1. Well posed problems

a) First order systems, Cauchy problem

\[ u_t = \sum_{j=1}^n A_j D_j u = P(D) u, \quad D_j = \frac{\partial}{\partial x_j} \]  
\[ u(x,0) = f(x), \quad x = (x_1, \ldots, x_n), \quad -\infty < x_j < \infty \]

\[ u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad A_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \]

Well posed

1) There is an estimate

\[ \| u(\cdot,t) \| \leq K e^{\alpha(t-t_0)} \| u(\cdot,t_0) \| \]

2) Perturbation by lower order terms

(If (1) is well posed so is \( \tilde{u}_t = P(D) \tilde{u} + Bu \), for any bounded operator \( B \))

\[ \sum \varphi_j = 1, \quad \varphi_j \in C_0 \infty \]

\[ u^{(j)} = \varphi_j u \]

\[ u^{(j)}_t = P(D) u^{(j)} + \text{lower order} \]

localised problems.

1) principle of frozen coefficients

2) Reduction to halfplane problem and Cauchy problems.
Theorem 1. \( A_j = A_j^* \) the problem is well posed.

If not let \( A_j \) be constant.

Simple wave solutions
\[
    u(x, t) = e^{i \langle \omega, x \rangle} \hat{u}(\omega, t), \quad \omega = (\omega_1, \ldots, \omega_n) \text{ real}
\]

Separation of variables
\[
    \hat{u}_t = i\omega \hat{P}(\omega') \hat{u}, \quad \hat{P}(\omega') = \sum A_j \omega_j', \quad \omega = \omega'/\mu
\]

Theorem 2. If the eigenvalues of \( P(\omega') \) are not real then (2) has solutions which grow like
\[
    |\hat{u}(\omega, t)| \sim e^{i \omega t}
\]

If the eigenvalues are real (weakly hyperbolic) stability to lower order perturbation is not guaranteed.

The problem is well posed if and only if the system is strongly hyperbolic.

In this case the principle of frozen coefficients hold. (Smaller norm assumptions for the symbol)

b) Second order systems
\[
    u_{tt} = P_0 u + P_2 u_t
\]

\[
    P_0 = \sum A_{ij} D_i D_j, \quad P_2 = \sum A_i D_i
\]

Simple wave solutions
\[
    u(x, t) = e^{i \langle \omega, x \rangle} \hat{u}(\omega, t)
\]

\[
    \hat{u}_{tt} = -i \omega^2 \hat{P}_0(\omega') \hat{u} + i \omega \hat{P}_2(\omega') \hat{u}_t
\]
Let \( \hat{u}_t = i\omega \hat{u} \)

\[
(\hat{u})_t = i\omega \left( \begin{pmatrix} \hat{P}_1(\omega) & \hat{P}_0(\omega) \\ 1 & 0 \end{pmatrix} \right) \hat{u}
\]

Linear order terms

\[
Q_u = \sum B_j D_j u + B_0 u_t + Cu
\]

2. Half-plane problems. (Cauchy problem is well posed)

\[
u_t = Au_x + Bu_y, \quad x > 0, t > 0, -\infty < y < \infty
\]

\[
u(x, y, 0) = f(x, y)
\]

\[
A = \begin{pmatrix} -A^R & 0 \\ 0 & A^R \end{pmatrix}, \quad A^R > 0
\]

\[\mathbb{A} = A^* \text{ if also } B = B^* \text{ and if the boundary conditions are dissipative then the problem is well posed in the same sense as before.}\]

\[\text{Lemma: The problem is not well posed if we can find a solution}\]

\[u = e^{st + i\omega y} \hat{u}(x), \quad \hat{u}(\infty), \text{ Re} s > 0, \hat{u}(0) = S\hat{u}^0(0)\]
\[ \hat{u} = A \hat{u}_x + i\omega B \hat{u} \]

\[ \hat{u}_x = A^2 (s \mathbf{I} - i\omega B) \hat{u} =: M \hat{u} \]

For Re \( s > 0 \), there are no eigenvalues \( \lambda \) of \( M \) with Re \( \lambda = 0 \). There are exactly as many \( \lambda \) with Re \( \lambda < 0 \) as the number of boundary conditions.

*General bounded solution*

\[ \hat{u} = \sum_{\text{Re} \gamma < 0} \gamma e^{\gamma x} \hat{e}_j \]  \( \text{(4)} \)

Introduce (4) into the boundary condition

\[ C(s, \omega) \hat{u} = 0 \]  \( \text{(5)} \)

**Theorem 3.** The problem is not well posed if (5) has a solution for some \( s, \omega, \text{Re} s > 0 \).

The problem is well posed if (5) has no solution for \( \text{Re} s \geq 0 \).

