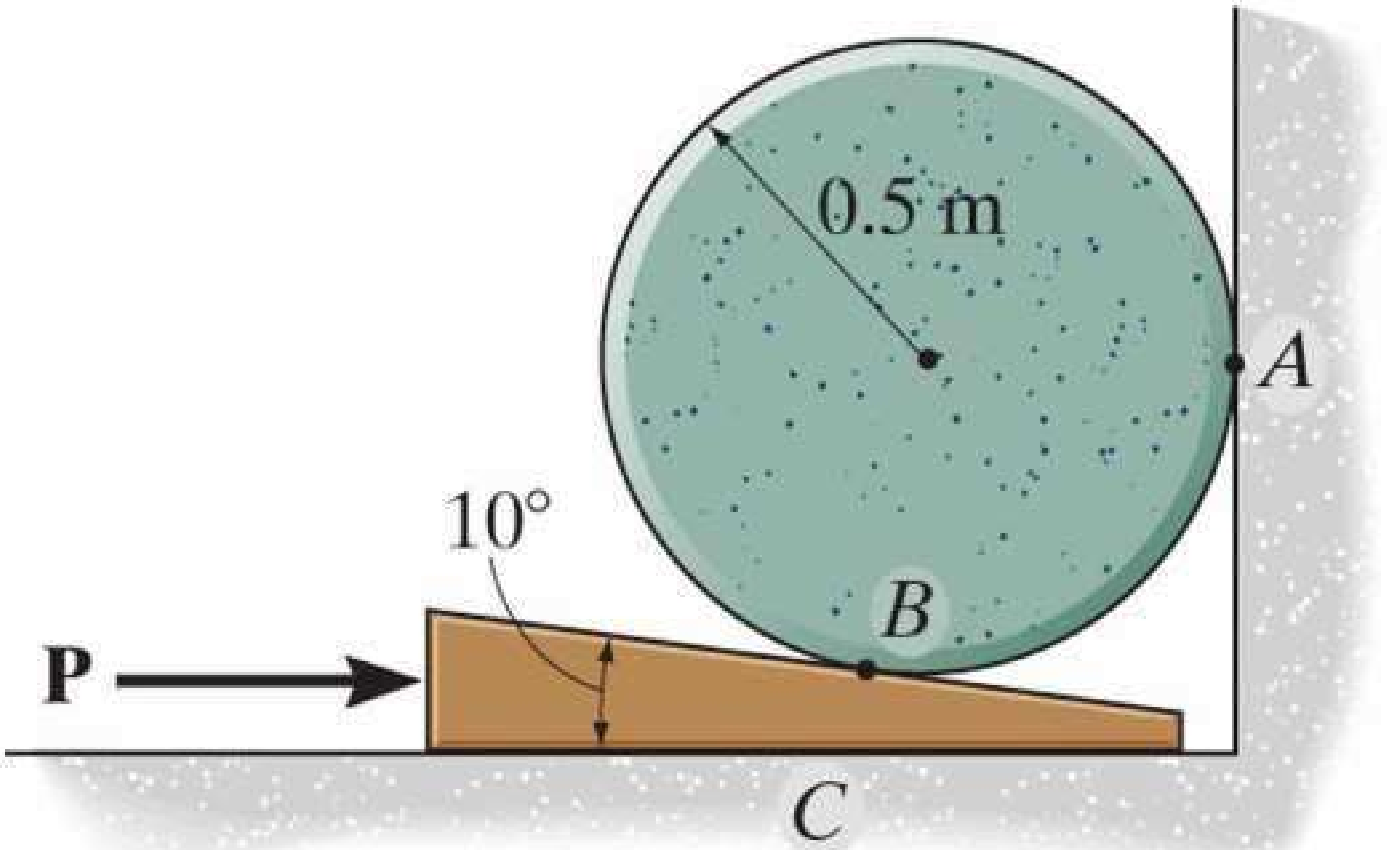
PHYS 170 Section 101
Lecture 18
October 19, 2018

Lecture Outline/Learning Goals

- 12.5 Curvilinear motion: rectangular components
- 12.6 Motion of a projectile
- Worked projectile problem
- 12.7 Curvilinear motion: normal and tangential components

What happens at *A* when the wedge first starts sliding at *B* and *C*?



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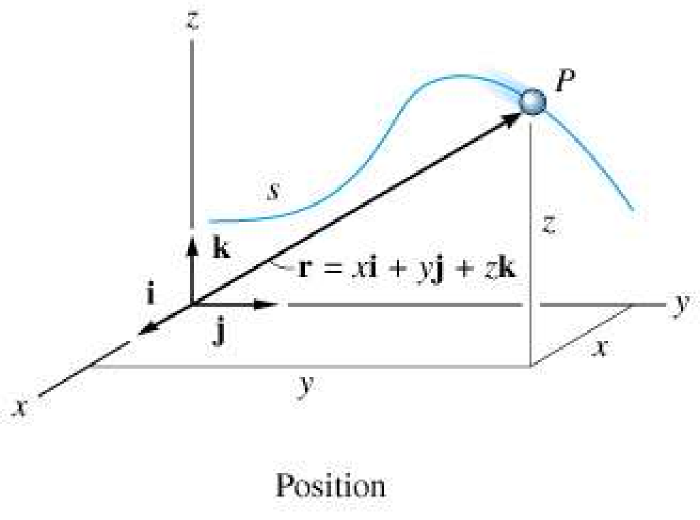
12.5 Curvilinear Motion: Rectangular Components

- Here we assume that the particle path is specified in a fixed Cartesian coordinate system (*x*,*y*,*z*)



POSITION

- Assume at some instant, *t*, that the particle is at point $P=P(x,y,z)$ along the path

