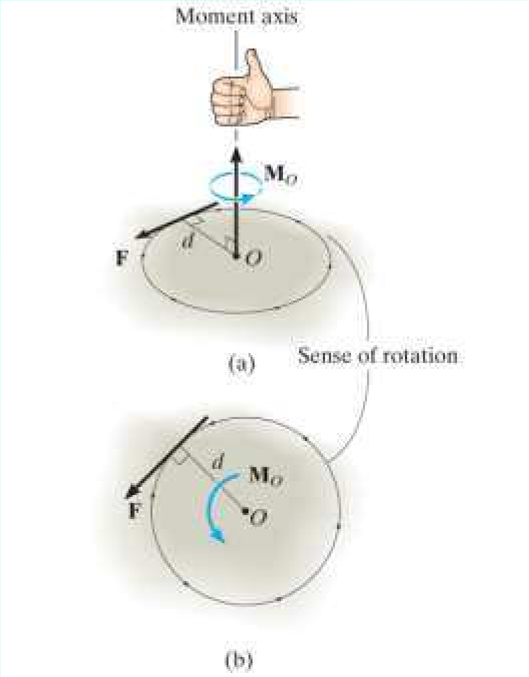
PHYS 170 Section 101
Lecture 9
September 24, 2018

Lecture Outline/Learning Goals

- General Definition of Moment (Scalar Formulation)
- Cross Product
- Moment of a Force: Vector Formulation
- Principle of Moments
- Moment of a Force about a Specified Axis

Moment of Force: General Case



- **Magnitude**

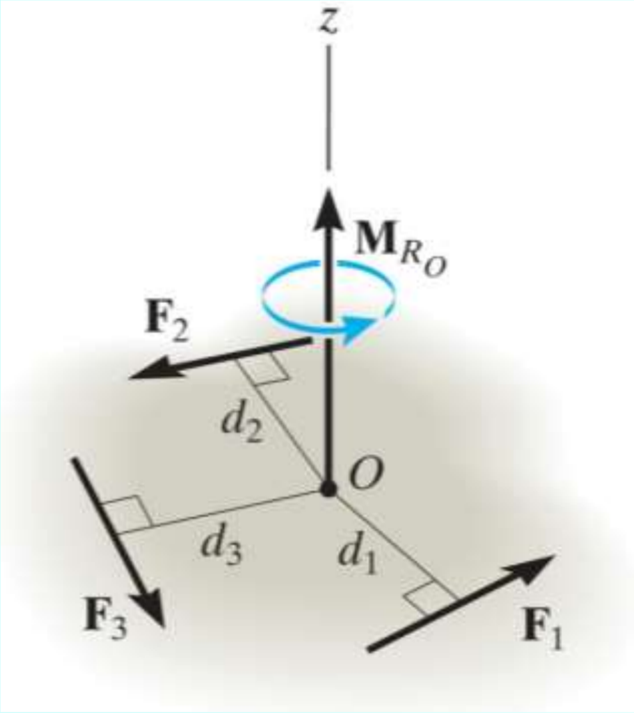
$$M_O = Fd$$

- *d* is known as the moment arm: again note that it is the perpendicular distance to the axis

- SI units of moments: $\mathrm{N \cdot m}$

- **Direction**
  - Another "right hand rule"
  - Curl fingers of right hand so that they follow sense of rotation (if rotation were possible)
  - Thumb then points in direction of moment (& with correct sense)

