Null Quasi-Spherical
Einstein characteristic code

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The Null Quasi-Spherical ansatz

The NQS coordinates \((z, r, \vartheta, \varphi)\) satisfy:

- The 3-surfaces \(z = \text{const.}\) are null,
- The 2-surfaces \((z, r) = \text{const.}\) are isometric to standard 2-spheres of radius \(r\),
- The coordinates \((\vartheta, \varphi)\) are standard spherical polars

General NQS metric:

\[
\begin{align*}
\text{ds}^2 &= -2u \, dz \,(dr + v \, dz) + 2\, |r \Theta + \bar{\beta} \, dr + \bar{\gamma} \, dz|^2 \\
\end{align*}
\]

where \(\Theta = \frac{1}{\sqrt{2}} \,(d\vartheta + i \sin \vartheta \, d\varphi)\) and \(\beta = \frac{1}{\sqrt{2}} \,(\beta^1 - i \, \beta^2)\)

\(\gamma = \frac{1}{\sqrt{2}} \,(\gamma^1 - i \, \gamma^2).\)
**NQS tetrad**

\[
\ell = \frac{\partial}{\partial r} - \frac{1}{r} (\bar{\beta} D_v - \beta D_{\bar{v}}) =: D_r
\]

\[
n = u^{-1} (D_z - v D_r)
\]

\[
m = \frac{1}{\sqrt{2} r} \left( \frac{\partial}{\partial \vartheta} - \frac{i}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right)
\]

The \( S^2 \) derivative operator \( \partial \) (edth) acts on a spin-\( s \) field

\[
\partial \eta = \frac{1}{\sqrt{2}} \sin^s \vartheta \left( \frac{\partial}{\partial \vartheta} - \frac{i}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) (\sin^{-s} \vartheta \eta),
\]

\[\text{div} \beta = \partial \bar{\beta} + \bar{\partial} \beta \] is the divergence of a vector field on \( S^2 \), and

\[r D_r = r \partial_r - \nabla_{\beta} = r \partial_r - (\beta \partial + \bar{\beta} \partial),\]

\[r D_z = r \partial_z - \nabla_{\gamma} = r \partial_z - (\gamma \partial + \bar{\gamma} \partial)\] are \( S^2 \)-covariant operators.
Define the auxiliary variables $H, J, K, Q, Q^\pm$ in terms of the metric parameters:

\[
H = u^{-1}(2 - \text{div } \beta)
\]

\[
J = v(2 - \text{div } \beta) + \text{div } \gamma
\]

\[
K = v\bar{\partial}\beta - \bar{\partial}\gamma
\]

\[
Q = rD_z\beta - rD_r\gamma + \gamma
\]

\[
Q^\pm = u^{-1}(Q \pm \bar{\partial}u)
\]
Hypersurface Equations

The Einstein tensor components $G_{\ell\ell}$, $G_{\ell m}$, $G_{\ell n}$ and $G_{mm}$ give equations involving only derivatives tangent to the null hypersurfaces:

\[ r\mathcal{D}_r H = \left( \frac{1}{2} \text{div} \beta - \frac{2|\bar{\partial}\beta|^2 + r^2 G_{\ell\ell}}{2 - \text{div} \beta} \right) H \]

\[ r\mathcal{D}_r Q^- = (\bar{\partial}\bar{\beta} - uH)Q^- + \bar{Q}^- \bar{\partial}\beta + 2\bar{\partial}\bar{\partial}\beta + u\bar{\partial}H - H\bar{\partial}u + 2r^2 G_{\ell m} \]

\[ r\mathcal{D}_r J = -(1 - \text{div} \beta)J + u - \frac{1}{2} u|Q^+|^2 - \frac{1}{2} u \text{ div}(Q^+) - ur^2 G_{\ell n} \]

\[ r\mathcal{D}_r K = \left( \frac{1}{2} \text{div} \beta + i \text{ curl} \beta \right) K - \frac{1}{2} \bar{\partial}\beta J + \frac{1}{2} u\bar{\partial}Q^+ + \frac{1}{4} u(Q^+)^2 \]
\[ + \frac{1}{2} ur^2 G_{mm} \]
The NQS evolution algorithm

Begin with the primary field $\beta$ on a null hypersurface $z = z_0$, and progressively solve the hypersurface constraint equations, viewed as radial ODE’s for the metric parameters:

1. $G_{\ell\ell}$ gives $H$, and thus $u = (2 - \text{div} \beta)/H$
2. $G_{\ell m}$ gives $Q^-$, and thus $Q$ and $Q^+$
3. $G_{\ell n}$ gives $J$
4. $G_{mm}$ gives $K$
5. Solving an elliptic system on $S^2$ determines $\gamma, v$ from $J, K$
6. Determine $\frac{\partial \beta}{\partial z}$ from $Q, \beta, \gamma$
7. Evolve $\beta$ to the next null hypersurface
Eliminating $v$ from the definitions of $J$ and $K$ gives an elliptic system for the vector field $\gamma$ restricted to the 2-sphere $(z, r) = \text{const}$.