- The particle position is then defined by the position vector, **r**

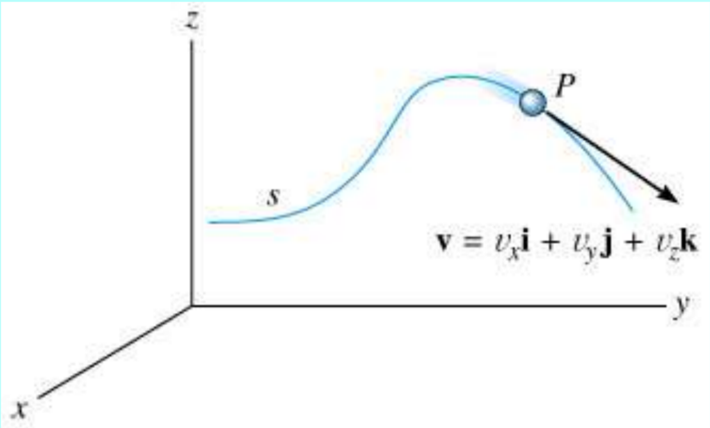
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- As usual, the **magnitude**, *r*, of the position vector is given by

$$r = \sqrt{x^2 + y^2 + z^2}$$

while the **direction** can be specified in terms of the unit vector $\mathbf{u}_r = \mathbf{r} / r$

VELOCITY



Velocity

- The particle velocity is the first time derivative of the position vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k})$$

- To evaluate this, we must apply the product rule for differentiation: for example

$$\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} + x\frac{d\,\mathbf{i}}{dt}$$

- Now, assuming that the coordinate system remains fixed, **i** is a **constant** vector, so

$$\frac{d\,\mathbf{i}}{dt} = \mathbf{0}$$

and introducing an “overdot” notation to denote differentiation with respect to time, we have

$$\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} = \dot{x}\,\mathbf{i} = v_x\,\mathbf{i}$$

$$\dot{x} = \frac{dx}{dt}$$

VELOCITY (continued)

- Treating the **j** and **k** components of the previous expression for **v** in a similar fashion we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$$

where

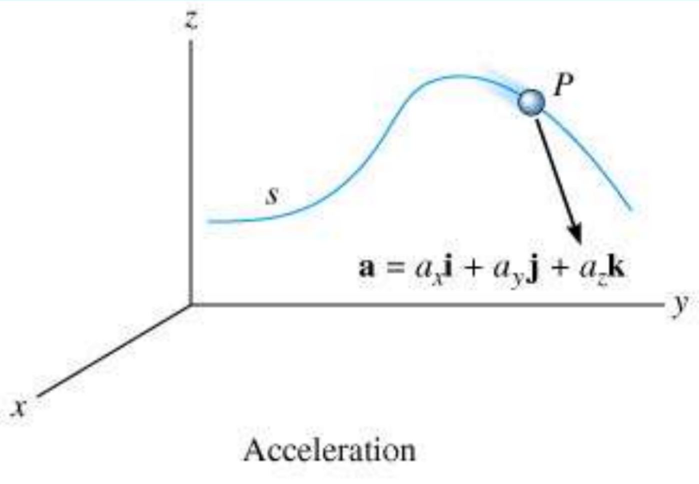
$$v_x = \dot{x} = \frac{dx}{dt} \qquad v_y = \dot{y} = \frac{dy}{dt} \qquad v_z = \dot{z} = \frac{dz}{dt}$$

- **Magnitude of velocity (speed)**

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- **Direction of velocity:** Given by components of unit vector $\mathbf{u}_v = \mathbf{v}/v$

- As discussed previously, this direction is **always tangent to the particle path**



ACCELERATION

- The particle acceleration is the first time derivative of the velocity vector, or the second time derivative of the position vector

- Using a development paralleling that used to derive the velocity, we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$$

where

$$a_x = \dot{v}_x = \ddot{x} = \frac{d^2x}{dt^2} = \frac{dv_x}{dt}$$

$$a_y = \dot{v}_y = \ddot{y} = \frac{d^2y}{dt^2} = \frac{dv_y}{dt}$$

$$a_z = \dot{v}_z = \ddot{z} = \frac{d^2z}{dt^2} = \frac{dv_z}{dt}$$

ACCELERATION (continued)

- **Magnitude of acceleration**

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

- **Direction of accleration:** Given by components of unit vector $\mathbf{u}_a = \mathbf{a} / a$

- As discussed previously, this direction will **not**, in general, be tangent to the particle path. Rather, it will be tangent to the **hodograph**

12.6 Motion of a Projectile

- **PROJECTILE MOTION:** Motion of a particle in the Earth's gravitational field (and remaining close to the surface of the Earth) – as discussed here, assumes no other forces act on particle (e.g. effects of air resistance neglected)

- Motion unfolds in a plane, so can be analyzed as a special case of curvilinear motion via rectangular components

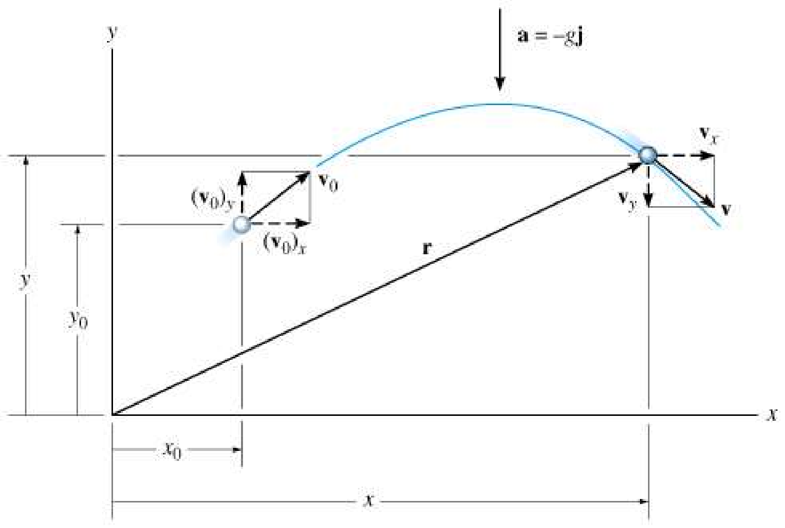
  – Two dimensional (2D): Adopt (x,y) coordinates with x, y axes oriented horizontally and vertically, respectively

