High order methods for multi-block evolutions in Numerical Relativity

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Motivation

• Need to have non-regular geometries, but simple enough that a semi-structured approach can be followed.

• Some examples: black hole excision (Choptuik's talk), outer spherical boundaries (e.g., compactified approach, Friedrich's talk), co-rotating coordinates.

• Break the domain into subdomains that are topologically cubes and glue them together.

• Numerical energy estimates through difference operators of arbitrary high order satisfying summation by parts (SBP) and penalty terms for the interfaces.

• As an extra, one gets some kind of (non nested) fixed adaptivity.
Matching technique and numerical stability: energy estimates for symmetric systems through penalty terms [Carpenter, Nordstrom and Gottlieb '98]

• Say you want to discretize the advection equation $u_t = cu_x$, in two domains. The Left one covers $(...,0]$, and the Right one $[0,...)$

• We use two fields to describe $u$, $u_L$ and $u_R$. At $x=0$ the two fields are defined, and the solution is multivalued.

\[
\frac{d}{dt} u_j^L = c D u_j^L - \frac{S_j^L}{h \sigma_{00}} (u_j^L - u_j^R) \\
\frac{d}{dt} u_j^R = c D u_j^R - \frac{S_j^R}{h \sigma_{00}} (u_j^R - u_j^L)
\]

• Now discretize using penalty terms:

• And use any operator $D$ satisfying the summation by parts property:

\[
(u, Dv)_\Sigma + (v, Du)_\Sigma = \frac{1}{2} uv_{|1}^L \\
(u, v)_\Delta x = \Delta x \sum_{i,j} <u_i, Hv_j>_\sigma_{ij}
\]
• Define the energy

\[ E = (u, u)_\Sigma^L + (u, u)_\Sigma^R \]

\[ (u, v)_\Sigma^L = h \sum_{i \in \Sigma} \sigma_i u_j v_j \quad (u, v)_\Sigma^R = h \sum_{i \in \Sigma} \sigma_i u_j v_j \]

Take its time derivative and use the SBP property to get

\[ \frac{d}{dt} E = (c - 2S^L)(u_0^L)^2 - (c + 2S^R)(u_0^R)^2 + 2(S^L + S^R)u_0^L u_0^R \]

• If \( c > 0 \), choosing \( S^L = c + \delta \), \( S^R = \delta \) gives

\[ \frac{d}{dt} E = -(u_0^L - u_0^R)^2 (c + 2\delta) \]

• And the energy estimate \( \frac{d}{dt} E \leq 0 \) follows if \( \delta \geq -\lambda/2 \)

• Using \( \delta = -\lambda/2 \) results in a “non-disipative” scheme, \( E = \text{constant} \).

Using \( \delta > -\lambda/2 \) “dissipates”, but only the difference between \( u_L \) and \( u_R \) at \( x = 0 \).

Using \( \delta > 0 \) any mismatch asymptotically decays to zero.

• Can do the same for any linear, variable coefficients symmetric hyperbolic system.
Very high order difference operators satisfying SBP, and associated dissipation
[Kreiss and Scherer '74, Strand '94, Mattsson, Svard and Nordstrom 2004]

- Diagonal and full restricted norms.

The norm is diagonal if \( \sigma_{ij} = \sigma_i \delta_{ij} \), full restricted if \( \sigma_{0i} = 0 \) for \( i \neq 0 \).

- In the diagonal (full restricted) case, the order of the derivative is 2n in the interior and n (2n-1) at and close to boundaries.

- There are some issues in the non-diagonal case.

- Derivatives with minimum bandwidth are not necessarily the optimal ones, as they might have a large spectral radius associated \( \rightarrow \) severe restrictions on the Courant limit.

- Inventory of high order derivatives we have analyzed/whose spectral radius we have “minimized” (notation: order in the interior – order at and close to boundaries):
  * 2-1, 4-2, 6-3, 8-4 (diagonal case)
  * 4-3, 6-5, 8-7 (full restricted case)

- Dissipations: need to be non-positive definite with respect to the SBP scalar product. Mattsson's solution: a prescription for all norms.
Computational infrastructure

- Parallel, modular infrastructure for the CACTUS framework (www.cactuscode.org)
- Uses Erik Schnetter's parallel driver for CACTUS CARPET (www.carpetcode.org).
- SBP thorns with all the derivatives and dissipations just described.
- Modular infrastructure: derivatives, geometries and equations being solved are completely independent of each other. Can choose at runtime different geometries, derivatives, etc.
- If you have a CACTUS code for a first order hyperbolic system, you can use the multipatch infrastructure essentially out of the box.
- Because of its modular nature, this infrastructure has opened the door to many applications (described below).
- The infrastructure allows for overlapping patches, but we haven’t exploited it so far.
Examples

