Black Hole “No-Hair” Theorems and Numerical Relativity

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What kinds of solutions to the Einstein equations might we expect to find?

- Black hole solutions
  
  These have a smooth event horizon, are non-singular outside the horizon and becoming asymptotically flat far from the horizon.

- “Soliton” or “Particle”-like solutions
  
  These are globally regular, i.e. they have no singularities and become asymptotically flat at large distances.

- Others
  
  - Cosmological
  
  - Gravitational waves
The No-Hair Conjecture

- Birkhoff – Schwarzschild is the unique spherically symmetric solution with asymptotic flatness.

- Israel – Schwarzschild is the unique solution for static, nonrotating black holes with asymptotic flatness.

- Robinson - Carter – Kerr(-Newman) is the unique solution for axisymmetric, stationary black holes with asymptotic flatness.

- Price's Theorem – Everything that can be radiated away will be radiated away in collapse.

⇒ Black holes are completely characterized by $M$, $J$, $Q_e$, and $Q_m$ (their gauge charges).
No Solitons

- Lichnerowicz – There are no globally regular solutions to the Einstein-Maxwell equations. (nonsingular and asymptotically flat)

- This was generalized to Kaluza-Klein and supergravity models.

- Deser - Coleman – There are no static Yang-Mills solutions in flat spacetime.

- Deser – There are no static soliton solutions of the Einstein-Yang-Mills equations in 2+1.

⇒ This is suggestive that there are no solitons in Einstein’s theory.
Nonetheless …

- Bartnik and McKinnon (1988) – There are particle-like (soliton) solutions to the EYM equations.

- Assume static, spherically symmetric, and an SU(2).

- The metric is
  \[ ds^2 = -A(r)^2 \mu(r) dt^2 + \frac{1}{\mu(r)} dr^2 + r^2 d\Omega^2 \]

- The SU(2) connection (matrix valued) is
  \[ A = a(r) \tau_3 dt + w(r) \tau_1 d\theta + (\cot \theta \tau_3 + w(r) \tau_2) \sin \theta d\phi \]

- Finite energy \( \Rightarrow a \equiv 0 \).

- BC’s for our ODE’s: regularity at the origin
  \[ w(r) \sim 1 - br^2 + O(r^4) \]
  and asymptotic flatness
  \[ w(r)^2 \to 1 \]

- The numerical problem is a simple shooting on \( b \).

- Infinite number of solutions characterized by the number of zeros of \( w(r) \): \( b_n \).
The first five BM solutions.
Energy density of the first five BM solutions.
Metric function $\mu(r)$ for the first five BM solutions
Black holes in EYM

• There also turn out to be black holes in addition to the BM “solitons.”

• The requirements are almost the same
  – static, spherically symmetric metric
  – spherically symmetric SU(2) gauge connection
  – asymptotic flatness

• Added assumption is the existence of a horizon at \( r_h > 0 \). The place where \( \mu(r_h) = 0 \).

• Again, simple shooting on \( w_h \equiv w(r_h) \) reveals an infinite number of discrete solutions characterized by the number \( (n) \) of zeros of \( w(r) \).

• These solutions are parameterized by \( r_h \), or correspondingly, the mass of the black hole: \( M(r_h) \).

• As with the BM solutions, the global YM charge vanishes.

• In the \( n \to \infty \) limit, this sequence of black hole solutions approaches the Reissner-Nordstrom solution.
The first five EYM black hole solutions.
Energy density of the first five EYM black hole solutions.
Metric function $\mu(r)$ for the first five EYM black holes


**Stability**

- These solutions – both soliton and black hole – are unstable in linear perturbation theory.

- For the $n^{th}$ solution, there are $2n$ unstable modes.

- There are $n$ modes unstable to gravitational perturbations and $n$ modes unstable to gauge field perturbations.

- Zhou and Straumann: BM solutions unstable either to dispersal of the fields to infinity or collapse to a Schwarzchild black hole.

- Choptuik *et al* showed that for $n = 1$ this is Type I critical behavior and that the model also has Type II critical behavior.
Einstein-Yang-Mills-some-kind-of-scalar

It is natural to generalize the EYM results to a broader class of theories which e.g. add some scalar field coupling.

Let’s consider two.

- Einstein-Yang-Mills-Dilaton

\[ \mathcal{L} = R - 2\nabla_\mu \phi \nabla^\mu \phi - e^{2\gamma \phi} F^a_{\mu \nu} F^{a \mu \nu} \]

where \( F^a_{\mu \nu} \) is the SU(2) Yang-Mills field strength and \( \gamma \) is a dimensionless coupling constant.