  – **No** acceleration in x-direction

$$a_x = 0$$

  – **Constant** acceleration in negative y-direction

$$a_y = -g \qquad g = 9.81 \text{ m/s}^2 \qquad g = 32.2 \text{ ft/s}^2$$



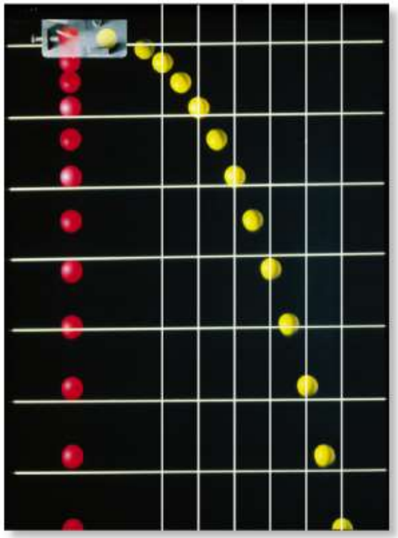
PROJECTILE MOTION (continued)

- Particle motion (trajectory) is determined by **initial conditions**

Initial position: $\mathbf{r} = x_0\,\mathbf{i} + y_0\,\mathbf{j}$

Initial velocity: $\mathbf{v} = (v_0)_x\mathbf{i} + (v_0)_y\mathbf{j}$

- Using results from rectilinear motion with constant acceleration separately in each of the two coordinate directions, we find the following



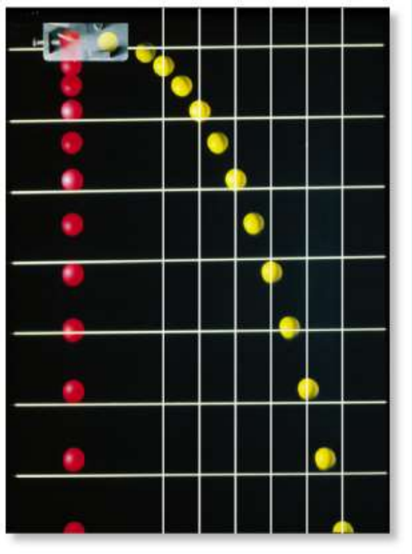
HORIZONTAL MOTION $(a_c = a_x = 0)$

$v = v_0 + a_c t:$	$v_x = (v_0)_x$
$x = x_0 + v_0 t + \frac{1}{2}a_c t^2:$	$x = x_0 + (v_0)_x t$
$v^2 = v_0^2 + 2a_c(s - s_0):$	$v_x = (v_0)_x$

- Note that the 1st and 3rd equations tell us the same thing, namely that the horizontal velocity component remains constant during the motion

VERTICAL MOTION $(a_c = a_y = -g)$

$v = v_0 + a_c t :$	$v_y = (v_0)_y - gt$
$y = y_0 + v_0 t + \frac{1}{2} a_c t^2 :$	$y = y_0 + (v_0)_y t - \frac{1}{2} g t^2$
$v^2 = v_0^2 + 2a_c (s - s_0) :$	$v_y^2 = (v_0)_y^2 - 2g(y - y_0)$



- Note that there are only 2 independent equations in the above set

- Thus there are a total of 3 independent equations for projectile motion (1 for horizontal motion, 2 for vertical motion), which means that in problems involving such motion, a maximum of 3 unknown quantities can be determined

The trajectory equation

Consider the trajectory of a projectile that is launched with an initial velocity $\vec{v}_0$ that makes an angle θ_0 with the horizontal. Then

$$(v_x)_0 = v_0 \cos\theta_0$$
$$(v_y)_0 = v_0 \sin\theta_0$$

The x and y coordinates of the projectile at time t are given by

$$x = x_0 + v_0 \cos\theta_0 \; t \qquad (1)$$

$$y = y_0 + v_0 \sin\theta_0 \; t - \frac{1}{2} g t^2 \qquad (2)$$

Now, solve (1) for t :

$$t = \frac{x - x_0}{v_0 \cos\theta_0} \qquad (3)$$

The trajectory equation

Now substitute the right hand side of (3) for t in (2). It is easy to show (EXERCISE) that the result can be written as

