Introduction to Numerical Field Theory III

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Lecture Summary

- Example error analysis using the advection equation
  - **Richardson expansions**
- Convergence tests
- Independent residual evaluation
- Convergence tests & independent residual evaluation problem for model problem (NKG)
- **Where to next?**
- **APOLOGIES: No Q-balls, but additional lecture on the topic will be available on-line eventually**

Sample Error/Convergence Analysis
The Advection Equation

- Let us consider the solution of the *advection equation*

$$u_t = a\,u_x \quad (a > 0) \qquad t \geq 0, \quad 0 \leq x \leq 1$$
$$u(0,x) = u_0(x)$$

with periodic boundary conditions: i.e. with $x = 0$ and $x = 1$ identified, and where $u_0(x)$ is the initial data function

- Note that $u_0(x)$ must be compatible with periodicity, i.e. we must have $u_0(1) = u_0(0)$

- Given the initial data fcn, we can immediately write down the full solution

$$u(t,x) = u_0((x + at) \bmod 1)$$

where mod is the modulus function that "wraps" $x + a\,t$, $t > 0$, onto the unit interval

- Due to the simplicity and solubility of this problem, we will see that we can perform a rather complete *closed-form* treatment of the convergence of a simple FDA of this eqn

- The point of this exercise, however, is not to advocate parallel closed-form treatments for more complicated (i.e. realistic and/or previously unsolved) problems

- Rather, the key idea to be extracted from what follows is that, in principle and, more importantly, in practice

The error, e^h, of an FDA is no less computable than the solution, u^h itself!

- This observation has *huge* ramifications, one of which is that there is no excuse for publishing solutions of FDAs that do not include error bars, or their equivalents.

- Let us first introduce some difference operators for the usual $O(h^2)$ centred difference approximations of ∂_t and ∂_x

$$D_t u_j^n \equiv \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}$$

$$D_x u_j^n \equiv \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

- Again, we take

$$\Delta x \equiv h$$

$$\Delta t \equiv \lambda \Delta x = \lambda h$$

and hold λ fixed as *h* varies, so that, as usual, our FDA is characterized by the *single* discretization/approximation parameter, *h*

- **First key idea behind error analysis:** Want to view the soln of the FDA as a *continuum* problem

- Therefore, we express both the difference operators and the FD solution as asymptotic series (in h) of differential operators, and continuum fcns, respectively

- **Exercise:** Show that we have the following expansions for D_t and D_x

$$D_t = \partial_t + \frac{1}{6}\lambda^2 h^2 \partial_{ttt} + O(h^4)$$

$$D_x = \partial_x + \frac{1}{6} h^2 \partial_{xxx} + O(h^4)$$

- In terms of the general abstract formulation introduced in the second lecture, we have

$$
\begin{aligned}
L\,u - f = 0 \quad &\Leftrightarrow \quad (\partial_t - a\partial_x)\,u = 0 \\
L^h\,u^h - f^h = 0 \quad &\Leftrightarrow \quad (D_t - aD_x)\,u^h = 0 \\
L^h\,u - f^h \equiv \tau^h \quad &\Leftrightarrow \quad (D_t - aD_x)u \equiv \tau^h = \frac{1}{6}h^2(\lambda^2\partial_{ttt} - a\partial_{xxx})u + O(h^4)
\end{aligned}
$$

- **Second key idea behind error analysis:**

Richardson expansions

- Appeal to LF Richardson's old (1910!) observation that the solution, u^h , of *any* FDA which

  1. Uses a uniform mesh structure with discretization parameter, h
  2. Is completely centred

  should have the following expansion in the limit of vanishing h

$$u^h(t,x) = u(t,x) + h^2 e_2(t,x) + h^4 e_4(t,x) + \cdots$$

$$u^h(t,x) = u(t,x) + h^2 e_2(t,x) + h^4 e_4(t,x) + \cdots$$

- In the above expansion, *u* is the continuum solution, satisfying $Lu = 0$, while e_2, e_4 etc. are (continuum) fcns that ***do not depend on h!!***

- The above expansion is *the* key expansion from which almost all error analysis of FDAs derives

- In the case that the FDA is *not* completely centred, one expects a modified expansion

- In particular, for first order schemes, will have

$$u^h(t,x) = u(t,x) + h e_1(t,x) + h^2 e_2(t,x) + h^3 e_3(t,x) + \cdots$$

$$u^h(t,x) = u(t,x) + h^2 e_2(t,x) + h^4 e_4(t,x) + \cdots$$

- Also note that the existence of a Richardson expansion is completely compatible with the naïve assumption mentioned earlier, namely that

$$\tau_h = O(h^2) \;\rightarrow\; e^h \equiv u - u^h = O(h^2)$$

- However, the Richardson form obviously contains *much* more information than "second order truncation error should imply second order solution error" or, more, generally, that "*p*-the order truncation error should imply *p*-th order solution error"

Richardson expansion dictates the precise form of the h dependence of u^h !!!

and is essentially the *only* basic relationship needed to fully analyze error in *arbitrary* FDAs of *arbitrary* systems of PDEs with smooth solutions!!!

- Let us now return to the advection equation to see how the error analysis flows from the Richardson expansion

- We start from the FDA $L^h u^h - f^h = 0$, and replace both L^h and f^h with continuum expansions

$$L^h u^h - f^h = 0 \;\rightarrow\; (D_t - aD_x)(u + h^2 e_2 + \cdots) = 0$$
$$\rightarrow \left(\partial_t + \frac{1}{6}\lambda^2 h^2 \partial_{ttt} - a\partial_x - \frac{1}{6}\partial_{xxx} + \cdots\right)(u + h^2 e_2 + \cdots) = 0$$

- Now, since h is *arbitrary*, the terms in the above must vanish order-by-order in h

- At $O(1)$ (zeroth-order), have

$$(\partial_t - a\partial_x)u = 0$$

which is simply a statement of the *consistency* of the FDA

- More interestingly, at $O(h^2)$ (second-order), we find

$$(\partial_t - a\partial_x)e_2 = \frac{1}{6}(a\partial_{xxx} - \lambda^2\partial_{ttt})u$$

- Viewing u as a "known" fcn, then this is simply a PDE for the leading order error function, e_2

- **Moreover, the PDE governing e_2 is of *precisely the same nature as the original* PDE, $(\partial_t - a\partial_x)u=0$**