**Well posed in what sense?**

\[ u_t = A u_x + B u_y + F \]  \( \text{(6)} \)

\[ u(x, y, 0) = 0 \quad u^T(0, y, t) = S u^T(0, y, t) \]

**Laplace - Fourier transform**

\[ (s \hat{u} - A \hat{u}_x - iB \omega) \hat{u} = \hat{F} \]  \( \text{(7)} \)

\[ \hat{u}^T(0) = S \hat{u}^T(0) \]

If (5) has no solution then (7) has a unique solution for \( s = i \xi + \eta, \eta > 0 \)
\[ \| \hat{u}(\cdot, \omega, s) \| \leq K(\gamma) \| \hat{F}(\cdot, \omega, s) \| \]
\[ \int_0^\infty e^{-2\gamma t} \| u(\cdot, \cdot, t) \|^2 dt \leq K(\gamma) \int_0^\infty e^{-2\gamma t} \| \hat{F}(\cdot, \cdot, t) \|^2 dt \]
\[ \lim_{\gamma \to \infty} K(\gamma) = 0 \]

(Assumption: (3) is strongly hyperbolic and the multiplicity of the eigenvalues of \( A \omega_1' + B \omega_2' \) is constant.)

Lower order terms, frozen coefficients.
3) Difference approximations for the wave equation.

\[ u_{tt} = \Delta u = u_{xx} + u_{yy} \]

\[ u(x,0) = f_1 \quad u_t(x,0) = f_2 \]

We discuss only Neumann conditions.

Relations between \( u \) at \( P_g \) and interior points approximate the boundary condition.
Main challenge: Stability
Main problem: Not is not aligned with the boundary (Some dissipation is needed).
Other methods: Finite element (variational principle) (unstructured meshes, implicit).

First order system:

\[ u_{tt} = u_{xx} \]

\[ D_{tt} D_{t} \tilde{u}(x, t) = D_{xx} D_{x} \tilde{u}(x, t), \quad (x, t) \text{ mesh point} \]

\[ u_{t} = u_{x} \implies \tilde{u}_{t} = D_{0x} \tilde{u} \]

\[ u_{t} = u_{x} \]

Spurious waves:

\[ \tilde{u}_{t} (x, v, t) = \frac{\tilde{u}(x, v, t) - \tilde{u}(x, v, t)}{2\Delta x} \]

\[ \tilde{u}(x, v, t) = (-1)^{v} \tilde{u}(x, v, t) \]

\[ \tilde{u}_{t} (x, v, t) = -D_{0} \tilde{u}(x, v, t) \quad (u_{t} = -u_{x}) \]

\[ u_{t} + (u_{x})^{2} = 0 \]

\[ \tilde{u}_{t} + D_{0} \tilde{u}^{2} = 0. \]

Smooth test \( \Rightarrow \) Estimate for \( u_{t} \) \( \quad \Delta u = F \quad F = u_{tt} \)
1D

\[ u_{tt} = u_{xx} \quad 0 \leq x \leq 1 \]

\[ x_v = (v - \alpha) h \]

\[ \tilde{u}_v(t) = \tilde{u}(x_v) \]

\[ \tilde{u}_{tt} = D_x D_{-x} u_v \quad v = 1, \ldots, N-1 \]

Dirichlet:

\[ (1 - \alpha) \tilde{u}_0 + \alpha \tilde{u}_2 + \beta (\tilde{u}_2 - 2 \tilde{u}_1 + \tilde{u}_0) = 0 \]

Neumann:

\[ D_x \tilde{u}_0 + h (\alpha - \frac{1}{2}) D_x^2 \tilde{u}_0 = 0 \]

\[ \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_N \end{pmatrix} = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & -2 & 1 & \cdots \\ \vdots & \vdots & \cdots & \cdots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_N \end{pmatrix} \]

Dirichlet:

\[ a = - \frac{2 - \alpha}{1 - \alpha + \beta} \quad b = \frac{1 - \alpha}{1 - \alpha + \beta} \quad \beta = \frac{1}{2} \]

Neumann:

\[ a = - \frac{1}{\frac{3}{2} - \alpha} \quad b = -a \]

2D

\[ u_{tt} = \Delta u \]

Mesh aligned with the boundary:

\[ \tilde{u}_{tt} = \Delta h \tilde{u} \]

\[ \tilde{u}_{tt} = D_x D_{-x} \tilde{u} - \frac{h}{(h_x)^2} \sin(h_x) \tilde{u} \]

No problem.
Discussion why there are problems if the mesh is not aligned with boundary.