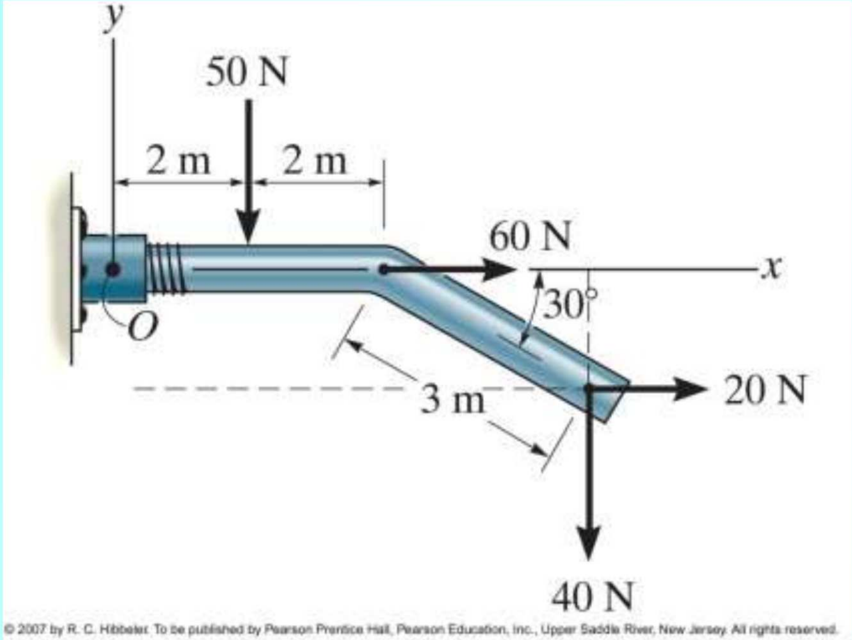
Resultant Moment of System of Coplanar Forces



- If system of forces is confined to *xy* plane, all moments about point *O* in that plane will be directed along *z* axis
- Thus all moments are collinear and can be added algebraically

$$\searrow + (M_R)_O = \Sigma Fd$$

- Note: ↘ is a facsimile of the "counterclockwise curl" used in the text and indicates the scalar sign convention: moments directed in +z direction are positive, those directed in –z direction are negative



- Example: Determine the resultant moment of the four forces acting on the rod about point *O*.
- Scalar sign convention: positive moments act in +z direction (out of plane of figure, counterclockwise)

- Note: in order to deduce moment arms, often useful to extend lines of action of forces (dotted lines)

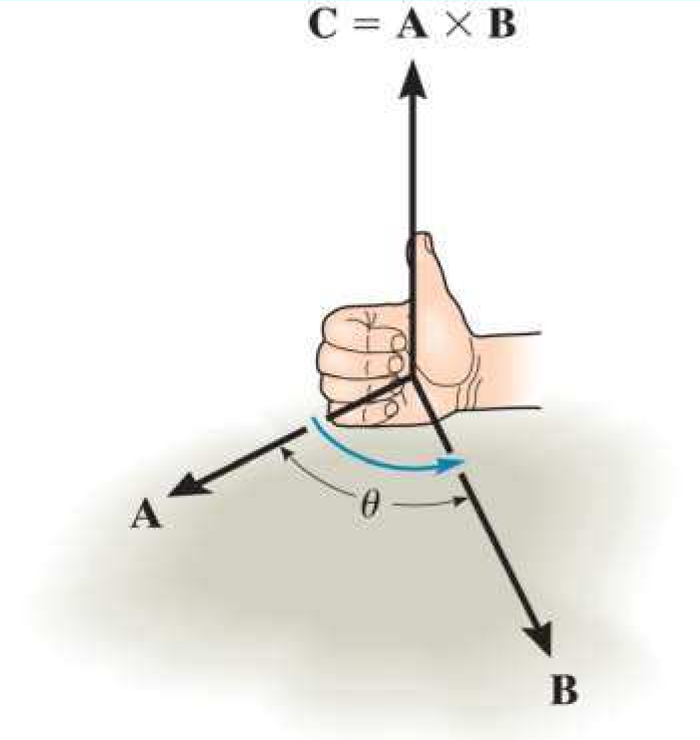
$$\searrow + (M_R)_O = \Sigma Fd$$

$$(M_R)_O = -50\text{N}(2\,\text{m}) + 60\text{N}(0) + 20\text{N}(3\sin 30^\circ\ \text{m})$$

$$-40\text{N}(4\text{m} + 3\cos 30^\circ\ \text{m})$$

$$= -334\text{N}\cdot\text{m} = 334\text{N}\cdot\text{m} \swarrow \quad \text{(clockwise)}$$