- Einstein-Yang-Mills-Higgs

\[ \mathcal{L} = \frac{1}{\alpha^2} R - F^a_{\mu \nu} F^{a \mu \nu} - 2D_{\mu} \phi^a D^{\mu} \phi^a - \frac{\beta^2}{\alpha^2} (\phi^a \phi^a - 1)^2 \]

where the Higgs field \( \phi^a \) is in the adjoint representation of SU(2) and \( \alpha \) and \( \beta \) are dimensionless coupling constants.
Einstein-Yang-Mills-Dilaton

- Various theoretical models (e.g. string theory, Kaluza-Klein, inflation, etc) suggest the existence of a massless, real scalar field – the “dilaton.”

- The limit $\gamma \to 0$ (with a constant $\phi$) recovers the Bartnik-McKinnnon solutions.

In addition, the limit $\gamma \to \infty$ leads to YMD uncoupled from gravity which also possesses soliton solutions.

- The particular value $\gamma = 1$ describes the low-energy limit of heterotic string theory.

- Both solitons and black hole solutions can be found with the coupling $\gamma$ describing a family of solutions.

- The assumptions and procedure are the same as before.

  - We shoot on a single parameter $b$, find an infinite set of discrete solutions ($n$) for a given $\gamma$ value.
First five regular solutions of EYMD with $\gamma = 1$
Dilaton field and metric function $\mu$ for regular solutions of EYMD with $\gamma = 1$
Energy density for regular solutions of \textbf{EYMD} with $\gamma = 1$
Black hole solutions of EYMD with $\gamma = 1$
Dilaton field and metric function $\mu$ for black hole solutions of EYMD with $\gamma = 1$
Stability of EYMD solitons and black holes

- All the solutions are also unstable in linear perturbation theory.

- Again, for the $n^{th}$ solution, there are $2n$ unstable modes with $n$ modes unstable to gravitational perturbations and $n$ modes unstable to gauge field perturbations.

- Conjecture: Like the BM soliton solutions, these will be unstable either to dispersal of the fields to infinity or collapse to a Schwarzschild black hole.

- In addition, if the BM solutions are any guide, and they belong in this family of solutions, this model should exhibit both Type I and Type II critical behavior. So we would have yet another parameterized family of critical solutions.
Einstein-Yang-Mills-Higgs

- These should describe gravitating monopoles and dyons for example, so we sort of expect to find these when gravity is “turned on.”

- In addition, one might expect that once these objects become sufficiently massive, they will “collapse” and form black holes.

- We again make the assumptions of spherical symmetry, staticity, SU(2) and asymptotic flatness. In addition, we make the ansatz (hedgehog) for the Higgs field of $\phi^a = \tilde{r}^a H(r)$.

- Again, assuming regularity at the origin leads to solitons and assuming the existence of a horizon (i.e. of $r_h$ such that $\mu(r_h) = 0$) leads to black hole solutions.

- The numerical problem is somewhat more difficult as we must search on two parameters.
**Gravitating monopoles**

- We again get an infinite number \((n = 0, 1, 2...)\) of monopole solutions each parameterized by \(\alpha\) and \(\beta\). The limit \(\alpha \to 0\) for \(\beta = 0\) and \(n > 0\) correspond to the BM solutions. The \(n = 0\) solutions can be thought of as the gravitating generalization of the t'Hooft-Polyakov monopole.

- The parameter \(\alpha\) is roughly the ratio of the monopole mass to the planck mass, so as it increases we expect solutions to no longer exist as they become unstable. Indeed we get RN + throat + smooth origin.

- All the excited monopole solutions exhibit this behavior as well i.e. they are unstable above a critical value of their mass \((\alpha_c)\).

In addition they are unstable to gravitational perturbations as well. So we can conjecture again that critical behavior will be present in this system as well.

- The lowest lying \((n = 0)\) monopole, is however stable to gravitational perturbations for \(\alpha\) less than its critical value.
Non-abelian black holes

- There are also magnetically charged black holes paramterized by their radius $r_h$ (or equivalently their mass $M(r_h)$).

- Again, $\beta$ can take on any value but black holes will only exist for a range of $\alpha$.

- In some regions we find colored black holes. In others we get abelian RN black holes. There are also cases where we get infinitely many – a veritable zoo.

- The majority of the solutions turn out to also be unstable to gravitational perturbations. (?)
Some conclusions and possible directions

- EYM, EYMD, and EYMH yield non-trivial solutions which can be characterized as solitons or black holes. However, the majority are unstable.

- Implications for “No-Hair Conjecture”

- Physical realization? – early universe?

- Just an introduction – much more
  - Dyonic configurations – magnetic and electric charge
  - Other gauge groups (SU(n), SO(n)...)
  - Axisymmetry, rotation ...
  - SUSY
  - Black holes supporting defects
  - Critical behavior
  - Evolve the time-dependent equations and consider non-linear stability.
  - Analytic results
  - Rigorous proofs of existence