Global convergence order \(-0.5781\)

Time = 0.06
Going beyond proof of concept

- Single distorted black hole simulations with fixed shift (Nis Dorband et al)
- Incorporating and coding better "driver" shift conditions into the Z4 system (Carlos Palenzuela et al) and into our current symmetric hyperbolic system for binary black hole evolutions.
- Accretion processes (Burkhard Zink et al)
- Revisiting Cauchy-perturbative matching (Enrique Pazos et al)
- High order multigrid elliptic solver, possibly for multi-block scenarios (Mark Miller et al).
- In the meantime using parallel, adaptative finite element solver to provide initial data (Matt Andersson et al)
- Visualization for multiple patches (Werner Benger et al).
- Do mesh refinement on each block/patch (Schnetter et al).
Other efforts I: Spectral Einstein Code (SpEC)

- Lawrence Kidder (Cornell), Harald Pfeiffer (Caltech), and Mark Scheel (Caltech)


- Domains can be overlapping or touching. Each individual domain mapped to cube or spherical shell.

- Basis functions are tensor products of Chebyshev, Fourier (for periodic dimensions) or spherical harmonics (for spheres).

- Uses first order strongly hyperbolic systems.

- Outgoing characteristic fields provide boundary conditions on incoming fields of neighboring domains. Use spherical excision boundaries and outer boundaries.

- Excision boundary is outflow boundary (no bc needed). Outer boundary use constraint-preserving boundary conditions.
Other efforts II: high order methods, high resolution shock capturing methods and overlapping patches

- Jonathan Thornburg and Ian Hawke (Albert Einstein Institute)
- Cactus code. Also uses Carpet as underlying parallel driver.

- Overlapping patches communicated through interpolation.
- Can therefore in principle handle first or second order formulations.
- Fourth order vacuum code.
- Uses BSSN formulation of the Einstein’s equations.
- HRSC code for fluid part.
Other efforts III: high order methods and overlapping, moving patches

- Gioel Calabrese (Southampton University) and Dave Neilsen (BYU).
- Wave equation in an axisymmetric boosted rotating black hole background.
- Fourth order code.
- Data between patches communicated via n-th order Lagrangian interpolation for all fields. Outer boundary conditions imposed through Olsson’s orthogonal projections. A pinch of artificial dissipation gives (experimental) stability.
High order numerical schemes:

• Let’s consider diagonal metrics: $\sigma_{ij} = \sigma_i \delta_{ij}$

• In the absence of boundaries standard centered operators of order $2n$ satisfy SBP.

• In the presence of boundaries these operators have to be modified at and near boundaries in order to satisfy SBP.

• The modification at the boundary can be shown to be, necessarily, of order $n$.

• Second and fourth order cases ($n=1$, $n=2$): there is a unique modification near boundaries.
• Sixth order case ($n=3$): mono-parametric family of modifications.
• Eighth order case ($n=4$): three-parametric family.

• The standard choice is to pick up a preferred operator by choosing the one that has the minimum bandwidth.
The spectral radius of the evolution equation and the region of absolute stability of the time integrator

- For an ordinary differential equation
  \[ \frac{d}{dt} u = cu \]
  the region of stability in complex space is the set of c’s for which no exponential growth occurs.

- For a differential equation, say
  \[ \frac{\partial}{\partial t} u = A \frac{\partial}{\partial x} u \]
  the maximum eigenvalue of A has to be inside this region of absolute stability, otherwise the scheme is numerically unstable.

- Let’s take a look at the spectrum for a toy model:
  \[ \frac{\partial}{\partial t} u = \frac{\partial}{\partial x} u \]
  in a periodic domain, divided by an interface, with penalties used for the matching.
Second order case. Maximum = 1.414

The spectrum is purely imaginary, as it should be. The maximum eigenvalue, 1.414, is associated with the operator near the boundary.

Adds a negative real part to the spectrum, but the maximum in the imaginary axis remains essentially unchanged.
Fourth and sixth order cases

Maximum 1.936

Minimum bandwith operator
Maximum 2.129
Eight-th order case

Minimum bandwidth operator:
Maximum 16.04!

Optimized operator:
Maximum 2.242