "This problem is solvable."

SOCIAL WORKER: "I don't know the answer, but I'm glad we discussed this important question."

ATTORNEY: "In the case of Smith vs State, 2 + 2 was declared to be 4."

TRADER: "Are you buying or selling?"

ACCOUNTANT: "What would you like it to be?"

Chapter 15—Kinetics of a Particle: Impulse and Momentum

© David J. Green/Alamy

15_COC01

The design of the bumper cars used for this amusement park ride requires knowledge of the principles of impulse and momentum.

15.1 Principle of Linear Impulse & Momentum

- Recall that in the previous chapter, we integrated the tangential component of Newton's second law with respect to displacement, and used the kinematic equation $a_t\, ds = v\, dv$ to derive one form of the principle of work & energy

$$\Sigma \int_{s_1}^{s_2} F_t\, ds = \Sigma U_{1-2} = \int_{v_1}^{v_2} v\, dv = \frac{1}{2} m {v_2}^2 - \frac{1}{2} m {v_1}^2$$

- We noted that the PWE was useful in solving problems involving force, velocity and displacement, and that in particular, its use could save time/effort when compared to a solution approach involving a direct treatment using the equations of motion

- We will now proceed along a similar route, but this time will integrate the second law with respect to time; this will yield a principle that is useful in solving problems involving force, velocity and **time**, again, particularly in terms of saving some effort relative to straightforward application of the equations of motion (however, due to time constraints won't actually solve any problems of this sort—in lectures, homework or exams)

Principle of Linear Impulse and Momentum (continued)

- We thus recall that the equation of motion for a particle of mass *m* is

$$\Sigma \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$$

where the velocity, **v**, and acceleration, **a**, must both be measured in an inertial frame

- We rearrange this equation as

$$\Sigma \mathbf{F}\, dt = m d\mathbf{v}$$

and integrate both sides over some **time** interval $t_1 \le t \le t_2$, such that

$$\mathbf{v}(t_1) = \mathbf{v}_1$$
$$\mathbf{v}(t_2) = \mathbf{v}_2$$

Principle of Linear Impulse and Momentum (continued)

- Thus, we have

$$\Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m \int_{\mathbf{v}_1}^{\mathbf{v}_2} d\mathbf{v}$$

or

$$\Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

- This last equation is one form of the **principle of linear impulse and momentum**

- Before discussing it in a little more detail, let us make a slight digression to define the concept of a particle's **linear momentum**

Principle of Linear Impulse and Momentum (continued)

Linear Momentum (defn): For a particle with mass, m, and velocity, $\mathbf{v}$ (as measured in an inertial frame), the linear momentum, $\mathbf{L}$, of the particle is

$$\mathbf{L} = m\mathbf{v}$$

- Note that $\mathbf{L}$ is a **vector** quantity, and that since m is a positive scalar, the direction of $\mathbf{L}$ will always be identical to that of $\mathbf{v}$

- **Units for Linear Momentum (no special units)**
  - **SI:** kg · m/s
  - **FPS:** slug · ft/s

Principle of Linear Impulse and Momentum (continued)

- Thus, we see that the equation

$$\Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

can also be written as

$$\Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 - \mathbf{L}_1 = \Delta\mathbf{L}$$

and thus relates the time integral of the resultant force acting on the particle to the change, $\Delta\mathbf{L}$, in the momentum of the particle over that time interval

- Perhaps not surprisingly, the time integral of a force is also generally identified as a quantity worthy of special consideration – and it known as the **linear impulse** of the force

Principle of Linear Impulse and Momentum (continued)

Linear Impulse (defn): Given a generally time-dependent force, $\mathbf{F}(t)$, and a time interval $t_1 \le t \le t_2$ over which the force acts, the linear impulse, $\mathbf{I}$, of the force over that time interval is given by