$$y(x) = a(x - x_0)^2 + b(x - x_0) + y_0$$

$$a = -\frac{g}{2v_0^2 \cos^2 \theta_0}$$

$$b = \tan \theta_0$$

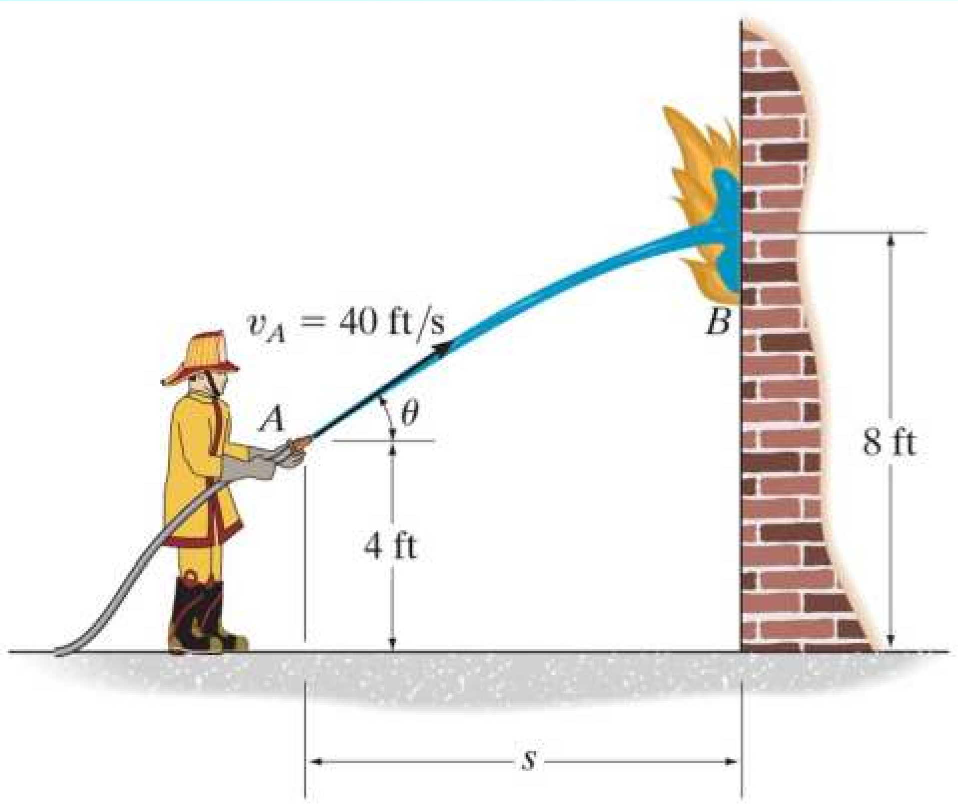
The above equation is known as the trajectory equation and can be used to solve trajectory problems in which the time and final velocity of the projectile do not explicitly appear.

Note that the equation involves four unknowns: $x_0,\ x,\ v_0,\ \theta_0$. We need to know three of these to determine the fourth. Also note that the equation is quadratic in v_0 and θ_0, so we can expect two solutions, in general, when solving for those unknowns. (May not both be physical.)

Problem 12-92 (page 48, 12th edition)

Water is discharged from the hose with a speed of 40 ft/s.

(1) Determine the two possible angles θ the firefighter can hold the hose so that the water strikes the building at B. Take $s = 20$ ft.



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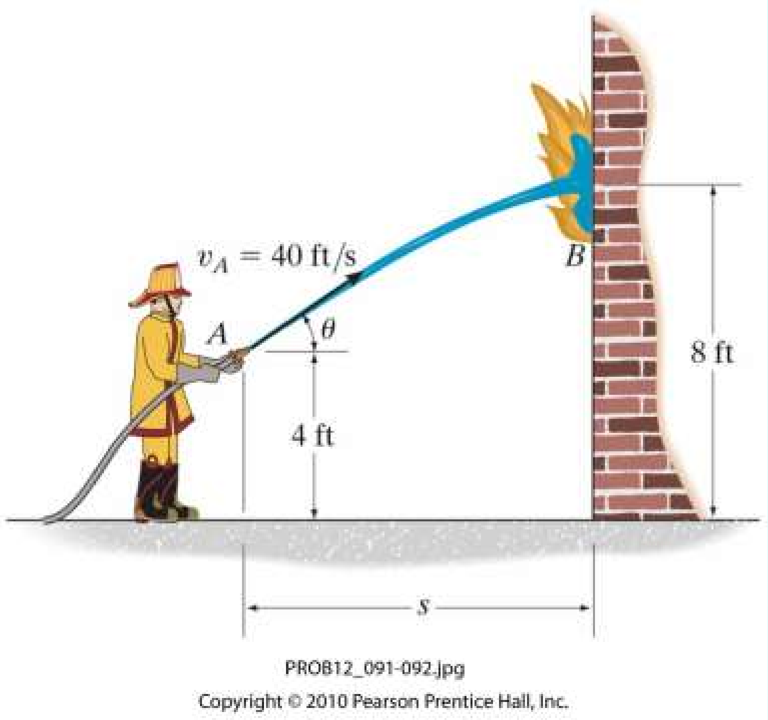
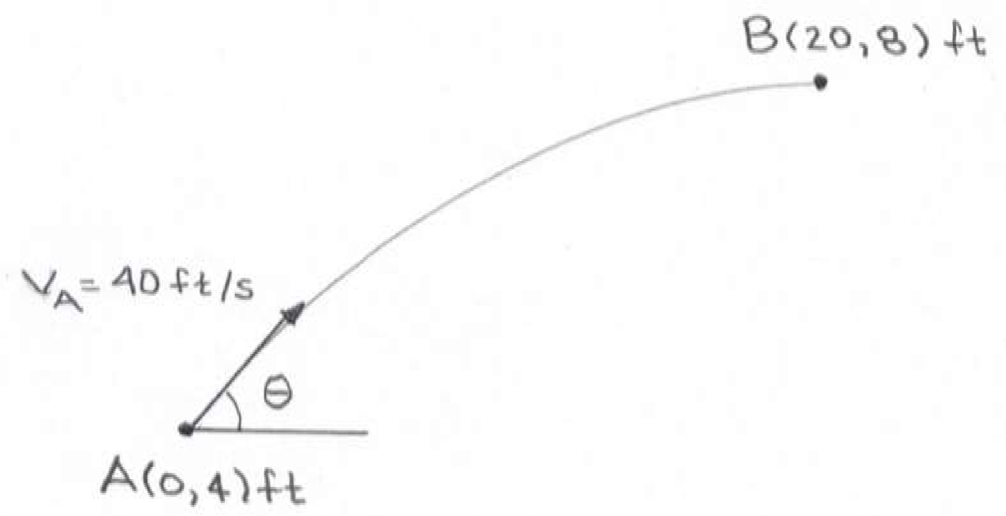
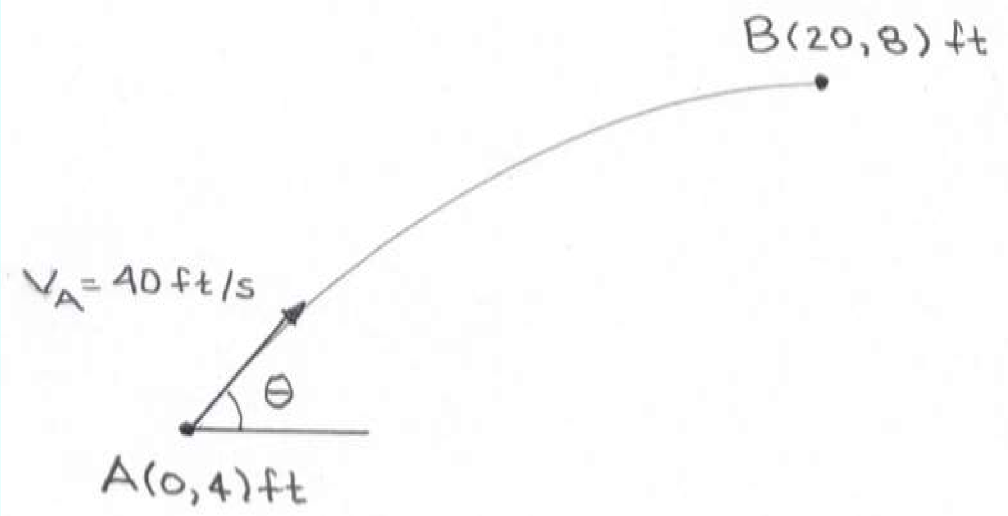