- In fact, we can solve

$$(\partial_t - a\partial_x)e_2 = \frac{1}{6}(a\partial_{xxx} - \lambda^2\partial_{ttt})u$$

explicitly for e_2

- Given the "natural" initial conditions

$$e_2(0,x) = 0$$

(i.e. we initialiaze the FDA with the exact solution so that $u^h = u$ at $t = 0$), and defining $q(x + at)$

$$q(x+at) = \frac{1}{6}a(1-\lambda^2 a^2)\partial_{xxx}u(t,x)$$

we have (**exercise:** verify the following)

$$e_2(t,x) = t\,q((x+at) \bmod 1)$$

$$e_2(t,x) = t\,q((x+at) \bmod 1)$$

- (In passing we note that, as is typical for *any* FDA of a time dependent program, we have *linear* growth of the finite difference error with time (to leading order in *h*)

- Now, the above analysis can be extended to higher order in *h*—what results when one does this is an entire (infinite) *hierarchy* of differential equations for *u* and the error functions e_2 , e_4 , e_6 …

- Indeed, it is extremely useful to keep the following view in mind

  When one solves an FDA of a PDE, one is *not* solving some system that is "simplified" relative to the PDE; rather, one is solving a much *richer* system consisting of an infinite hierarchy of PDEs, one for each fcn appearing in the Richardson expansion

Convergence Tests

- For general systems of PDEs we will *not* be able to solve the PDE that governs u, let alone the PDE that governs e_2—otherwise we wouldn't be considering the FDA in the first place, of course!

- However, it is precisely in this instance that the true power of Richardson's observation becomes most evident!

- The key observation is that by adopting the Richardson expansion as a "working hypothesis"—i.e. we assume that the solution of our PDEs *will* admit a Richardson expansion—and computing FD solns using the same initial data, but with differing values of h, we can learn a great deal about the error in our calculations

- The whole game of investigating the manner in which a particular FDA does or does not converge (i.e. by examining carefully what happens as h is varied) is called *convergence testing*

- It must be stressed at the outset that there are no "hard and fast" rules for convergence testing (i.e. what precisely constitutes "good" convergence, what precisely one should monitor in a given calulation …)

  Rather, one tends to tailor the tests to the specifics of the problem at hand, and being largely an empirical approach, one gains experience and intuition for convergence tests as one works through more and more problems

- However, we will proceed under the assumption that the Richardson expansion, in some form, *always* underlies convergence analysis of smooth solutions deriving from FDAs of PDEs

- Should a Richardson expansion *not* underly a FDA (and I have had arguments with referees about this point, especially in the context of turbulent flows and the like, then all bets re convergence are off, to adopt the vernacular and, indeed, one should be suspicious of the efficacy of the FDA approach for the PDE(s) of interest---and, yes, this means that fully developed turbulent hydrodynamics has been, and continutes to be a real bear!

- A simple example of a convergence test, and the one most commonly used in numerical relativity, for example, is as follows

- We compute three distinct FD solns u^h , u^{2h} and u^{4h} at resolutions h, $2h$ and $4h$, respectively, but using the same initial data (as naturally expressed on the 3 distinct FD meshes)

- We also will assume (not least since it is the usual case) that the FD meshes "line up", i.e. that the $4h$ grid points are a subset of the $2h$ points which, in turn, are a subset of the h points

- Thus, for example, the $4h$ points constitute a common set of events (t^n , x_j) at which specific grid fcn values can be directly (i.e. no interpolation required) and meaningfully compared to one another

- The Richardson expansion tells us that we should expect

$$u^h = u + h^2 e_2 + h^4 e_4 + \cdots$$
$$u^{2h} = u + (2h)^2 e_2 + (2h)^4 e_4 + \cdots$$
$$u^{2h} = u + (4h)^2 e_2 + (4h)^4 e_4 + \cdots$$

- We then compute a quantity, $Q(t)$, which we will call a *convergence factor* , as follows

$$Q(t) \equiv \frac{\| u^{4h} - u^{2h} \|_2}{\| u^{2h} - u^h \|_2}$$

where $\| \cdot \|_2$ is any suitable discrete spatial norm, such as the l_2 norm (RMS value)

$$\| u^h \|_2 = \left(J^{-1} \sum_{j=1}^{J} u_j^h \, ^2 \right)^{1/2}$$

- It is simple to show (**exercise**) that if the FD scheme is converging we should find

$$\lim_{h \to 0} Q(t) = 4$$

- In practice, one can use additional levels of discretization, 8*h*, 16*h*, etc. to extend this test to look for "trends" in $Q(t)$ and, in short, to convince oneself (and, with luck, others), that the FDA really is converging

- Additionally, once convergence of an FDA has been established, then the point-wise subtraction of *any* two solns computed at different resolutions immediately provides an estimate of the level of error in both

- For example, assume that our FD solution u^h is converging as $O(h^2)$, and that we have computed u^{2h} as well

- Then Richardson tells us that we have

$$u^{2h} - u^h = (u + (2h)^2 e_2 + \cdots) - (u + h^2 e_2 + \cdots)$$

$$= 3h^2 e_2 + O(h^4) \sim 3e^h \sim \frac{3}{4} e^{2h}$$

Independent Residual Evaluation

- The astute student will note that the following question concerning convergence testing generically arises

  "OK, so you've established that u^h is converging as h tends to 0—great, but how do you know that what u^h is converging to is u, the desired solution of the continuum problem?"

- The pièce de resistance of convergence testing—and the technique that allows me to claim that I can determine whether anybody's solution to any set of differential equations using any FDA is converging to the desired solution—is independent residual evaluation (MWC 1992)

- Even better, after this lecture you will, in principle at least, have this ability as well!

- As is the case for virtually everything that I have discussed in these lectures, the basic idea underlying independent residual evaluation is very simple

- Again, we start with the abstract description of our continuum PDE(s)

$$Lu - f = 0$$

and the corresponding FDA

$$L^h u^h - f^h = 0$$

- We further assume that we have implemented a program to solve the FDA (again, for the sake of discussion, we will assume that the FDA is $O(h^2)$ accurate), and that we have determined that it is converging to *some* continuum fcn as $O(h^2)$

- Again, however, a successful convergence test does *not* guarantee that the continuum fcn we are computing in the limit is actually u

- Note that implicit in the implementation of the solution of the FDA is the fact that, particularly for multi-dimensional work and/or implicit and/or multi-component FDAs, considerable "work" (i.e. analysis and coding) is generally involved in setting up and solving the algebraic eqns for u^h

- To establish whether our FD soln is or is not converging to u, we consider a distinct (i.e. independent) discretization of the PDE, which we denote by

$$\hat{L}^h \hat{u}^h - f^h = 0$$

- The *only* thing that we need from this FDA for the purposes of the independent residual test is the new FD operator $\hat{L}^h$

- As was the case for L^h, we can expand this independent FD operator in powers of the mesh spacing

$$\hat{L}^h = L + h^2 \hat{E}_2 + h^4 \hat{E}_4 + \cdots$$

where $\hat{E}_2$, $\hat{E}_4$, … are higher order (involve higher order derivatives than L) differential operators

- To perform the independent residual test, we simply take the new operator and apply it to our putative FD soln, u^h, and investigate what happens as h tends to 0

- If u^h *is* converging to the continuum solution, u, we will have

$$u^h = u + h^2 e_2 + O(h^4)$$

- By applying the independent discretization to our numerical solution we will thus compute

$$\begin{aligned}\hat{L}^h u^h - f^h &= (L + h^2\hat{E}_2 + O(h^4))(u + h^2 e_2 + O(h^4)) - f \\ &= Lu - f + h^2(\hat{E}_2 u + Le_2) + O(h^4) \\ &= O(h^2)\end{aligned}$$

Thus, if u^h *is* converging to u, the independent residual calculation *will* produce a residual-like quantity that converges quadratically as h tends to 0!!!!