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}(t)\, dt$$

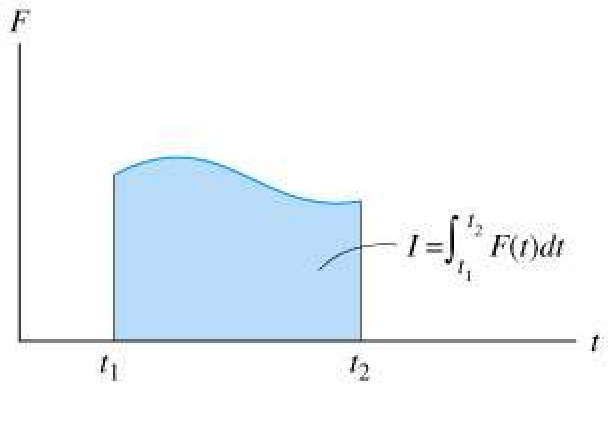
- Note that $\mathbf{I}$ is also a **vector** quantity, but one whose direction will not usually be able to be identified immediately (i.e. without explicitly performing the integral)

- **Units for Linear Impulse (no special units)**
  - **SI:** $\mathrm{N} \cdot \mathrm{s}$
  - **FPS:** $\mathrm{lb} \cdot \mathrm{s}$

EXERCISE: Show that the units for linear momentum and linear impulse are equivalent in both the SI and FPS systems

Principle of Linear Impulse and Momentum (continued)

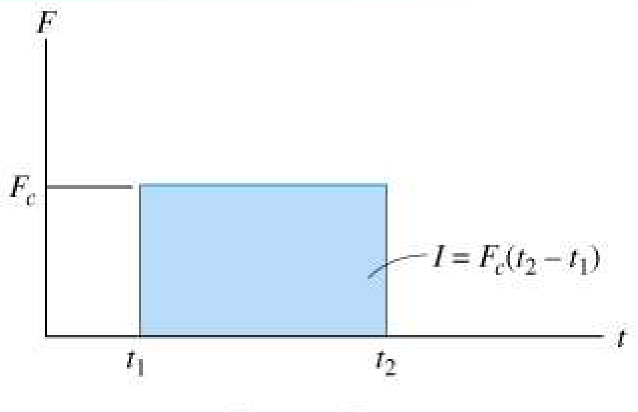
- **Linear Impulse interpretation:** Since $\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}(t)\, dt$, and assuming that the force acts in a single direction, we see that the magnitude of the impulse can be interpreted as the area under the $F(t)$ curve as shown in the figure



Variable Force

- For the special, but important case, where a force $\mathbf{F}_c$ has both constant magnitude and direction, the impulse becomes

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}_c\, dt = \mathbf{F}_c (t_2 - t_1)$$



Constant Force

Principle of Linear Impulse and Momentum (continued)

- Now, recall that we had

$$\Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

- We now rearrange this equation to get a version of the principle of linear impulse and momentum that is especially convenient for problem solving

$$m\mathbf{v}_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2$$

- In words: The initial momentum of a particle plus the total impulse acting on the particle is equal to the final momentum of the particle

Scalar Equations

- The two versions of the principle of linear impulse and momentum are **vector** equations

- As usual, we can also formulate a scalar version of the equation, which in the case that the vectors are expressed in terms of their components in a rectangular (Cartesian) inertial coordinate system (x,y,z) is

$$m(v_x)_1 + \Sigma \int_{t_1}^{t_2} F_x dt = m(v_x)_2$$

$$m(v_y)_1 + \Sigma \int_{t_1}^{t_2} F_y dt = m(v_y)_2$$

$$m(v_z)_1 + \Sigma \int_{t_1}^{t_2} F_z dt = m(v_z)_2$$

- Thus, as is to be expected, the principle applies separately in each of the three rectangular directions x, y and z.

15.2 Principle of Linear Impulse & Momentum for a System of Particles

DIGRESSION: Center of Mass for a System of Particles (Statics, Ch 9)

- Have very briefly referred to this concept (center of mass) previously in the course: since it is quite useful in understanding the nature of the (total) linear momentum of a system of particles, we now need to give it a precise definition

Center of Mass (defn): Consider a system of particles with masses m_i and position vectors $\mathbf{r}_i$. Then the **center of mass** of the system is a location given by a position vector, $\mathbf{r}_G$, which can be computed using

$$\mathbf{r}_G = \frac{\Sigma m_i \mathbf{r}_i}{\Sigma m_i}$$

DIGRESSION: Center of Mass for a System of Particles (continued)