Solution strategy:

Use trajectory equation. Will get a nonlinear equation for θ which can be solved using solver function on TI graphing calculators.



Trajectory equation ($\theta_0 = \theta$)



$$y(x) = a(x - x_0)^2 + b(x - x_0) + y_0$$

$$a = -\frac{g}{2v_0^2 \cos^2 \theta_0}$$

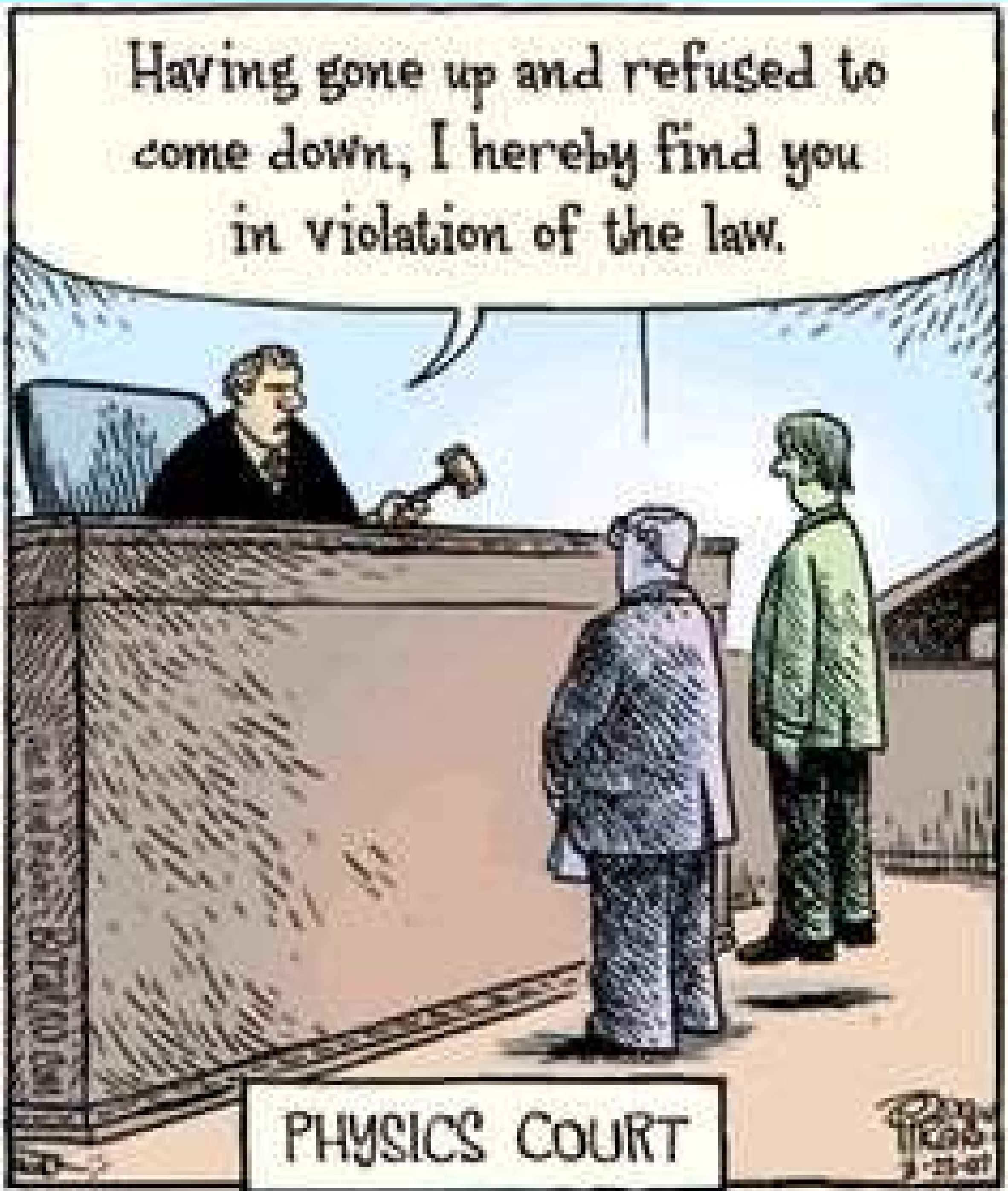
$$b = \tan \theta_0$$

Data: $v_0 = 40$ ft/s $(x_0, y_0) = (0,4)$ ft $(x, y) = (20,8)$ ft

$$8 = -\frac{32.2(20)^2}{2(40)^2 \cos^2 \theta_0} + 20 \tan \theta_0 + 4$$

Use solver (may have to experiment a little with initial guess to get both solutions):