$$\partial \gamma + \frac{\partial \beta}{2 - \text{div} \beta} \text{div} \gamma = J \frac{\partial \beta}{2 - \text{div} \beta} - K.$$  

The right hand side is known from solving the hypersurface constraint equations, so we have an elliptic system for $\gamma$. The remaining metric parameter $v$ is then determined, by

$$v = \frac{J - \text{div} \gamma}{2 - \text{div} \beta}.$$  

The primary field $\beta$ is evolved using $Q$:

$$r \frac{\partial \beta}{\partial z} = Q + r \frac{\partial \gamma}{\partial r} + \nabla \gamma \beta - \nabla \beta \gamma - \gamma.$$
The Bianchi II (conservation law) identity $F_{ab;b}^b = 0$ for a symmetric tensor $F_{ab}$ gives equations for the components $F_{mm}, F_{nm}, F_{nn}$ (in NP notation with $\kappa = 0$)

$$0 = \text{Re } \rho F_{mm},$$

$$D_\ell(F_{nm}) = (2\rho + \bar{\rho} - 2\bar{\varepsilon}) F_{nm} + \sigma F_{n\bar{m}} + D_{\bar{m}}F_{m\bar{m}} + (\bar{\pi} - \tau) F_{m\bar{m}},$$

$$D_\ell(F_{nn}) = 2 \text{Re}(\rho - 2\varepsilon) F_{nn} - \text{Re } \mu F_{m\bar{m}} + D_m F_{n\bar{m}} + D_{\bar{m}} F_{nm} + \text{Re}((2\beta + 2\bar{\pi} - \tau) F_{n\bar{m}}).$$

If $\rho \neq 0$ and $F_{nm} = F_{nn} = 0$ on a boundary surface transverse to the null hypersurface, then $F_{m\bar{m}} = F_{nm} = F_{nn} = 0$ everywhere on the null hypersurface. Thus the constraint (subsidiary) equations are propagated by the evolution.
Boundary (Subsidiary) Equations

The Einstein components $G_{nn}, G_{nm}$ yield the evolution ($\frac{\partial}{\partial z}$) relations:

$$r \mathcal{D}_z (J/u) = v^2 r \mathcal{D}_r (J/(uv)) + \left( \frac{1}{2} J - v \right) J/u$$

$$+ 2u^{-1} |K|^2 - \nabla Q^+ v - \Delta v + ur^2 G_{nn}$$

$$r \mathcal{D}_z Q^+ = (v r \mathcal{D}_r + J + \bar{\partial} \bar{\gamma} - v \bar{\partial} \bar{\beta}) Q^+ - K \bar{Q}^+$$

$$+ 2u^{-1} r \mathcal{D}_r (u \bar{\partial} v) - (2 + i \text{curl} \beta) \bar{\partial} v$$

$$+ 2\bar{\partial} K + \bar{\partial} J - 2u^{-1} \bar{\partial} u J - 2ur^2 G_{nm}$$

These equations constrain the boundary conditions for the fields $J/u$ and $Q^+$. At non-boundary points they provide compatibility conditions on the $z$-derivatives.
Free Boundary Data

- The Hypersurface Equations require boundary data (initial conditions) for $H, Q^-, J, K$.
- The Boundary Equations constrain the $z$-evolution of the boundary data for $J/u$ and $Q^+$. 
- The $z$-evolution of $\beta$ is determined everwhere from $Q$. 
- Consequently, the boundary data for $u, K$ are unconstrained (free).
- $u$ determines the starting sphere for the “next” null hypersurface, hence $u$ represents gauge freedom.
- $K$ describes the outgoing radiation (ingoing shear) and is free geometric data.
Aspects of the Numerical Methods

- 8th order Runge-Kutta for the radial integration of the null hypersurface constraint ODEs, with 256 radial steps, rescaled to reach $I^+$. 

- FFT and projection to spin-weighted spherical harmonics used to minimise polar problems and to compute angular derivatives. Resolution is $L = 7, 15$ or $31$. 

- Preconditioned conjugate gradient method to solve the elliptic system on $S^2$ for $\gamma$. 

- 4th order Runge-Kutta for the time evolution with timestep $\Delta z = 0.05$. 
Infalling radial coordinate

Use a radial grid variable $n = 0, \ldots, n_\infty = 256$ and the Schwarzschild radius function

$$r = r(z, n) = 2M \phi^{-1}(\exp(-z/4M)\phi(f(n)/2M)),$$

where $\phi(x) := (x - 1)e^x$, $x \geq 0$. Then $n = \text{const.}$ defines infalling radial curves.