- As is typically the case for a vector formula such as the above, we can also write an equivalent definition in terms of the x, y and z components of the various position vectors, i.e.

$$x_G = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

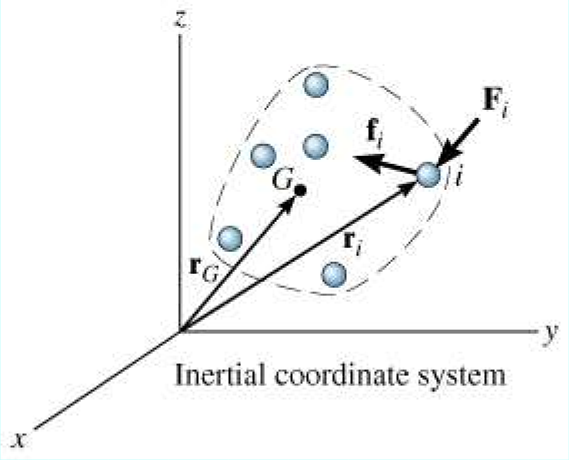
$$y_G = \frac{\Sigma m_i y_i}{\Sigma m_i}$$

$$z_G = \frac{\Sigma m_i z_i}{\Sigma m_i}$$

- Finally, note that we can take time derivatives of the above formulae, to get, for example, an equation for the velocity, $\mathbf{v}_G$, of the center of mass of a system

$$\mathbf{v}_G = \frac{d\mathbf{r}_G}{dt} = \frac{\Sigma m_i (d\mathbf{r}_i / dt)}{\Sigma m_i} = \frac{\Sigma m_i \mathbf{v}_i}{\Sigma m_i} = \frac{\text{total linear momentum}}{\text{total mass}}$$

Principle of Linear Impulse & Momentum for a System of Particles



- Now consider a system of particles as shown in the figure, where we have

  $\mathbf{F}_\mathbf{i}$ = resultant external force acting on particle i

  $\mathbf{f}_i$ = resultant internal force acting on particle i

  $\mathbf{r}_i$ = position vector of particle i

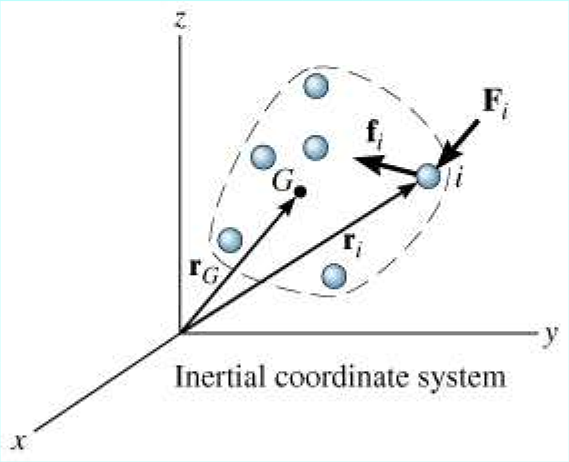
  and for the system as a whole

  $\mathbf{r}_G$ = position vector of center of mass

- Applying Newton's second law to each particle individually, and then summing over all of the particles we have (note that following the text's convention, summation symbols generally apply to all terms with subscripts that follow them)

$$\Sigma\ \mathbf{F}_\mathbf{i} + \mathbf{f}_\mathbf{i} = \Sigma m_i \frac{d\mathbf{v}_i}{dt}$$

Principle of Linear Impulse & Momentum for a System of Particles (cont.)



- Now, since the internal forces occur in action-reaction pairs, $\Sigma \mathbf{f}_i = 0$, so we have

$$\Sigma \mathbf{F_i} = \Sigma m_i \frac{d\mathbf{v}_i}{dt}$$

or

$$\Sigma \mathbf{F_i} dt = \Sigma m_i d\mathbf{v}_i$$

- Integrating this last equation, precisely as we did previously for the case of a single particle, we get the principle of linear impulse and momentum for a system of particles

$$\Sigma m_i (\mathbf{v}_i)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt = \Sigma m_i (\mathbf{v}_i)_2$$

(Total) Initial linear momentum of system + Sum of impulses due to all external forces = (Total) final linear momentum

Principle of Linear Impulse & Momentum for a System of Particles (cont.)

