Second-order in space systems

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• Second-order systems in GR

• Definitions of hyperbolicity

• Finite differencing
Gauge, constraints, and different formulations

Each gauge freedom generates one constraint.

Ingredients of a formulation of the Einstein equations:

- Well-posedness depends on the gauge choice.
- It can be achieved by adding constraints.
- ... and introducing redundant variables and their definition constraints.

Historical confusion between introducing some redundant variables to obtain hyperbolicity, and reducing to first order to prove it.
Some formulations

**ADM**: 2nd order, 12 variables $\gamma_{ij}$ and $K_{ij}$

**KST**: 1st order, 18 auxiliary variables

$$d_{kij} \equiv \gamma_{ij,k}$$

**NOR**: 2nd order, 3 redundant variables

$$f_i \equiv \gamma_{jk} \gamma_{ij,k}$$

**BSSN**: 2nd order, $K_{ij} \rightarrow (K, \tilde{A}_{ij})$, $\gamma_{ij} \rightarrow (\phi, \tilde{\gamma}_{ij})$, plus $\tilde{\Gamma}^i \sim f_i$.

"**BSSN-C**": BSSN with algebraic constraints $\text{tr}\tilde{A}_{ij} = 0$ and $\det\tilde{\gamma}_{ij} = 1$ imposed continuously: equivalent in the principal part to a variant of NOR.

**Z4**: 2nd order, redundant variables $Z^\mu \sim \Box x^\mu$. 
Reduction to first order

CG & Martin-Garcia

Matrix notation: \( u \) is a vector of variables. Not all variables have second space derivatives: 

\[
\begin{align*}
u & \equiv (v, w), \\
\dot{u} & = \partial \partial v + \partial w + \text{lower order terms}
\end{align*}
\]

Reduction \( d_i \equiv v, i \) is possible only for

\[
\begin{align*}
\dot{v} & = A^i_1 v, i + A_2 w + \text{l.o.} \\
\dot{w} & = B^{ij}_1 v, ij + B^i_2 w, i + \text{l.o.}
\end{align*}
\]

(Counterexample \( \dot{v} = v'' \).)

Parameterise all ambiguities \( v, i \) or \( d_i \), and \( d_{i,j} \) or \( d_{j,i} \). Hyperbolicity of the second-order system should be defined independently of these reduction parameters.

Evolution of auxiliary constraints \( v, i - d_i \) and \( d_{i,j} - d_{j,i} \) closes \( \Rightarrow \) we can restrict to the second order system.
Strong hyperbolicity

**Definition:** Second order system strongly hyperbolic \( \iff \) there is a reduction that is strongly hyperbolic.

**Theorem:** \( \iff \)

\[
\mathcal{A} \equiv \begin{pmatrix} B^n_2 & B^{nn}_1 \\ A_2 & A^n_1 \end{pmatrix}
\]

is uniformly diagonalisable for all \( n_i \), where \( A^n_1 \equiv A^i_1 n_i \) etc.

**Lemma:** \( \iff \) second order system has a complete set of characteristic variables of the form \( w + \partial v \).

**Lemma:** \( \iff \) pseudo-differential reduction strongly hyperbolic
Idea of proof

A particular choice of reduction parameters gives the principal part

\[ \dot{v} \approx 0 \]
\[ \dot{w} \approx B_1^{ij} d_{k,i} + B_2^i w_i \]
\[ \dot{d}_j \approx \left( \delta_j^i A_1^k + i \mu \epsilon_{jik} \right) d_{k,i} + A_2 w_i \]

With \( d_i \equiv (d_n, d_A) \), the principal part neglecting transverse derivatives is

\[ \dot{w} \approx B_1^{nn} d_{n,in} + B_2^n w_{n,n} + B_1^{nB} d_{B,n} \]
\[ \dot{d}_n \approx A_1^n d_{n,n} + A_2 w_{n,n} + A_1^B d_{B,n} \]
\[ \dot{w}_A \approx i \mu \epsilon A_1^n B d_{B,n} \]

The lower diagonal block is diagonalisable with eigenvalues \( \pm \mu \). Then the entire matrix is diagonalisable if (and in fact only if) the upper diagonal block \( \mathcal{A} \) is diagonalisable. (We choose \( \mu \) large enough so that \( \pm \mu \) are not eigenvalues of \( \mathcal{A} \).)
Symmetric hyperbolicity

Definition: Second order system symmetric hyperbolic ⇔ there is a reduction that is symmetric hyperbolic.

Theorem: ⇔ (1)

\[(\mathcal{H}A)^\dagger = \mathcal{H}A\]

for all \(n_i\), and (2)

\[H > 0\]

where

\[\mathcal{H} \equiv \begin{pmatrix} K & L^n \\ L^n & M^{nn} \end{pmatrix}, \quad H \equiv \begin{pmatrix} K & L^i \\ L^i & M^{ij} \end{pmatrix},\]

Theorem: ⇔ second order system admits a conserved energy \(\epsilon\) quadratic in \((w, v, i)\) and conserved in the sense that

\[\dot{\epsilon} = \phi^i, i\]
Idea of proof

Step 1:

\[(\mathcal{H}A)^\dagger = \mathcal{H}A\]

is necessary for

\[(HP^i)^\dagger = HP^i\]

for any first-order reduction (with principal part \(P^i\)).

Step 2: We can find reduction parameters, which depend on \(H\), such that this condition is also sufficient.

Step 3: The energy for the second-order system is also the energy for the reduction (with \(v_i \leftrightarrow d_i\)).
Constraint evolution

**Theorem:** Vector of constraints

\[ c ≡ C^{ij}_1 v_{,ij} + C^i_2 w_{,i} + \text{l.o.} = 0 \]

compatible with the evolution equations, and main system strongly hyperbolic \(⇒\)

Constraint system strongly hyperbolic, and characteristic variables in the direction \(n_i\) given by

\[ c \simeq \partial_{n} u + \partial_{A} \ldots \]

where the \(u\) are some of the characteristic variables of the main system (and \(c\) and \(u\) have the same speed).
Finite differencing

- Interior: Semidiscrete version of some symmetric hyperbolic systems is unstable when using standard centered differences: shifted wave equation with \( \beta > 1 \) using \((\dot{\phi}, \phi)\) but not \((\Pi, \phi)\). Z4 but not NOR/BSSN.

- Boundaries: Summation by parts operators do not give a conserved semi-discrete energy even for the shifted wave equation in \((\Pi, \phi)\) form (too many separate summations by part required).

- Ad-hoc finite differencing methods give stable excision and timelike boundaries for the shifted wave equation with 2nd and 4th order accuracy (Calabrese & CG, in progress).
First order or second order?

- Astrophysics simulations so far only in BSSN.

- In testbeds, first-order formulations seem to require fine-tuning of parameters.

- First order allows “standard” finite differencing treatment of excision, outer, and multipatch boundaries, using summation by parts and projection methods.

- Second order simpler and probably more accurate (phase error, error growth from non-principal terms) but require
  - stable heuristic boundary treatments,
  - stable overlapping multipatch schemes.