- Conversely, assume that there is a problem in the derivation and/or implementation of the (original) FDA, $L^h u^h - f^h = 0$, but that there is still convergence, i.e., for example

$$u^{2h} - u^h \to 0 \quad \text{as} \quad h \to 0$$

- Then, we must be able to write something like

$$u^h = u + e_0 + h e_1 + h^2 e_2 + \cdots$$

where the crucial fact/observation is that
the error, $u^h - u$, must have an $O(1)$ component, e_0

- In this case we will compute

$$\begin{aligned}\hat{L}u^h - f^h &= (L + h^2\hat{E}_2 + O(h^4))(u + e_0 + he_1 + h^2e_2 + \cdots) \\ &= Lu - f + Le_0 + hLe + O(h^2) \\ &= Le_0 + O(h)\end{aligned}$$

- The probability that Le_0 vanishes is many, many, many orders of magnitude smaller than that of Canada winning the World Cup in anyone in this lecture room's lifetime, and we will thus *not* observe the expected convergence

- Rather, as we make the grid finer and finer, we will observe that $\hat{L}u^h - f^h$ is tending to a *finite* (i.e. $O(1)$) quantity—a sure sign that something is wrong!

- Again, the astute student will say

  Wait a second! How do I know that a measurement of an $O(1)$ independent residual doesn't indicate that I've screwed up the implementation of the independent residual calculation, rather than the orginal FDA?

- The answer, of course, is "I don't!"

- However, a key point in this regard is that because $\hat{L}^h$ is only used *a posteriori* on a computed solution—and, in particular, is *never* used to compute $\hat{u}$—it is a relatively easy matter to ensure that $\hat{L}^h$ has been implemented in an error-free fashion, ideally using symbolic computing (Maple, Mathematica, etc.)

- Also importantly, many of the restrictions one must place on the "real" discretization $L^h u^h = f^h$ such as the need for the FDA to be stable, or the ease of solution of the resulting algebraic eqns, do not apply to $\hat{L}^h$

- Indeed, I emphasize to students that an independent residual is analogous to the scaffolding that a contractor constructs when putting up a new multi-story structure: contractors who put the scaffolding up *after* construction has started aren't likely to have successful careers

- That is, especially for complicated systems of PDEs (which is most of the PDEs of current interest in theoretical physics), **one should *always* code the independent residual *first***

- **Psychological/sociological note:**

  Implementing the IR first is the sort of manuever that doesn't come naturally for most physicists since, as is the case of scaffolding, the independent residual appears to the non-practitioner (i.e. someone who has actually determined one or more previously uncomputed solutions to one or more PDEs) to be wasted effort since it doesn't show up in the finished product. That is, training in traditional theoretical physics favours, quick-and-dirty, one-off, do-it-as-many-ways-as-you-can (if you're of the Feynman camp)

- This seat-of-the-pants, shoot-from-the-hip approach doesn't tend to work well in computational physics. Again, with current computer languages, no detail is irrelevant, and the sooner one adopts a rigorous, engineering approach to one's numerical analysis, the better.

- **Psychological/sociological note (cont.):**

- Moreover, I am happy to report that I have observed strong correlation of
  - The quality of student vis a vis solving PDEs using FDAs
  - The likelihood that that student will take my advice concerning when the IR should be coded seriously

Independent Residual for NKG eqn

- Recall that our model PDE for the (complex) nonlinear Klein Gordon field is

$$\nabla^{\mu}\nabla_{\mu}\phi = -\phi_{tt} + 3\frac{\partial}{\partial(r^3)}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{dV}{d\phi^*} = c_1\phi + c_2 \mid \phi \mid \phi + c_3 \mid \phi \mid^2 \phi$$

  and that I left it as an exercise for you to show that this could be re-written as

$$r\nabla^{\mu}\nabla_{\mu}\phi = -(r\phi)_{tt} + (r\phi)_{rr} = r\frac{dV}{d\phi^*} = c_1(r\phi) + c_2 \mid \phi \mid (r\phi) + c_3 \mid \phi \mid^2 (r\phi)$$

- Introducing the usual second-order ($O(h^2)$) approximations to the second derivatives ∂_{tt} and ∂_{rr}

$$r\nabla^{\mu}\nabla_{\mu}\phi = -(r\phi)_{tt} + (r\phi)_{rr} = r\frac{dV}{d\phi^*} = c_1(r\phi) + c_2 \mid \phi \mid (r\phi) + c_3 \mid \phi \mid^2 (r\phi)$$

- Introducing the usual second-order ($O(h^2)$) approximations to the second derivatives ∂_{tt} and ∂_{xx} in the form of difference operators D_{tt} and D_{rr}

$$D_{tt} = \frac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{\Delta t^2}$$