4.2 Cross Product

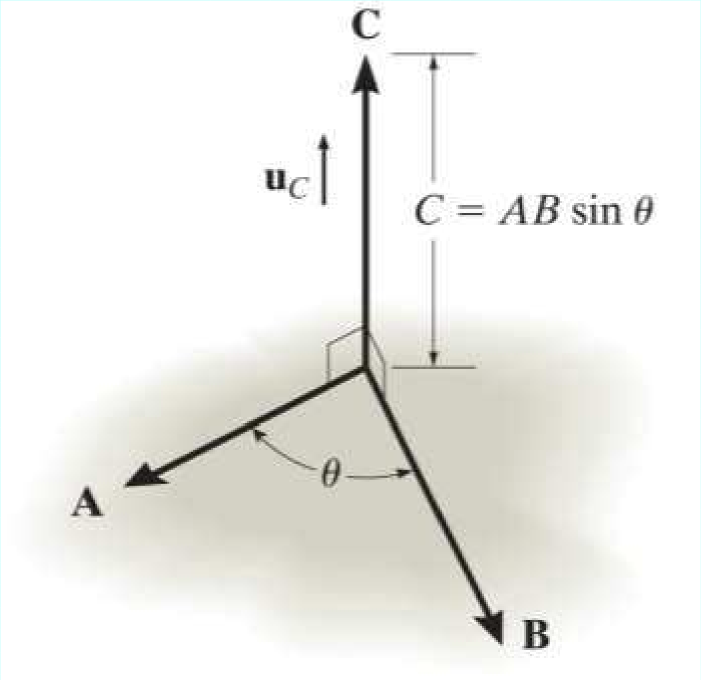


- In order to compute moments for general 3D cases, need to consider second type of vector multiplication: cross product

- Cross product of two vectors **A** and **B** is another vector **C**

- Notation

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$



- Magnitude of **C**

$$C = AB\sin\theta \qquad (0^\circ \le \theta \le 180^\circ)$$

- Direction of **C**
  - Perpendicular to plane containing **A** and **B**
  - Given by yet another right hand rule with fingers of right hand rotating **A** into **B**