$\theta = \theta_0 = 23.8^\circ$ $\theta = \theta_0 = 77.5^\circ$


Having gone up and refused to come down, I hereby find you in violation of the law.
PHYSICS COURT

12.7 Curvilinear Motion: Normal & Tangential Components

MOTIVATION

- Have previously discussed curvilinear motion in **rectangular components**; i.e. using a Cartesian coordinate system whose orientation and origin remains fixed as the particle moves along a path

- In particular, the Cartesian unit vectors, **i**, **j** and **k** do **not** vary with time

- We now proceed to discuss curvilinear motion using two different types of coordinate systems (**tangential/normal** and **polar**) whose origins and orientations are generally **not** fixed, but rather vary with time (and with the motion of the particle)

- In particular, the **unit vectors** associated with these coordinates will generally **depend on time**, and this fact must be taken into account in the description of the motion.

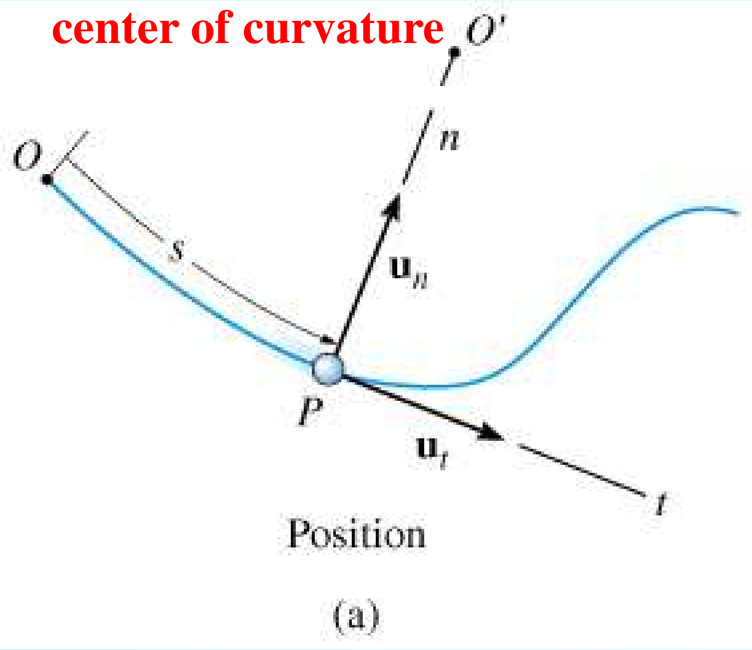
MOTIVATION (continued)

- The resulting formulas for velocity and acceleration are more complicated that for the Cartesian case, but **the analysis of many problems is nonetheless simplified by the use of such coordinates**

- The first case we will consider are tangential/normal coordinates which are especially convenient when the path along which a particle is moving is known (e.g. car moving along a curved road)

- We will restrict our attention to the case of 2D, or planar, motion (refer to the text for the brief discussion of the extension to 3D)

- As should now be familiar, we approach the kinematics of a particle in tangential/normal components by discussing the particle **position**, **velocity** and **acceleration** in turn



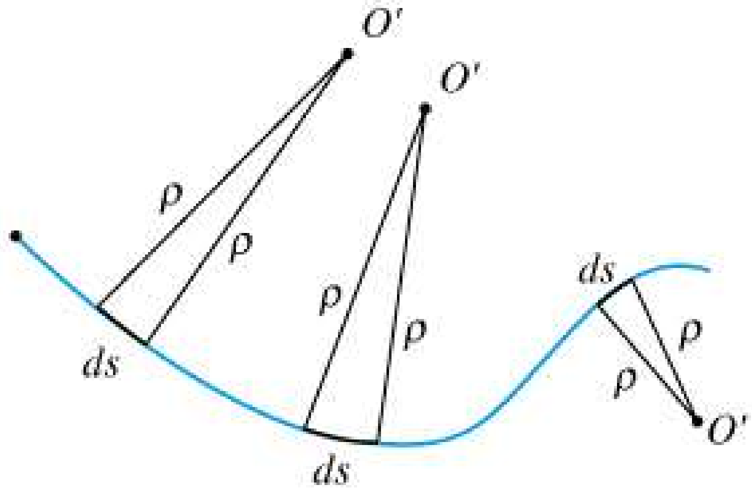
Position

(a)

POSITION

- Consider a particle moving along a path as shown in Fig (a), such that at some instant of time it is located at point *P*, which is at position *s* along the path relative to an origin *O,* also on the path
- At this instant we construct a coordinate system (*t,n*) (for **tangential, normal**), which has an (instantaneous) origin at the particle position, *P*

- The ***t* axis** is **tangent** to the curve, and has positive sense in the direction of increasing *s*
- Associated with this direction is a **unit vector** $\mathbf{u}_t$
- The ***n* axis** is perpendicular to the *t* axis, and has positive sense towards the **center of curvature**, *O'*, of the path at point *P*
- Associated with this direction is a **unit vector** $\mathbf{u}_n$



Radius of curvature

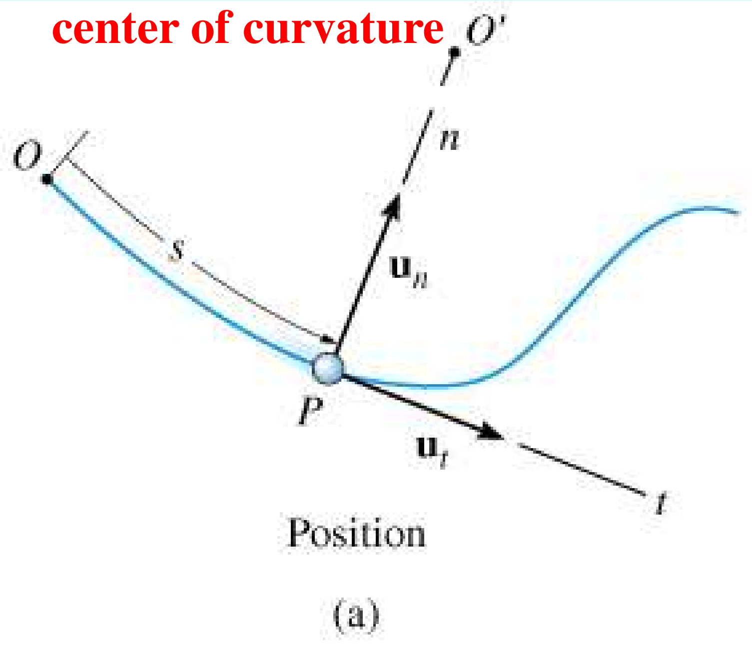
(b)