$$D_{rr} = \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta r^2}$$

the following is our independent residual, R_I

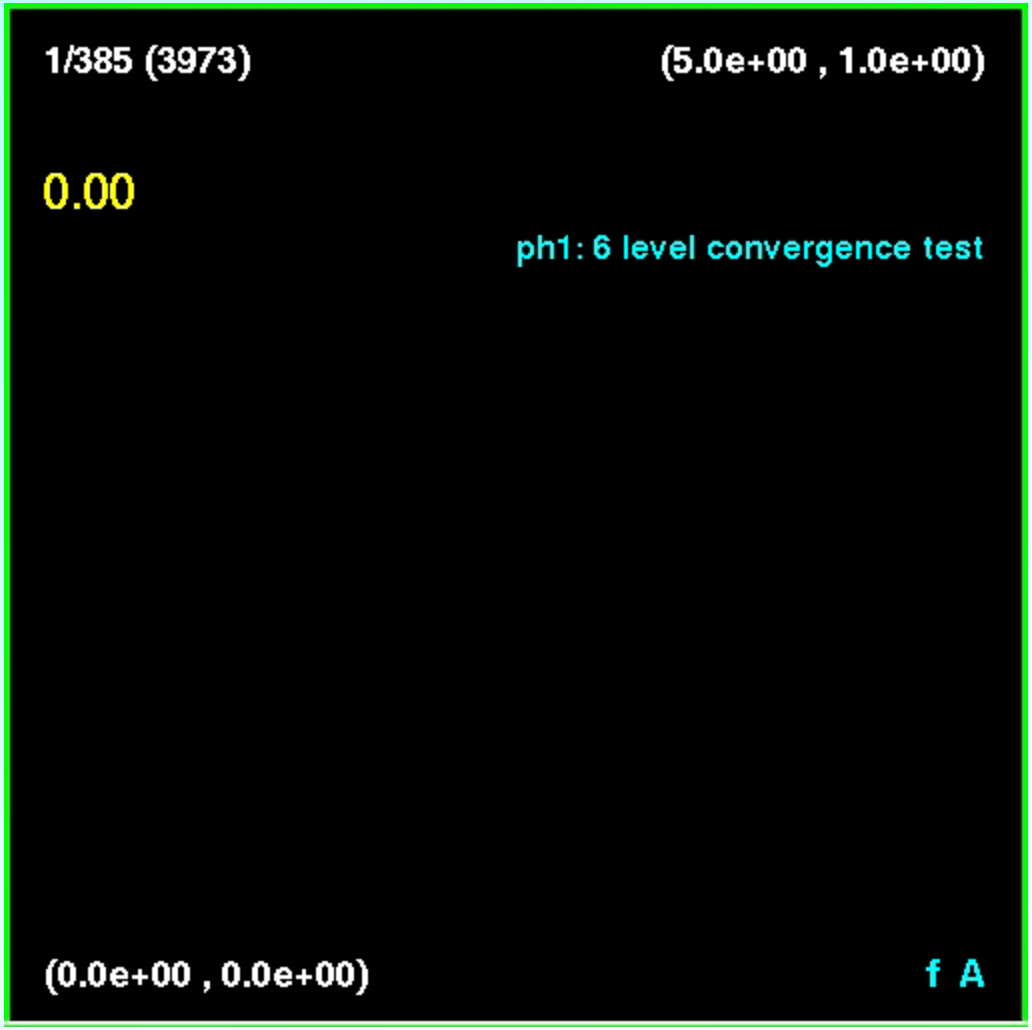
$$R_I = -D_{tt}(r\phi_j^n) + D_{rr}(r\phi_j^n) - r\frac{dV}{d\phi^\star}$$

$$= -D_{tt}(r\phi_j^n) + D_{rr}(r\phi_j^n) - c_1\, r_j\phi_j^n + c_2 \mid \phi_j^n \mid r_j\phi_j^n + c_3 \mid \phi_j^n \mid^2 r_j\phi_j^n$$

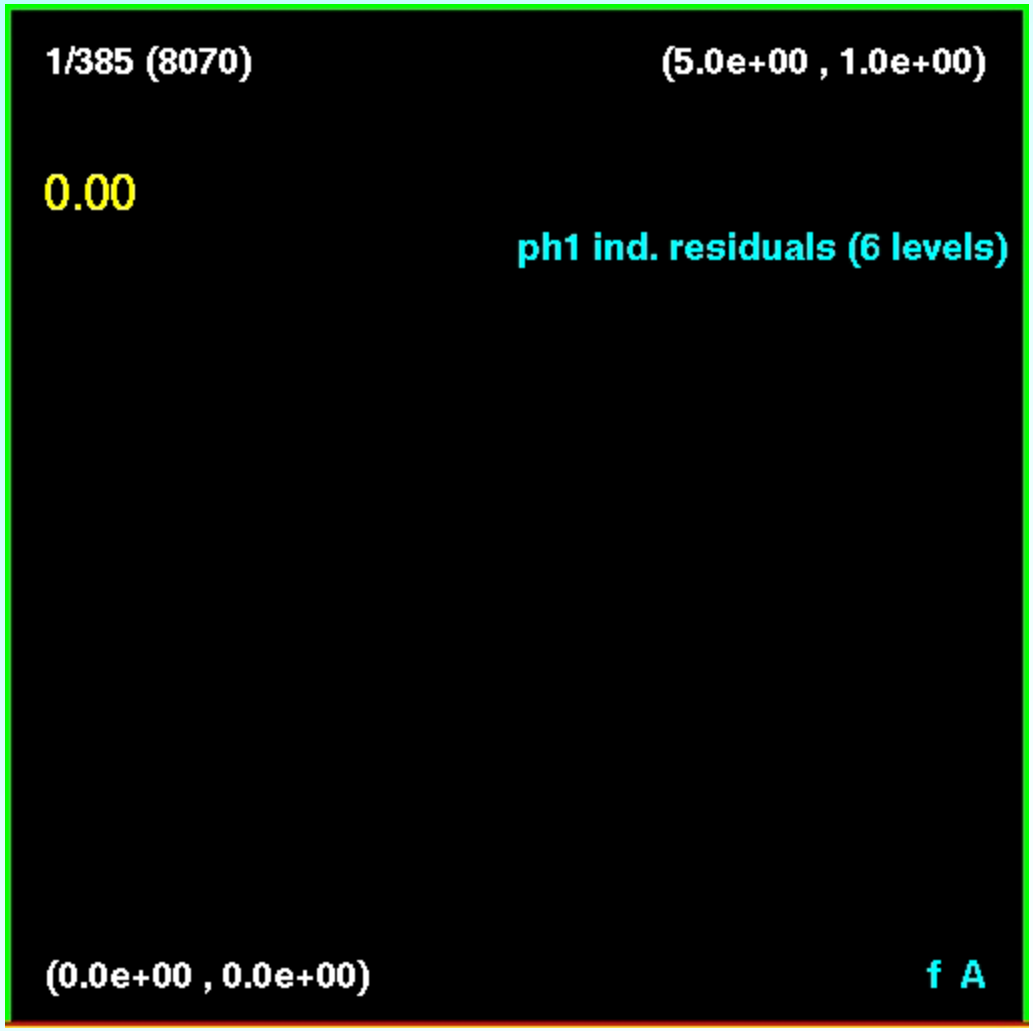
- Note that we are not evaluating independent residuals for the boundary conditions—in practice we should *definitely* do this particularly if the BC's are *differential* (Dirichlet conditions can be checked by inspection!)