Thus can write

$$\mathbf{C} = (AB\sin\theta)\mathbf{u}_c$$

where $\mathbf{u}_c$ is the unit vector in the direction of **C**

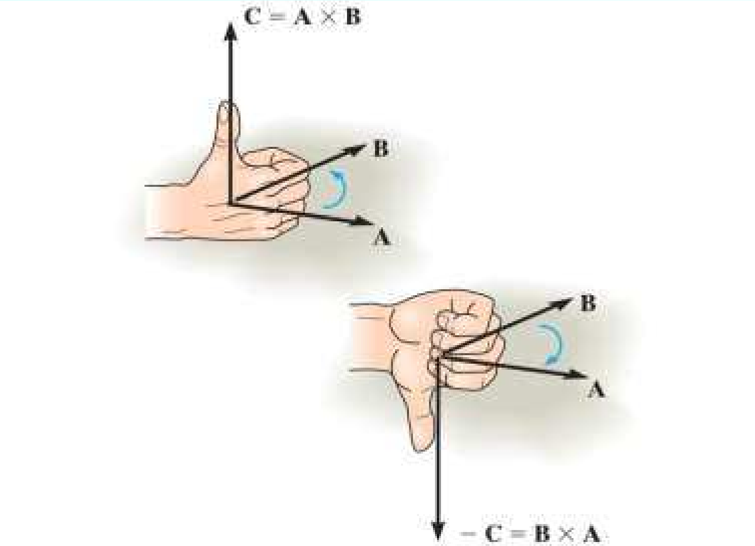
Cross Product: Laws of Operation

1. NOT commutative!!

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

Instead

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$



2. Multiplication by scalar

$$a(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})a$$

3. Distributive law

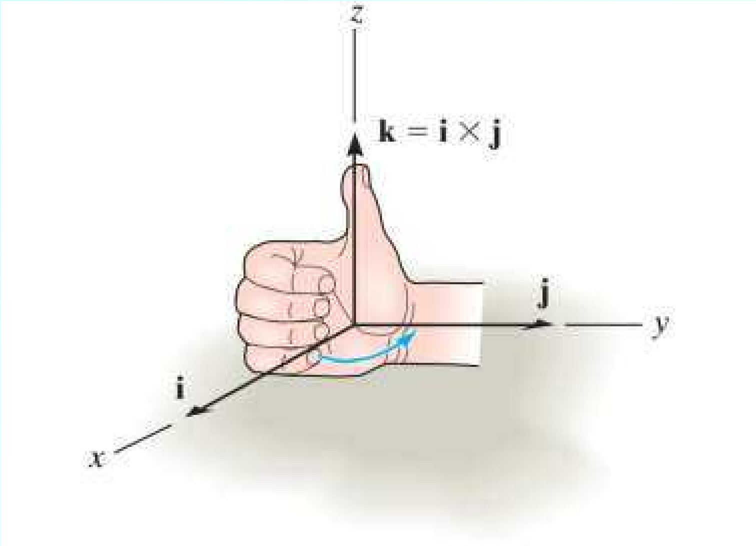
$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$$

Cross Product: Cartesian Vector Formulation

- Consider cross product of unit vectors **i** and **j**. Magnitude of cross product is

$$|\mathbf{i}\times\mathbf{j}| = |\mathbf{i}||\mathbf{j}|\sin\theta = (1)(1)\sin(90) = 1$$

By the right hand rule, direction is **+k**

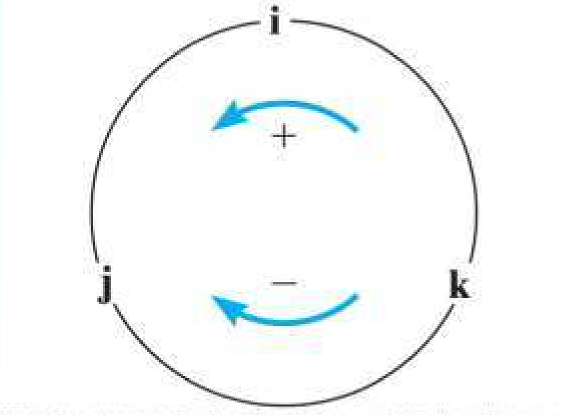


- Thus

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}$$

- Can repeat this for all possible combinations using the three unit vectors **i**, **j** and **k** to find:

$$\begin{array}{lll} \mathbf{i}\times\mathbf{j} = \mathbf{k} & \mathbf{i}\times\mathbf{k} = -\mathbf{j} & \mathbf{i}\times\mathbf{i} = \mathbf{0} \\ \mathbf{j}\times\mathbf{k} = \mathbf{i} & \mathbf{j}\times\mathbf{i} = -\mathbf{k} & \mathbf{j}\times\mathbf{j} = \mathbf{0} \\ \mathbf{k}\times\mathbf{i} = \mathbf{j} & \mathbf{k}\times\mathbf{j} = -\mathbf{i} & \mathbf{k}\times\mathbf{k} = \mathbf{0} \end{array}$$



- Can now work out cross product for general vectors **A** and **B** given in Cartesian form

$$\mathbf{A}\times\mathbf{B} = (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\times(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})$$

$$= A_xB_x(\mathbf{i}\times\mathbf{i}) + A_xB_y(\mathbf{i}\times\mathbf{j}) + A_xB_z(\mathbf{i}\times\mathbf{k})$$

$$+ A_yB_x(\mathbf{j}\times\mathbf{i}) + A_yB_y(\mathbf{j}\times\mathbf{j}) + A_yB_z(\mathbf{j}\times\mathbf{k})$$

$$+ A_zB_x(\mathbf{k}\times\mathbf{i}) + A_zB_y(\mathbf{k}\times\mathbf{j}) + A_zB_z(\mathbf{k}\times\mathbf{k})$$

$$= (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

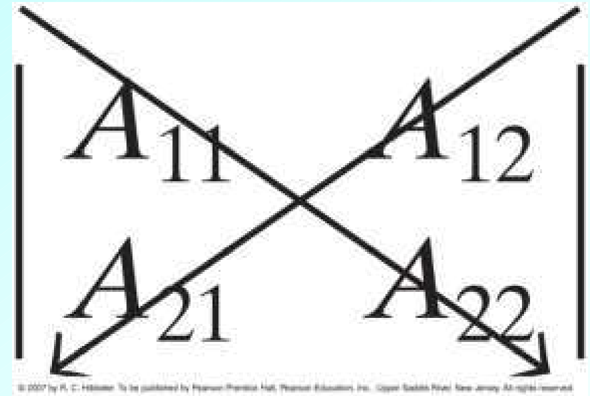
$$= (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