- Last equation from previous slide

$$\Sigma m_i (\mathbf{v}_i)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt = \Sigma m_i (\mathbf{v}_i)_2$$

- Now from our discussion of the center of mass, we had

$$\mathbf{v}_G = \frac{\Sigma m_i \mathbf{v}_i}{\Sigma m_i}$$

so denoting the sum of all of the particles masses, Σm_i , as m, we have

$$m\mathbf{v}_G = \Sigma m_i \mathbf{v}_i$$

and we can then write the principle of linear impulse & momentum for a system of particles in the form

$$m(\mathbf{v}_G)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt = m(\mathbf{v}_G)_2$$

Principle of Linear Impulse & Momentum for a System of Particles (cont.)

- Last equation from previous slide

$$m(\mathbf{v}_G)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt = m(\mathbf{v}_G)_2$$

Interpretation:

- $m\mathbf{v}_G = \Sigma m_i \mathbf{v}_i$ where $m = \Sigma m_i$: Total linear momentum of a system of particles can be viewed as the linear momentum of an "aggregate" particle having a mass equal to the sum of all particle masses in the system, and a velocity given by the velocity of the center of mass of the system

- The change in linear momentum of this aggregate particle is equal to the total impulse due to all the **external** forces that act on the system during the time interval under consideration

15.3 Conservation of Linear Momentum for a System of Particles

- We have just derived the following equation for a system of particles

$$\Sigma m_i (\mathbf{v}_i)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt = \Sigma m_i (\mathbf{v}_i)_2$$

- In the analysis of the dynamics of systems of particles, it is often the case that the sum of all the **external impulses** acting on the system is **zero** – this will happen, for example, if there are no external forces acting on the system

- The above equation then becomes

$$\Sigma m_i (\mathbf{v}_i)_1 = \Sigma m_i (\mathbf{v}_i)_2$$

and this is known as the **conservation of linear momentum** since it states that the linear momentum of the system is constant in time (again, assuming that there is no net external impulse on the system during the motion)

Conservation of Linear Momentum for a System of Particles (cont.)

- We can also write our equation for the conservation of linear momentum in terms of center-of-mass quantities. Specifically,

$$m(\mathbf{v}_G)_1 = m(\mathbf{v}_G)_2$$

or, given that the total mass, *m*, of the system remains constant (which we are assuming here)

$$(\mathbf{v}_G)_1 = (\mathbf{v}_G)_2$$

so that conservation of linear momentum is equivalent to the statement that the velocity of the center of mass of the system is constant (in both magnitude and direction) over the time interval in question

Conservation of Linear Momentum for a System of Particles (cont.)

- As usual, when solving problems, it will often be convenient/useful to use a component (scalar) form of conservation of linear momentum. Thus, in terms of the *x*, *y* and *z* components of initial and final velocities of particles, we have

$$\Sigma m_i (v_{ix})_1 = \Sigma m_i (v_{ix})_2$$
$$\Sigma m_i (v_{iy})_1 = \Sigma m_i (v_{iy})_2$$
$$\Sigma m_i (v_{iz})_1 = \Sigma m_i (v_{iz})_2$$

- In general, these equations will be able to applied independently of one another

- As is the case for the conservation of mechanical energy, the conservation of linear momentum can be a powerful tool in the solution of certain types of problems

Problem Solving Using Conservation of Linear Momentum

When can we use conservation of linear momentum?

1. Can **always** apply it to a system of particles when there are no external impulses acting on the system (or, more generally, can apply it in one or more coordinate directions when there are no external impulses acting in those directions)

2. As discussed in more detail in the text, can also apply it (approximately) when there **are** impulses, but these impulses can be neglected – a typical case is when the time period being considered is very short, and the forces in question are not very large – the ignorable forces in such an instance are often called **non-impulsive** (see worked examples in text for more details)

- In working **any** problem using conservation of linear momentum, we need to be careful to be consistent with the signs of velocities – if senses are known, can work problem using those senses, otherwise, a positive sense should be assumed