Compactify $\mathcal{I}^+$ by $n_\infty - n = O(r^{-1/2})$, with

$$f(n) = f_1(\nu)/(1 - \nu)^2,$$

where $\nu = n/n_\infty$, $f_1$ monotone on $[0, 1]$. 
Figure 1: Evolution of $r_\beta$ for $0 \leq z \leq 55$. Observe that the infalling grid tracks the dynamical evolution. $n = 0$ is the past horizon $r = 2M$, $n = 256$ is future null infinity $\mathcal{I}^+$. 
Figure 2: Schwarzschild spacetime in Kruskal-Szekeres coordinates.
Numerical convergence tests

Refine code parameters:

- radial resolution $n_\infty = 128, 256, 512, 1024$, shows 8-th order accuracy in radial integrations;
- angular resolution $L = 7, 15, 31$;
- timestep $\Delta z = 0.01, 0.05, 0.1$, shows 4-th order accuracy in timestep;

or vary initial field strength $\beta(z = 0)$:

- weak field run_150, with 1% of the total energy as radiation
- intermediate field run_160, with 20% radiation
- strong field run_170, with 50% radiation
Figure 3: Convergence of $\beta$ with increasing radial resolution: weak field solutions with $n_\infty = 256, 512$ compared to $n_\infty = 1024$. The error decreases by approximately a factor of $2^8$ on doubling the radial resolution.
Figure 4: Convergence of $\beta$ with decreasing time step: weak field solutions for $\Delta z = 0.1, 0.05$, compared against $\Delta z = 0.025$. Where the error is not dominated by the radial discretisation error, the curves show a decrease in error which is consistent with 4th order convergence.
Figure 5: Effect of spectral resolution on constraint quantity $|r^2 G_{nn}|_{z=10}$ at times $z = 10, 20, 30, 40$, for strong (top 4 curves), intermediate (middle 4 curves) and weak (bottom 4 curves) fields.
Accuracy Conclusions

For the data studied (pure $l = 2$ initial $\beta$ with Gaussian profile centered at $r = 20M$), the solutions are

- relatively insensitive to the timestep $\Delta z$;
- improved by increasing $n_\infty$;
- fundamentally limited by the spectral resolution: $L = 15$ corresponds to solving the $L = 10$–truncated Einstein equations.
**Geometric consistency tests**

- Evaluate the constraint equations
  
  - $G_{nn} = G_{nm} = 0$ ("subsidiary" equations)
  
  - $G_{m\overline{m}}$ ("trivial" equation).

- Test the Trautman-Bondi mass loss formula (for $\frac{d}{dz}m_{Bondi}$).

- Test peeling behaviour $\Psi_k = O(r^{k-5})$ for the Weyl curvature components $\Psi_k$, $k = 0, \ldots, 4$. 
Hawking and Bondi Mass

The Hawking mass of the \((z, r) = \text{const.}\) 2-spheres is

\[
m_H(z, r) = \frac{1}{2} r \left( 1 - \frac{1}{8\pi} \oint_{S^2} HJ \right)
\]

where the integral is over the unit 2-sphere and

\[
\oint_{S^2} HJ = \oint_{S^2} \frac{1}{u} (2 - \text{div} \beta)(\text{div} \gamma - v(2 - \text{div} \beta))
\]

The Bondi mass of the null hypersurface is

\[
m_B(z) = \lim_{r \to \infty} m_H(r, z)
\]

and the Trautman-Bondi mass-loss formula is

\[
\frac{d}{dz} m_B(z) = \frac{1}{16\pi} \lim_{r \to \infty} \oint_{S^2(z,r)} H|K|^2.
\]
Figure 6: Difference between $\frac{d}{dz} m_B(z)$ calculated by numerical differentiation, and from the Trautman-Bondi mass-loss formula.
Example: Peeling obstruction

Under generic asymptotic behaviour ($r^\beta$ bounded at scri), we find that $\Psi_0 = O(r^{-4})$, not $O(r^{-5})$ as predicted by the peeling hypothesis.

\begin{align*}
\text{run}_\text{160}: & \quad |r^4\Psi_0|_{t \leq 10} \\
\text{run}_\text{802}: & \quad |r^4\Psi_0|_{t=4}
\end{align*}

Figure 7: Comparison of $r^4\Psi_0$ shows peeling and non-peeling behaviour
Website demonstrations

1. \( r/\beta \) for \texttt{run\_150}, \( z = 0..55 \) — (a) 2D plot with mpeg; (b) 3D surface plot

2. Spectral decay for (a) \texttt{run\_150} with \( l = 0..15 \), (b) \texttt{run\_170} with \( l = 0..10 \), to estimate relative accuracy by \(|l = 10| : |l = 2|\)

3. Hawking mass for \texttt{run\_170}

4. \( dm/dz \) for \texttt{run\_170}

5. Weyl spinor \( r^5\Psi_0 \) for \texttt{run\_160}, \texttt{run\_802}