- Can write this in a more compact form as the determinant of a 3 x 3 matrix:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Review: Calculating Determinants

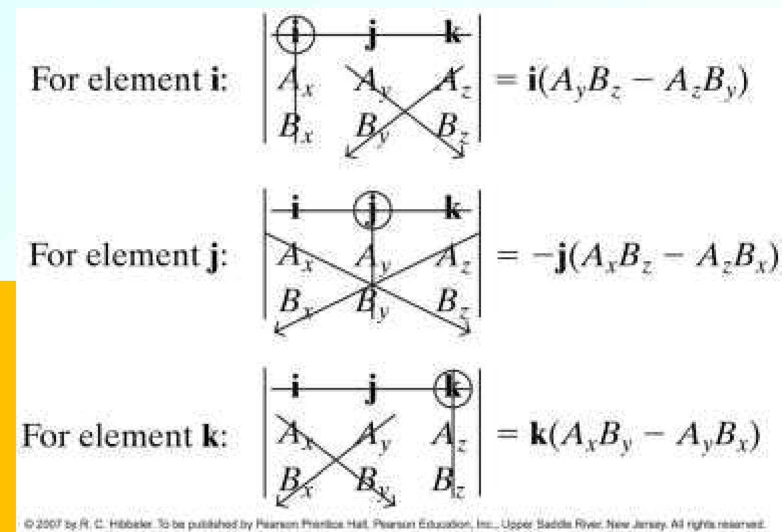
- 2 x 2 case



$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

- 3 x 3 case

Note "-" sign!!



$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

Sample calculation of a cross product

Take

$$\vec{A} = -\vec{i} + 5\vec{j} + 3\vec{k}$$
$$\vec{B} = 10\vec{i} - 20\vec{j} + 5\vec{k}$$

Compute $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & 3 \\ 10 & -20 & 5 \end{vmatrix}$$
$$= \left(25 - (-60)\right)\vec{i} - \left(-5 - 30\right)\vec{j} + \left(20 - 50\right)\vec{k}$$
$$= 85\vec{i} + 35\vec{j} - 30\vec{k}$$

Exercise: Show that

$$(\vec{A} \times \vec{B}) \cdot \vec{A} = 0$$
$$(\vec{A} \times \vec{B}) \cdot \vec{B} = 0$$

Interpretation?

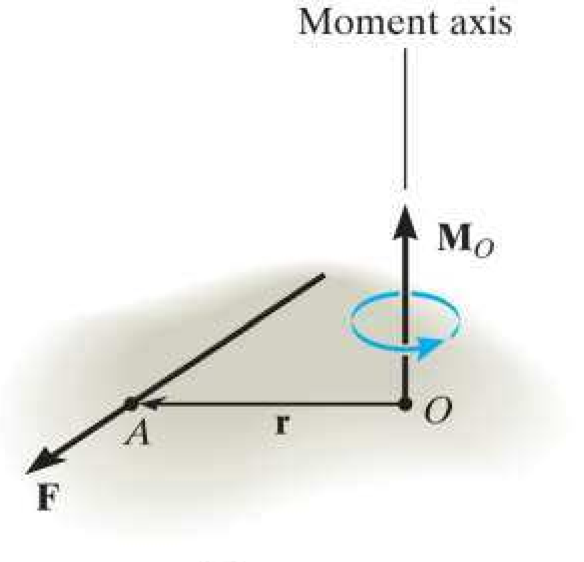
What do you get if you cross a mosquito and a mountain climber?

$\times$

$= ?$

No one knows. You can't cross a vector with a scalar.