DIGRESSION: RADIUS OF CURVATURE CENTER OF CURVATURE

- We can view the curved particle path as being comprised of differential arc segments *ds*, each of which can be identified as an arc of a circle with radius, ρ , known as the **radius of curvature**, and with a center, O' , known as the **center of curvature**, as shown in Fig. (b)

- **NOTE:**
  - For a precisely circular path with radius, *R*, we have $\rho = R$
  - In the limit of a **straight** path, we have $\rho \to \infty$
  - In 2D it is often convenient and/or possible to express the particle path as *y = f(x)*. In such a case, the radius of curvature is given by

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|}$$

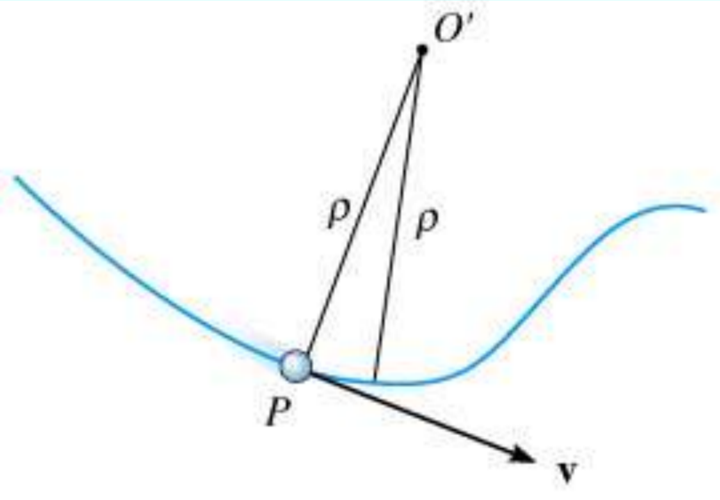


POSITION (cont)

- Thus, n axis is always positive on the **concave** side of the path
- Crucial to observe that as the particle moves along the path, the (t,n) coordinate origin follows the particle, and the coordinate axes **rotate** so that t and n **always** coincide with the tangent and normal directions (as defined above)
- Since the coordinate system moves with the particle, there is no need to write down expressions for the **position vector** which in effect **is always the 0-vector!**
- However, bear in mind that the particle's position along the curve (i.e. arc length position) is always given by

$$s = s(t)$$

which we are assuming here to be a given function of time (and don't confuse time t, with tangential coordinate, t !)



Velocity

(c)

VELOCITY

- We have $s = s(t)$, and we have previously seen that the particle's velocity vector, **v**, is always tangent to the particle path

- We also know that the magnitude of the particle velocity, or the speed of the particle, is given by

$$v = \frac{ds}{dt} = \dot{s}$$

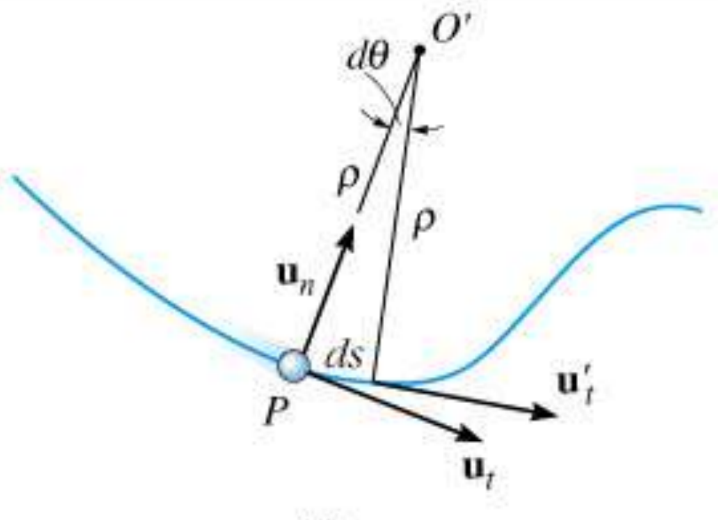
- We thus have

$$\mathbf{v} = v\,\mathbf{u}_t$$

where

$$v = \frac{ds}{dt} = \dot{s}$$

- To emphasize, the particle velocity has **no** normal component, but only a tangential component



(d)

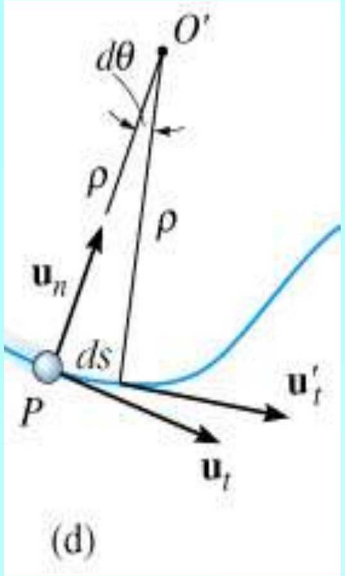
ACCELERATION

- The acceleration is the first time derivative of the velocity, so we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(v\mathbf{u}_t\right) = \dot{v}\mathbf{u}_t + v\dot{\mathbf{u}}_t$$