**Halfplan problem**

\[ u_{tt} = \Delta u \]

**No energy estimate**

\[ u = e^{s t + i w y} \hat{u}(x) \]

\[ \hat{u}_{xx} = (\omega^2 + s^2) \hat{u}, \text{ for } x \geq 0 \]

\[ \hat{u}_x = i \omega \hat{u}, \text{ for } x = 0 \]

\[ u = \sigma_1 e^{-s \sqrt{\omega^2 + s^2} x} + \sigma_2 e^{s \sqrt{\omega^2 + s^2} x} \quad (1) \]

\[ \sigma_2 = 0, \quad -\sigma_1 (\sqrt{\omega^2 + s^2} - i \omega) = 0 \Rightarrow \sigma_1 = 0 \]

There are no solutions for Re(s) > 0.
There are solutions
\[ u = \delta_1 e^{-\sqrt{\omega + i \varepsilon} x} \] for Re s = 0

\[ s = i \omega \sqrt{1 + \varepsilon}, \quad \sqrt{\omega^2 + \varepsilon^2} = \varepsilon i \omega \]

Highly oscillatory for large \( \varepsilon \)

Strip problem

\[ u_x = 0 \quad \text{at } x = 0, \quad x = 1 \]

\[ u_x = u_y \]

\[ u = e^{s t + i \omega y} \hat{u}(x), \quad \hat{u}(x) \text{ given by (1)} \]

\[ \text{Re s} \approx \frac{1}{\varepsilon} \log (\varepsilon |1|), \quad e^{(\text{Re s} t)} = (\varepsilon |1|)^{\frac{t}{\varepsilon}} \]

Highly oscillatory instability can be controlled by adding a dissipative term to the differential equation.

\[ u_{tt} = \Delta u + 2 \alpha u u_{xy} \quad 0 < \alpha < 1 \]

Highly oscillatory instability problem is the only well-posed problem.
\[ u_{tt} = \Delta u, \quad x > 0 \]
\[ u_x = d u_y y \quad \text{at} \ x = 0 \]

\[ u = e^{st + i \omega y} \hat{u}(x), \quad \text{Re} \ s > 0 \]

\[ \hat{u}(x) = e^{-\sqrt{\omega^2 + s^2} x} \]

\[ -\sqrt{\omega^2 + s^2} = -\alpha \omega^2 \]

1) If \( \alpha < 0 \) there are no solutions for \( \text{Re} \ s \geq 0 \).

2) If \( \alpha > 0 \) then \( s \sim \alpha \omega^2, -\sqrt{\omega^2 + s^2} \sim \alpha \omega^2 \).

Boundary layer instability cannot be controlled by dissipative terms in the differential equation.
Modified equation

\[ u_t + D_0 u = \nu D_+ D_- u \quad u_t + u_x = u_{xx} \]

\[ u_t + u_x + \frac{1}{6} h^2 u_{xxx} = \nu u_{xx} + \frac{\nu}{12} h^2 u_{xxx} + O(h^4) \]

\[ w_t + w_x + \frac{1}{6} h^2 w_{xxx} = \nu \frac{\partial}{\partial x} \quad |w_t| \ll 1 \]

We use \( \Box \) to discuss the behavior of \( \Box \)

Modified problem

\[ u_{tt} = \Delta u \quad x \geq 0, \quad -\infty < y < 0 \]

\[ u_x = \beta_1 h^2 u_{yyy} + \beta_2 h^3 u_{yyyy}, \quad x = 0 \]

\[ u_{tt} = \Delta u - \alpha h^3 \Delta (\varphi(x) \Delta u_e) \quad \varphi(x) \]

\[ u_x = \beta_1 h^2 u_{yyy} + \beta_2 h^3 u_{yyyy} + \gamma h^3 u_{yyyyy}, \quad y > 0, \alpha > 0 \]

The code is very robust. The amount of dissipation is very small and one can calculate for long times. However, if there are corners we had to make additional decisions how to calculate tangential derivatives.
We eliminate the ghostpoints
\[ \Delta_h u \rightarrow A u \]
\[ u_{tt} = A u - \alpha h^3 A^* A u_t \]
(Away from the boundary \( A = \Delta_h, A^* A = \Delta_h^2 \))
\[ D_t D_t u_t = A u - \alpha h^3 A^* A D_t u_t \]
Higher order method away from the boundary.
Maxwell's equations.
\[ E_{tt} = \Delta E, \quad E = \begin{pmatrix} E_x \\ E_y \end{pmatrix} \]
\[ \text{div } E = 0 \text{ (no problem)} \]

Elastic wave equation. Model problem (halfplan)
\[ u_{tt} = \Delta u + 2\alpha u_{xy} \quad \alpha f < 1, \quad x > 0, \]
\[ u_x + \alpha u_y = 0 \quad x = 0 \]
only well posed if \( \alpha = \alpha \)
\[ u_{tt} = 2ax t + b u_{xx}, \quad b > -a^2. \]
\[ u_{tt} = 2au_{xx} + bu_{xx} \]