Convergence Tests & Independent Residual Evaluation for Model Problem

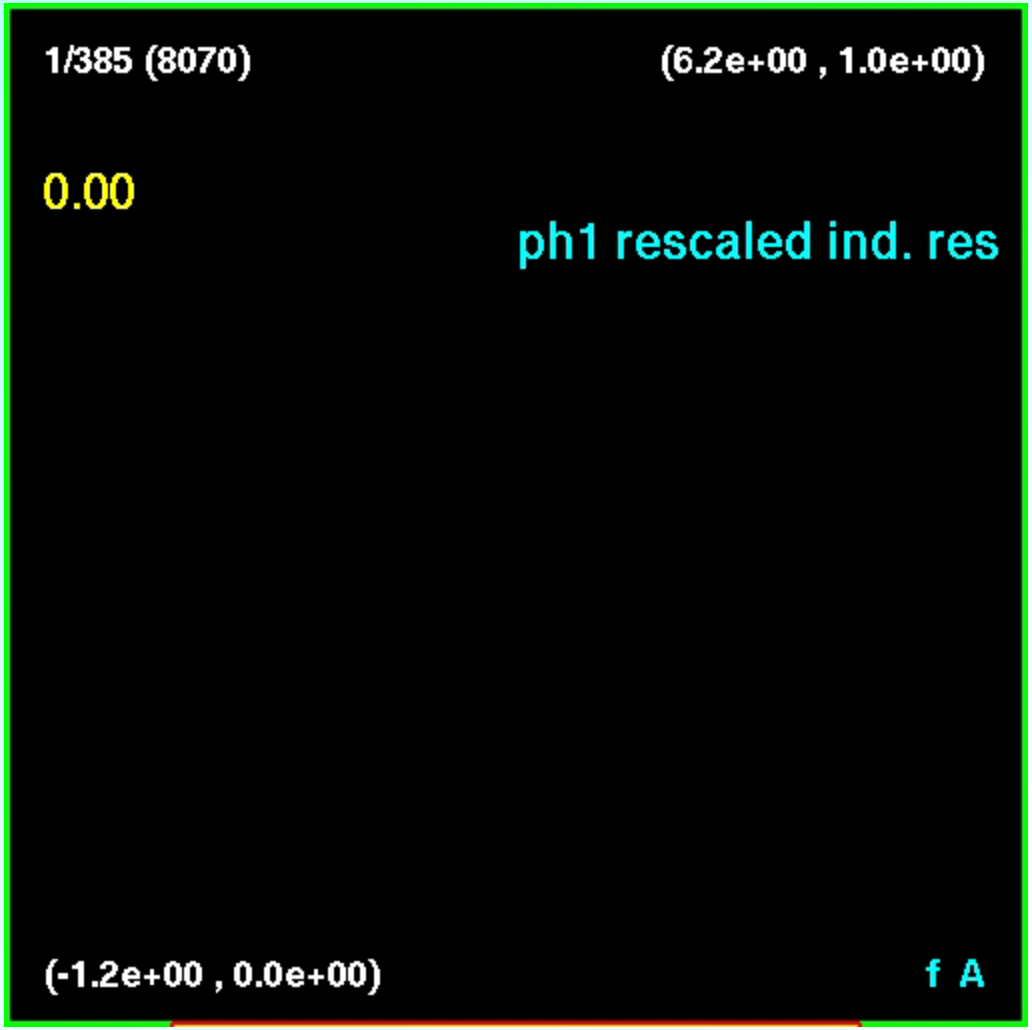
6-level Convergence Test of φ_1



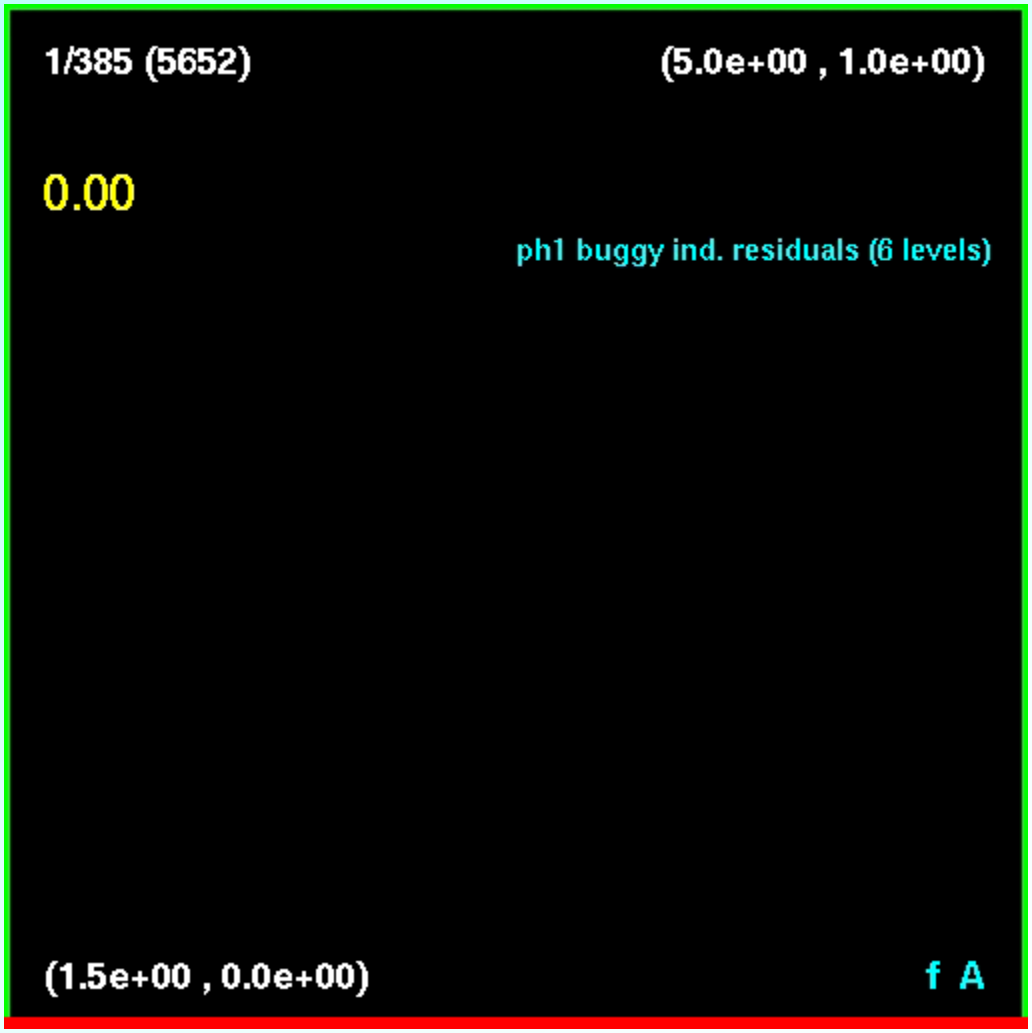
6-level Independent Residual Test of φ_1



6-level Scaled Independent Residual Test of φ_1



Actual failed IR test (Tue AM)
Forgot factor of *r* multiplying dV/dφ* (i.e. difference solution was correct, IR was wrong!)