4.3 Moment of a Force – Vector Formulation



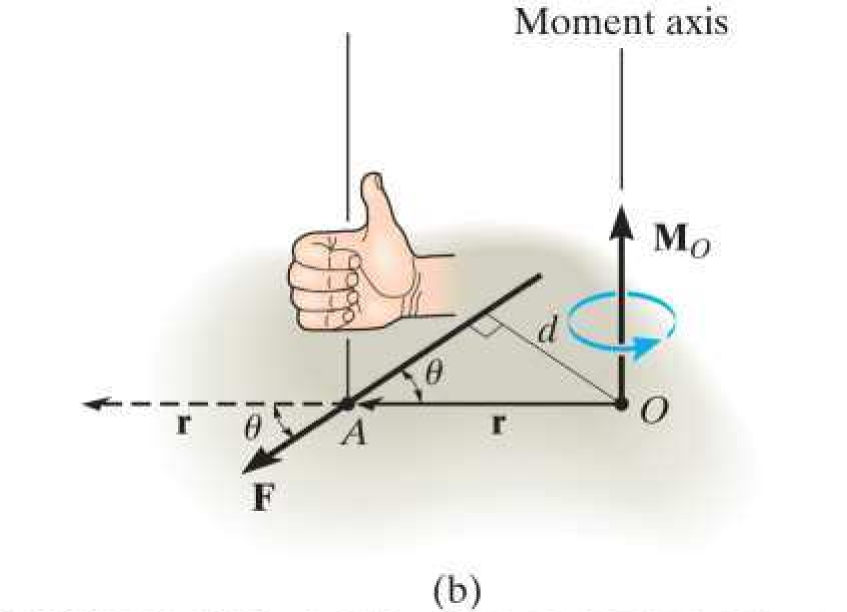
(a)

- Moment of force **F** about moment axis passing through *O* and perpendicular to plane containing *O* and **F** is given by

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

- Note: **r** is a position vector drawn **from** *O* to **any** point lying on the line of action of **F**

Moment of a Force: Magnitude & Direction



(b)

- **Magnitude:** Treat **r** as a "sliding vector" to move it to line of action of **F** so that angle θ is determined properly

  From definition of magnitude of cross product, magnitude of moment is

$$M_O = rF\sin\theta$$

- But this can be written as

$$M_O = F(r\sin\theta) = Fd$$

  where *d* is the moment arm, which agrees with our original definition

- **Direction:** Again, apply right hand rule, rotating **r** (sliding **r** as needed so that its tail intersects line of action of **F**) into **F** with fingers of right hand. Thumb points in direction of moment, which is perpendicular to both **r** and **F** (and thus to the plane that contains both **r** and **F)**

Cartesian Vector Formulation

- Establishing a right-handed *x*, *y*, *z* coordinate system we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (r_y F_z - r_z F_y)\,\mathbf{i} - (r_x F_z - r_z F_x)\,\mathbf{j} + (r_x F_y - r_y F_x)\,\mathbf{k}$$

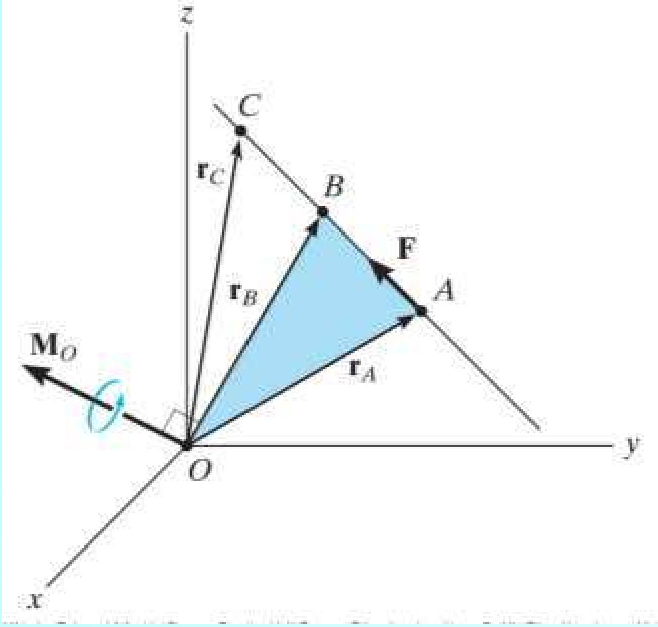
where

r_x, r_y, r_z : are the components of a position vector from *O* to any point on the line of action of the force

F_x, F_y, F_z : are the components of the force

- Use of above expression is recommended practice when working with general 3D forces and position vectors