- As mentioned previously, and in contrast to the rectangular/Cartesian case, here we do **not** have $\dot{\mathbf{u}}_t = \mathbf{0}$ in general, since the (t,n) coordinates and associated unit vectors translate and rotate as time passes and the particle moves

- However, we can compute $\dot{\mathbf{u}}_t$ by first noting that as the particle moves, $\mathbf{u}_t$ remains a unit vector (i.e. its length remains 1), but its direction changes, as shown in Fig. (d)



(d)

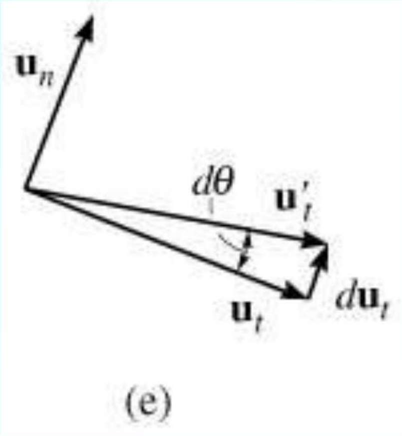
ACCELERATION (continued)

- Over an infinitesimal time interval *dt*, we have a change in $\mathbf{u}_t$ of $d\mathbf{u}_t$ as shown in Fig. (e)

- We see from the Fig. (e)

Magnitude of $d\mathbf{u}_t$: $du_t = (1)\, d\theta = d\theta$

Direction of $d\mathbf{u}_t$: $\mathbf{u}_n$



(e)

- Therefore we have

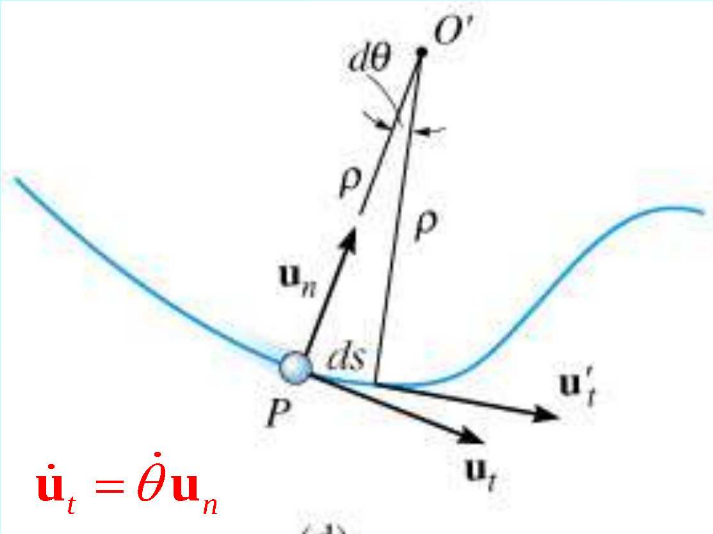
$$d\mathbf{u}_t = d\theta\, \mathbf{u}_n$$

and we can now compute $\dot{\mathbf{u}}_t$

$$\dot{\mathbf{u}}_t = \frac{d\mathbf{u}_t}{dt} = \frac{d\theta}{dt}\mathbf{u}_n = \dot{\theta}\,\mathbf{u}_n$$

- We're almost done with the derivation of the acceleration components in normal coordinates; the last step involves rewriting $\dot{\theta}$ in terms of v and ρ

ACCELERATION (continued)



(d)

- Referring back to Fig. (d), we see that

$$ds = \rho\, d\theta$$

so

$$d\theta = \frac{ds}{\rho}$$

$$\frac{d\theta}{dt} = \dot{\theta} = \frac{1}{\rho}\frac{ds}{dt} = \frac{\dot{s}}{\rho} = \frac{v}{\rho}$$

- Recall that we had

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(v\mathbf{u}_t\right) = \dot{v}\mathbf{u}_t + v\dot{\mathbf{u}}_t$$

so we have

$$\mathbf{a} = \dot{v}\mathbf{u}_t + v(\dot{\theta}\mathbf{u}_n)$$

$$= \dot{v}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n$$

ACCELERATION (cont)

- In summary, we can write the acceleration in the form

$$\mathbf{a} = a_t \mathbf{u}_t + a_n \mathbf{u}_n$$

where a_t and a_n are called the **tangential** and **normal** components of the acceleration, respectively, and are given by

$$a_t = \dot{v} \qquad \text{or} \qquad a_t\, ds = v\, dv$$

NOTE: These are the (differential) equations for rectilinear motion. For the special case that $\dot{v} = \text{constant}$, the equations for constant-acceleration rectilinear motion also apply

and

$$a_n = \frac{v^2}{\rho}$$

- Since a_t and a_n are mutually perpendicular components, we have that the magnitude, a, of the acceleration is given by

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}}$$

ACCELERATION (cont)

- It is instructive to consider the following special cases

1. **Straight-line motion:** In this case $\rho \to \infty$, so

$$a_n = \frac{v^2}{\rho} \to 0$$

   and the magnitude of the acceleration is

$$a = a_t = \dot{v}$$

2. **Motion at constant speed:** In this case $\dot{v} = 0$, so the magnitude of the acceleration is

$$a = a_n = \frac{v^2}{\rho}$$

   and since $\mathbf{a}_n = a_n \mathbf{u}_n$ always acts towards the center of curvature, this component is sometimes called the **centripetal** acceleration

ACCELERATION (cont)

- Finally, from the above special cases, as well as

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{v}\mathbf{u}_t + v\dot{\mathbf{u}}_t$$

$$= \dot{v}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n$$

$$= a_t\mathbf{u}_t + a_n\mathbf{u}_n$$

one can deduce that the tangential and normal acceleration components have the following interpretations

Tangential component: $\mathbf{a}_t = a_t\mathbf{u}_t$: Change in magnitude of velocity

Normal component: $\mathbf{a}_n = a_n\mathbf{u}_n$: Change in direction of velocity