Where to from here?

- Will be setting up a web page (which will be accessible via my home page -> "Previous Teaching" (google "choptuik" to locate my home page) that contains
  - These lectures
  - Links to further associated material
  - Links to instructions for installing software, including all of the code used to generate results shown in these lectures
  - Additional lecture on *Q*-balls
  - Problems/exercises
    - Will happily look at any code you write in instances where you are having difficulty
    - Will happily "grade" any solutions you submit (can't guarantee a very quick turn around, although I will try to be responsive

Where to from here?

UBC of course!!

Think about joining us if you're considering pursuing graduate studies outside of Brazil

Excerpts from Richardson's 1910 Paper
Phil. Trans. Roy. Soc. **210** 307—357

extrapolate in the manner explained below.

It is known* that when central differences are used, the expansions of the differential coefficients of a function in terms of its differences contain only alternate powers of the co-ordinate difference h. The same is true for partial differential coefficients and for products of differential coefficients. Consequently the error of the representation of any differential expression by central differences is of the form $h^2F_2(x, y, z)$ $+h^4F_4(x, y, z)+$ terms in higher powers of h^2, where F_2, F_4, &c., are independent of h.†

Next, as to the error of the finite-difference-integral ϕ. This is the infinitesimal integral of a differential equation having the error $h^2F_2(x, y, z)+h^4F_4(x, y, z)+$, &c. Let $\wp$ be the integral of the correct differential equation. Then, if we write

$$\phi(x, y, z) = \wp(x, y, z)+m\psi_1(x, y, z)+m^2\psi_2(x, y, z)+ \text{ terms}$$

in higher powers of m, it follows that a differential expression of any order and degree for ϕ differs from the corresponding one for $\wp$ by

$m \times$ (a function of the differential coefficients of $\wp$ and of ψ_1)+ terms in m^2, m^3, &c.,

provided only that m is independent of the co-ordinates. Now, identifying m with h^2, it follows that: *the errors of the integral and of any differential expressions derived from it, due to using the simple central differences of* § 1·1 *instead of differential coefficients, are of the form*

$$h^2f_2(x, y, z)+h^4f_4(x, y, z)+h^6f_6(x, y, z)+, \text{ &c.}$$

proportional to h^2. Peculiarities present themselves on the boundary, but it is easy to see that errors will be of the form $h^2f_2+h^4f_4+$, &c., provided that in passing from one table to another each part is either kept infinitesimally correct, or else is worked by differences whose size in the one table bears a constant ratio to that in the other.

An extrapolation can only be made where the tabular points of the several tables coincide with one another. It is conceivable that in the future some method will be found of defining a continuous function in terms of the discrete body and boundary values, so that this continuous function shall have an error of the form $h^2f_2(x, y, z)+h^4f_4(x, y, z)+$, &c., everywhere. Extrapolation would then be possible everywhere.

An excellent illustration is afforded by Lord RAYLEIGH's account of the vibration of a stretched string of beads ('Sound,' vol. I., § 121). He gives the frequency of the fundamental for the same mass per unit length concentrated in various numbers of beads. This is reproduced below in the table. The co-ordinate difference h is inversely as one plus the number of beads, not counting beads at the fixed ends.

Number of free beads + one . .	2	3	4	5	10	20	40	∞
Ratio of frequency to that of continuous string	·9003	·9549	·9745	·9836	·9959	·9990	·9997	unity
Error in representation of continuous string by string of beads	·0997	·0451	·0255	·0164	·0041	·0010	·0003	·0000
Ratio of error to square of co-ordinate difference × a constant	·3988	·4059	·4080	·4091	·4107	·4111	·4112	·4112

The degree of constancy of the last line shows that if we found the frequency for one bead and for three, then extrapolation, on the assumption that the error is proportional to h^2, would give us the frequency for the continuous string to about one part in 1000; which is as near as we could get by twenty beads and no extrapolation. While extrapolation from the exact solutions for four beads and for nine would leave an error of only one in 50,000. Other examples of extrapolation will be found in § 3·1.

§ 2. *Procedure when the Conditions allow the Integral to be Marched out from a Part of the Boundary.*

§ 2·0. *Historical.*—Step-by-step arithmetical methods of solving ordinary difference equations have long been employed for the calculation of interest and annuities. Recently their application to differential equations has been very greatly improved by the introduction of rules allied to those for approximate quadrature. The papers referred to are:—

RUNGE, "Über die numerische Auflösung von Differentialgleichungen," 'Math. Ann.,' Bd. 46. Leipzig, 1895.