Principle of Transmissibility



- In equation for moment about *O* due to force **F**

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

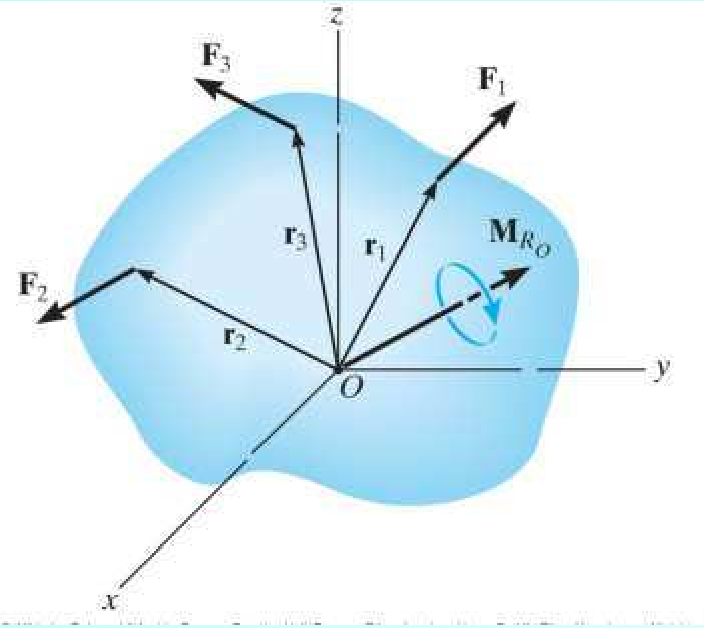
position vector **r** can join *O* and *any* point on the line of action of **F**

- Thus,

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \mathbf{r}_B \times \mathbf{F} = \mathbf{r}_C \times \mathbf{F} \quad \text{etc.}$$

- Therefore, when being used to compute a moment, a force can be treated as a sliding vector and can be relocated so that its tail is at an arbitrary point on its line of action
- This is known as the principle of transmissibility and will be used in our future discussion of equivalent systems

Resultant Moment of System of Forces



- When more than one force acts on a body, resultant moment of the forces about a point *O* is determined by a vector sum of the individual moments about *O* due to the individual forces, i.e.

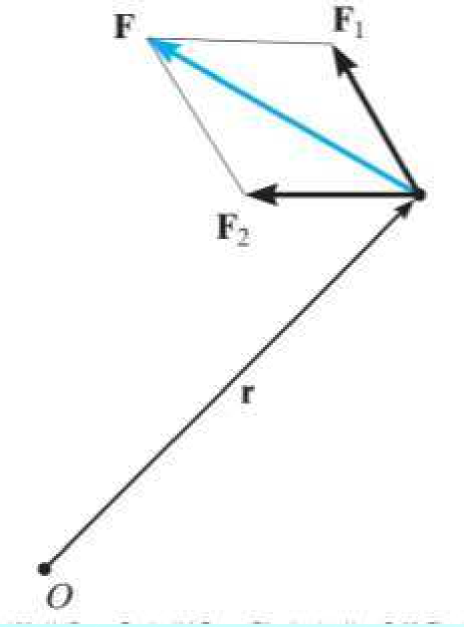
$$(\mathbf{M_R})_O = \Sigma(\mathbf{r} \times \mathbf{F})$$

- So in the example pictured above, we have

$$(\mathbf{M}_R)_O = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3$$

- Note that in this general case the resultant moment will not necessarily be perpendicular to any of the forces or position vectors!

4.4 Principle of Moments



- Principle of Moments: Moment of a force about a point is equal to the (vector) sum of the moments of the force's components about the point

- Proof is a direct consequence of the distributive property of the cross product. Considering the figure, for example, we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 = \mathbf{M}_{1O} + \mathbf{M}_{2O}$$

- Text notes that can often use this principle to make calculations of moments easier, especially when all of the forces and position vectors lie in a plane, and it is worth studying the text's worked examples to see this, as well as to try a few problems
- However, as was the case for force equilibria, we will be focusing attention on three-dimensional problems, where the Cartesian vector approach is almost always most straightforward