Excerpts from Richardson's 1910 paper

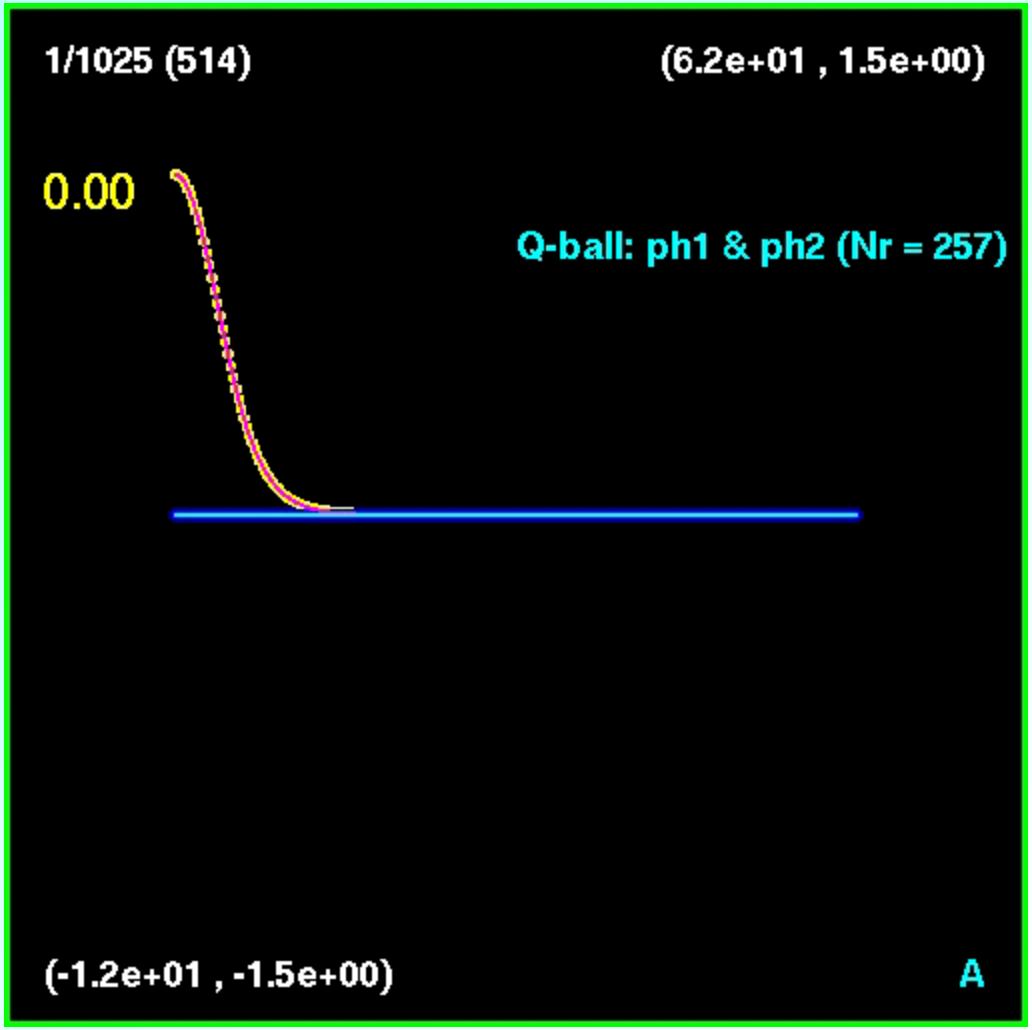
§ 3·2·5. *Routine of Approximation.—Time and Cost.*—To anyone setting out on a problem I offer the following experience as a guide in forming estimates:—It was found convenient to enter certain stages on a table with large squares, each divided into compartments. Thus for $\frac{\delta^4\phi}{\partial x^4} + 2\frac{\delta^4\phi}{\delta x^2 \delta y^2} + \frac{\delta^4\phi}{\delta y^4} = 0$, one of the squares is shown in the annexed table. All the quantities in it refer to the central point of the square. The intermediate stages are done on rough paper and thrown away. So far I have paid piece rates for the operation $\delta_x^2 + \delta_y^2$ of about $\frac{n}{18}$ pence per co-ordinate point, n being the number of digits. The chief trouble to the computers has been the intermixture of plus and minus signs. As to the rate of working, one of the quickest boys averaged 2,000 operations $\delta_x^2 + \delta_y^2$ per week, for numbers of three digits, those done wrong being discounted.

TABLE.

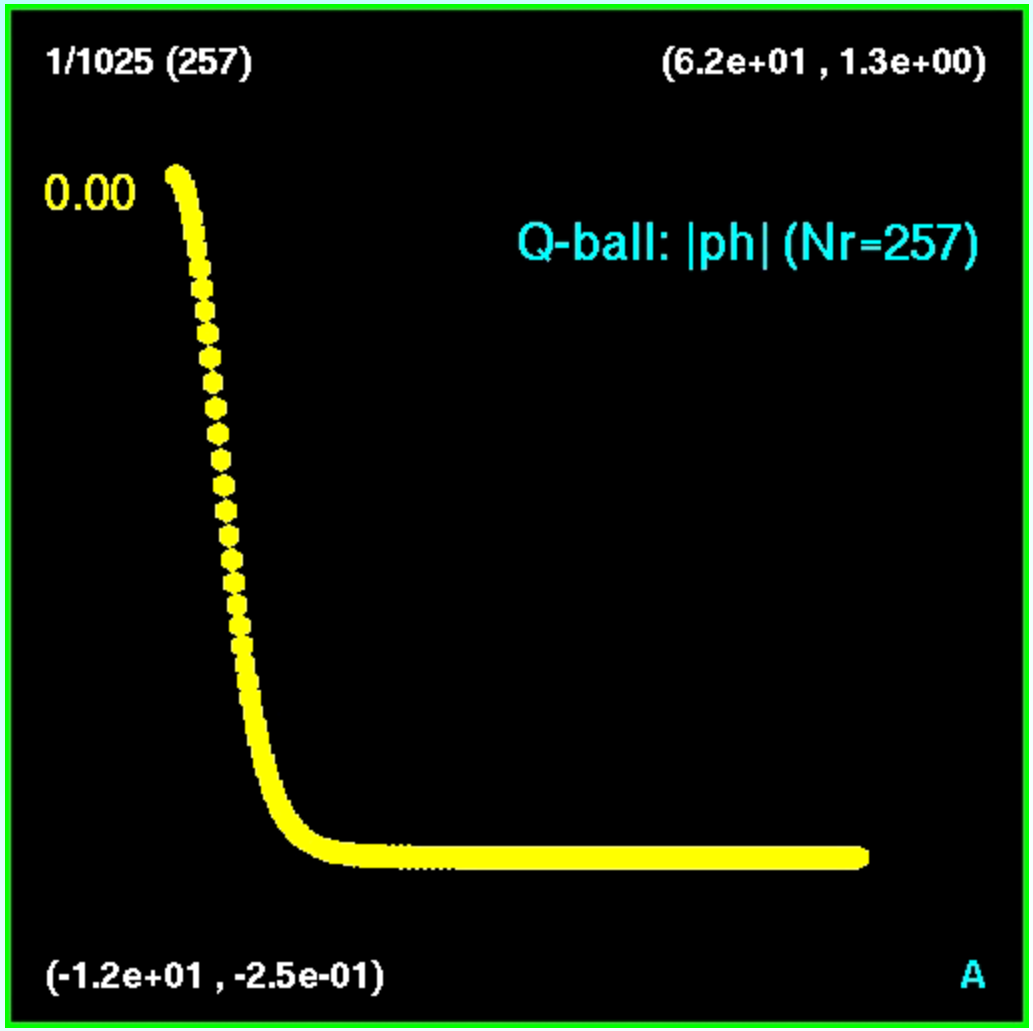
ϕ_1	$\nabla^2\phi_1$	$\nabla^4\phi_1$
ϕ_2	$\nabla^2\phi_2$	$\nabla^4\phi_2$
ϕ_3	&c.	

Q-Ball Results

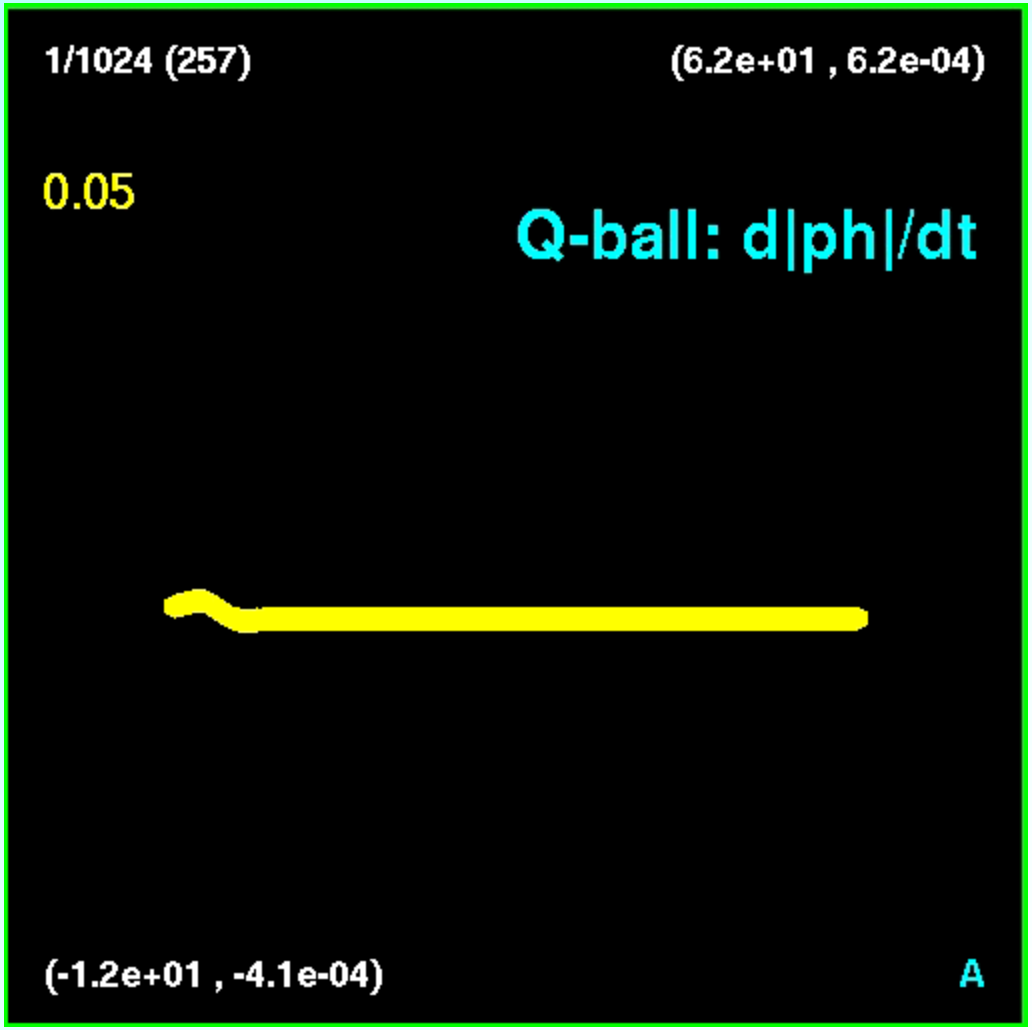
Evolution of real and imaginary field components



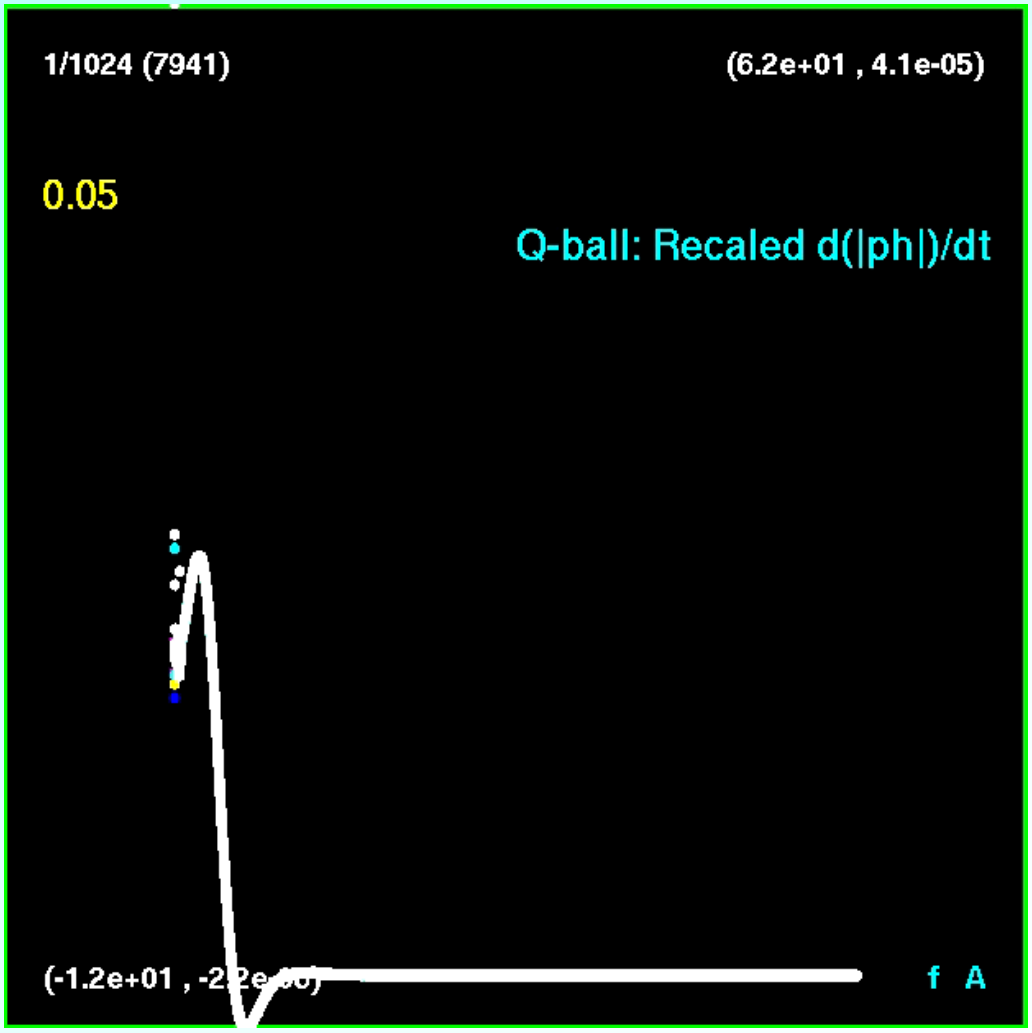
Evlolution of field modulus



Evolution of d(phi)/dt (level 8, $N_x = 257$)



6-level recaled d(phi)/dt



Rescaled independent residuals for φ_1